14.3 and 14.4 Divisor Methods and Which Method is Best

We have used the standard divisor, *s*, to represent the average district population. We will use *s* for all apportionment methods to calculate the quota.

The divisor methods will also use an adjusted divisor, *d*, to calculate an adjusted quota. The adjusted quota combined with the appropriate rounding rules for each method will give the final apportionment for divisor methods.

Jefferson Method

Step 1 Compute the standard divisor.
Step 2 Compute the quota for each "state" (group).
Step 3 Round each quota *down*.
Step 4 If the total number of seats is not correct, call the current apportionment N, and find new divisors, $d_i = \frac{p_i}{N_i+1}$, that correspond to giving each state one more seat.
Step 5 Assign a seat to the state with the *largest d*. (Notice that divisor methods look at the entire number of *d* rather than the fractional part of the number.)

Repeat Steps 4 and 5 until the total number of seats is correct. The last d_i used is the adjusted divisor, *d*.

Example

Let's return to the splitting of the 36 silver coins. Use the Jefferson method to distribute the coins.

$$s = \frac{14900}{36} \approx 413.8888889$$

inc, so pick largest

$d_i = \frac{p_i}{N_i + 1}$

N

Person	Cont.	q	Rounded quota	d_i	Jefferson App.
Doris	\$5900	5900/413.89 14.2550	14	$\frac{5900}{14+1} = 393.\overline{3}$	14
Mildred	\$7600	18.3624	18	$\frac{7600}{18+1} = 400$ * +1	19
Henrietta	\$1400	3.3826	3	$\frac{1400}{3+1} = 350$	3
TOTAL	**\$14,900**		35		36

36−35 = 1 coin left to apportion

$d =$ 400

Example

When the ladies opened the bag of coins, they discovered that there were 37 coins. Use the Jefferson method to apportion the coins.

$s = \frac{14900}{37} \approx 402.7027$

inc, so pick largest $\lfloor q \rfloor$ $d_i = \frac{p_i}{N_i+1}$

inc, so pick largest

Person	Cont.	q	Rounded quota	d_i	Next App.	Next d_i	Jefferson App.
D	\$5900	$\frac{5900}{402.7027}$ = 14.6510	14	$\frac{5900}{14+1}$ = 393.33	14	☆ +1 393.33	15
M	\$7600	18.8725	18	$\frac{7600}{18+1}$ 400 ☆ +1	19	$\frac{7600}{19+1}$ = 380	19
H	\$1400	3.4765	3	$\frac{1400}{3+1}$ 350	3	350	3
TOTAL	**\$14,900**		35		36		37

37−35 = 2 coins left to apportion

$d =$ 393.33

		$\frac{p_i}{d}$	$\lfloor \frac{p_i}{d} \rfloor$
D	5900	15.000	15
M	7600	19.32	19
H	1400	3.56	3

Webster Method

Step 1 Compute the standard divisor.

Step 2 Compute the quota for each “state” (group).

Step 3 Round each quota *to the nearest integer.*

Step 4 If the total number of seats is not correct, call the current apportionment *N*, and find new divisors.

If the number of seats needs to increase, use $d_i^+ = \frac{p_i}{N_i+0.5}$.

If the number of seats needs to decrease, use $d_i^- = \frac{p_i}{N_i-0.5}$.

Step 5 Adjust the seats according to *d*.
If the number of seats needs to increase, assign a seat to the state with the largest d_i^+.
If the number of seats needs to decrease, remove a seat from the state with the smallest d_i^-.
Repeat Steps 4 and 5 until the total number of seats is correct.
The last d_i used is the adjusted divisor, *d*.

Example

Let's return to the splitting of the 36 silver coins. Use the Webster method to distribute the coins.

$$s = \frac{14900}{36} \approx 413.8888889$$

inc, so pick largest

[q] $d_i^{+} = \frac{p_i}{N_i + 0.5}$

Person	Cont.	q	Rounded quota	d_i	Webster App.
Doris	\$5900	14.2550	14	$\frac{5900}{14+0.5} = 406.896$	14
Mildred	\$7600	18.3624	18	$\frac{7600}{18+0.5} = 410.811$ ☆	19
Henrietta	\$1400	3.3826	3	$\frac{1400}{3+0.5} = 400$	3
TOTAL	**\$14,900**		35		36

36−35 = 1 coin left to apportion

$d =$ 410.811

Example

Apportion the regions below using the Webster method for a house size of 16.

$s = \frac{45,173}{16} = 2823.3125$

decrease, so pick smallest

pop/s [9] $d_i^- = \frac{p_i}{N_i - 0.5}$

Region	Pop.	q	Rounded quota	d_i	Webster App.
Beach	28,204	$\frac{28204}{2823.3125} =$ 9.990	10	$\frac{28204}{10-0.5}$ 2968.842	10
Forest	11,267	3.991	4	$\frac{11267}{4-0.5} =$ 3219.143	4
Plains	4,240	1.502	2	$\frac{4240}{2-0.5} =$ $2826.\overline{6}$ ☆ −1	1
Swamp	1,462	0.518	1	$\frac{1462}{1-0.5} =$ 2924	1
TOTAL	45,173		17		16

16 − 17 = −1 seats to apportion

$d = 2826.\overline{6}$

Hill-Huntington Method

The Hill-Huntington method does a great job of keeping the relative differences of representative share $\left(\text{i.e., } \frac{\text{apportionment}}{\text{population}}\right)$ and district population $\left(\text{i.e., } \frac{\text{population}}{\text{apportionment}}\right)$ stable between states. It also ensures that every group gets at least one representative, so it favors small states. Since 1941, the Hill-Huntington method with a house size of 435 has been used to apportion the House of Representatives.

Step 1 Compute the standard divisor.

Step 2 Compute the quota for each "state" (group).

Step 3 Round each quota *according to the geometric mean* of $\lfloor q \rfloor$ and $\lceil q \rceil$, $q^* = \sqrt{\lfloor q \rfloor \lceil q \rceil}$.

Step 4 If the total number of seats is not correct, call the current apportionment N, and find new divisors.
If the number of seats needs to increase, use $d_i^+ = \frac{p_i}{\sqrt{N_i(N_i+1)}}$.
If the number of seats needs to decrease, use $d_i^- = \frac{p_i}{\sqrt{N_i(N_i-1)}}$.

Step 5 Adjust the seats according to *d*.
If the number of seats needs to increase, assign a seat to the state with the largest d_i^+.
If the number of seats needs to decrease, remove a seat from the state with the smallest d_i^-.
Repeat Steps 4 and 5 until the total number of seats is correct.
The last d_i used is the adjusted divisor, *d*.

Example

Let's return to the splitting of the 36 silver coins. Use the Hill-Huntington method to distribute the coins.

$$s = \frac{14900}{36} \approx 413.8888889$$

inc, so pick largest

$\sqrt{\lfloor q \rfloor \lceil q \rceil}$

$d_i^{+} = \frac{p_i}{\sqrt{N_i(N_i+1)}}$

Person	Cont.	q	q^*	Rounded quota	d_i	HH App.
Doris	\$5900	14.2550	$\sqrt{14 \cdot 15} =$ 14.4914	14	$\frac{5900}{\sqrt{14 \cdot 15}} =$ 407.139	14
Mildred	\$7600	18.3624	$\sqrt{18 \cdot 19}$ 18.4932	18	$\frac{7600}{\sqrt{18 \cdot 19}} =$ +1 410.961	19
Henrietta	\$1400	3.3826	$\sqrt{3 \cdot 4} =$ 3.4641	3	$\frac{1400}{\sqrt{3 \cdot 4}} =$ 404.145	3
TOTAL	**\$14,900**			35		36

36−35 = 1 coin left to apportion

$d =$ 410.961

Example

Apportion the regions below using the Hill-Huntington method for a house size of 16.

dec, so pick smallest

$$s = \frac{45{,}173}{16} = 2823.3125$$

$d_i = \frac{p_i}{\sqrt{N_i(N_i - 1)}}$

Region	Pop.	q	q^*	Rounded quota	d_i	HH App.
Beach	28,204	9.990	$\sqrt{9 \cdot 10} =$ 9.4868	10	$\frac{28204}{\sqrt{10(9)}} =$ ☆ −1 2972.963	9
Forest	11,267	3.991	$\sqrt{3 \cdot 4} =$ 3.4641	4	$\frac{11267}{\sqrt{4(3)}} =$ 3252.503	4
Plains	4,240	1.502	$\sqrt{1 \cdot 2} =$ 1.4142	2	$\frac{4240}{\sqrt{2(1)}} =$ 2998.133	2
Swamp	1,462	0.518	$\sqrt{0 \cdot 1} =$ 0	1	$\frac{1462}{\sqrt{1(0)}} =$ undefined	1
TOTAL	**45,173**			17		16

16 − 17 = −1 coin left to apportion

$d =$ 2972.963

A ***paradox*** is a statement that is seemingly contradictory or opposed to common sense and yet is perhaps true.

Possible Issues - Alabama Paradox (Section 14.2)

The ***Alabama paradox*** occurs when a state loses a seat as the result of an increase in the house size.

Example

When you used the Hamilton method to apportion 36 silver coins to the ladies, Doris received 14, Mildred received 18, and Henrietta received 4 coins. Assume that they found an extra coin and use the Hamilton method to apportion 37 silver coins to the ladies.

$$s = \frac{14{,}900}{37} \approx 402.703$$

L2L

Person	Contribution	q	Rounded quota	Hamilton Apportionment	from page 3 of Notes w/ 36 coins
Doris	\$5900	14.6510	14 +1	15	14 Gained 1
Mildred	\$7600	18.8725	18 +1	19	18 Gained 1
Henrietta	\$1400	3.4765	3	3	4 lost 1
TOTAL	**\$14,900**		35	37	36

37-35=2 coins left to apportion

What information tells you that the Alabama paradox occurred in this example? An extra coin was found (house size increased), but Henrietta lost a coin (seat) even though no other changes occurred in the problem.

Possible Issues - Population Paradox (Section 14.2)

Consider two numbers, A and B, where $A > B$.
The ***absolute difference*** between the two numbers is $A - B$

The ***relative difference*** between the two numbers is $\frac{A-B}{B} \times 100\%$

The ***population paradox*** occurs when there are a fixed number of seats and a reapportionment causes a state to lose a seat to another state even though the percent increase in the population of the state that loses the seat is larger than the percent increase of the state that wins the seat.

Example

Earlier, we apportioned 100 council members to four districts. Initially, North had 42 seats, South had 27 seats, East had 30 seats, and West had 1 seat. Ten years later, the county reapportioned the 100 council seats using new population data. Use the Hamilton method for this apportionment

$s = \frac{65910}{100} = 659.1$

$\lfloor q \rfloor$

District	Population	q	Rounded quota	Hamilton Apportionment	
North	28,140	$\frac{28140}{659.1} = 42.6946$	42	+1 43	Gained 1
South	17,450	26.4755	26	26	Lost 1
East	19,330	29.3279	29	29	Lost 1
West	990	1.5020	1	+1 2	Gained 1
TOTAL	65,910		98	100	

100 − 98 = 2 seats left to apportion

Compare the districts' populations.

District	Initial Pop.	New Pop.	Absolute Difference	Relative Difference
North	27,460	28,140	28140 − 27460 = 680	$\frac{680}{27460} \cdot 100\% =$ 2.476%
South	17,250	17,450	17450 − 17250 = 200	$\frac{200}{17,250} \cdot 100\% =$ 1.1594%
East	19,210	19,330	19330 − 19210 = 120	$\frac{120}{19210} \cdot 100\% =$ 0.6247%..
West	1000	990	1000 − 990 = decrease 10	$\frac{10}{990} \cdot 100\% =$ Decrease $1.\overline{01}\%$

Did the population paradox occur? Yes

Explain what information helped you determine whether or not the population paradox occurred.

South and East both lost a seat. One of them lost a seat to West even though they had a larger increase than West (in fact, West had a decrease).

Possible Issues – New States Paradox (Section 14.2)

The ***new states paradox*** occurs in a reapportionment in which an increase in the total number of states (with a proportionate increase in representatives) causes a shift in the apportionment of existing states.

Example

A country has two states, Solid and Liquid. Use Hamilton's method to apportion 12 seats for their congress

$s = \frac{203995}{12} \approx 16999.58$

larger fractional portion gets extra seat

Lqd

State	Population	q	Rounded quota	Hamilton Apportionment
Solid	144,899	$\frac{144,899}{16999.58} = 8.524$	8	+1 9
Liquid	59,096	3.476	3	3
TOTAL	203,995		11	12

12 − 11 = 1 seat left to apportion.

Another state, Plasma, wants to join. If there are 38,240 people in that state, how many representatives should they receive?

$\frac{38,240}{S} = \frac{38,240}{16999.58} = 2.25$ They will add 2 representatives

Use Hamilton's method to apportion the seats for their congress (the 12 original seats plus the additional seats that were added when Plasma joined).

$s = \dfrac{242{,}235}{\underset{\text{orig}}{12} + \underset{\text{new}}{2}} = \dfrac{242{,}235}{14} = 17302.5$

Lq⌋

State	Population	q	Rounded quota	Hamilton Apportionment
Solid	144,899	$\frac{144{,}899}{17302.5} = 8.374$	8	8
Liquid	59,096	3.415	3	+1 4
Plasma	38,240	2.210	2	2
TOTAL	242,235		13	14

14 − 13 = 1 seat left to apportion

What information tells you that the new states paradox occurred in this example?

We added a new state (Plasma) with the proportionate number of representatives, but Solid lost a seat to Liquid. The existing apportionment shifted when the new state was added.

Possible Issues – Quota Condition (Section 14.3)

Example

A school offers four different art classes with the enrollments shown below. Ten new teachers will be hired according to an apportionment using Jefferson's method. Determine who gets the new teachers.

$s = \frac{1192}{10} = 119.2$

inc, so pick largest

L2) $d_i = \frac{p_i}{N_i + 1}$

Class	Enrollment	q	Rounded quota	d_i	Next App.	Next d_i	Jefferson App.
Ceramics	785	$\frac{785}{119.2} =$ 6.586	6	$\frac{785}{6+1} =$ 112.14 ☆	+1 7	$\frac{785}{7+1} =$ 98.13	+1 8
Painting	152	1.275	1	$\frac{152}{1+1} =$ 76	1	76	1
Dance	160	1.342	1	$\frac{160}{1+1} =$ 80	1	80	1
Theatre	95	0.797	0	$\frac{95}{0+1} =$ 95	0	95	0
TOTAL	1192		8		9		10

10−8 = 2 seats left to apportion

The ***quota condition*** says that the number assigned to each represented unit must be the standard quota, q, rounded up or rounded down.

What information tells you that the quota condition was violated in this example? Ceramics is larger than quota rounded up

Comparing Methods

Balinski and Young found that no apportionment method that satisfies the quota condition is free of paradoxes.

- Divisor methods are free of the paradoxes, but they can violate the quota condition.
- Hamilton's method may have paradoxes but does not violate the quota condition.

Sample Exam questions

Sample exam questions are likely to focus on performing all four apportionment methods and recognizing each of the four issues (three paradoxes and the quota condition).