

## CHAPTER 13 – FAIR DIVISION

We have three goals for “fairness”.

- A fair division procedure is *equitable* if each player believes he or she received the same fractional part of the total value.
- A fair division procedure is *envy-free* if each player has a strategy that can guarantee him or her a share of whatever is being divided that is, in the eyes of that player, at least as large as that received by any other player, no matter what the other players do.
- A fair division procedure is said to be *Pareto-optimal* if it produces an allocation of the property that no other allocation can make one player better off without making some other player worse off.

### *Adjusted Winner Procedure:*

The adjusted winner procedure can be used to divide items between two parties. It achieves all three goals for fairness.

- Step 1** Each party distributes 100 points over the items in a way that reflects their relative worth to that party.
- Step 2** Each item is initially given to the party that assigns it more points. Each party totals up the number of points it has received. If there is a tie, the item goes to the party with fewer points.
- Step 3** If the number of points for each party is equal, the procedure is complete. Otherwise, the party with more points is called the “initial winner” and the other party is called the “initial loser”.
- Step 4** Calculate the *point ratio* for each item that belongs to the initial winner. The point ratio is  $\frac{\text{the initial winner's point value for the item}}{\text{the initial loser's point value for the item}}$ .
- Step 5** Move items from the initial winner to the initial loser in increasing order of point ratio. Stop when you get to an item whose move will cause the initial winner to have fewer points than the initial loser. This item will need to be shared.
- Step 6** Let  $x$  represent the fractional part of the shared item that will be transferred from the initial winner to the initial loser. Set the initial winner’s total points after the sharing of the item equal to the initial loser’s total points after the sharing of the item.

**Step 7** Solve the equation and state the final division of items between the two parties. Notice that the parties now have an equal number of points of value.

Example

Kim and Andy will split 5 items using the adjusted winner procedure with the point values listed below. How are the items distributed?

**Step 1**

Item	Kim	Andy
Dresser	5	35
Necklace	30	10
Hat Box	30	5
Table	5	10
China Cabinet	30	40

**Step 3**

The initial winner is

The initial loser is

**Step 2**

**Step 4**

Item	

**Steps 5, 6, 7**

Example

Rock and Tina will split 4 items using the adjusted winner procedure using the point values listed below. How are the items distributed?

**Step 1**

Item	Rock	Tina
Bar Bells	25	25
Elliptical	50	5
Stair Master	10	60
Treadmill	15	10

**Step 3**

The initial winner is

The initial loser is

**Step 2****Step 4**

Item	

**Steps 5, 6, 7**

Example

Onza and Clyde will split some TAMU yearbooks using the adjusted winner procedure using the point values listed below. How are the items distributed?

**Step 1**

Item	Onza	Clyde
Aggieland '76	11	13
Aggieland '92	2	3
Aggieland '64	15	18
Longhorn '48	12	8
Longhorn '42	18	21
Aggieland '49	35	29
Aggieland '63	7	8

**Step 3**

The initial winner is

The initial loser is

**Step 2****Step 4**

Item	

**Steps 5, 6, 7**

### ***The Knaster Inheritance Procedure***

The Knaster inheritance procedure can be used to divide items among more than two parties. This procedure allocates the items one at a time but requires the parties to have a large amount of cash available.

**Step 1** The  $n$  heirs – independently and simultaneously – submit monetary bids for the item.

**Step 2** The high bidder is awarded the item and places  $\left(\frac{n-1}{n}\right)$  (bid) in a holding account.

**Step 3** Each of the other heirs withdraws  $\frac{1}{n}$  of *his or her own bid* from the holding account.

**Step 4** The money left in the holding account is divided equally among all  $n$  heirs.

**Step 5** The final division of items and cash for the heirs is stated.

### **Example**

Shelly, Toni, and Melissa receive a bookshelf. To decide who gets the bookshelf they use the Knaster Inheritance Procedure. Shelly bids \$120, Toni bids \$160, and Melissa bids \$110. What are the results of the division?

**Step 2** \_\_\_\_\_ gets the bookshelf and places  
\_\_\_\_\_ in a holding account.

**Steps 3-4** Shelly

Toni

Melissa

**Step 5** Shelly

Toni

Melissa

Example

Marcia, Jan, Cindy, and Bobby receive a boat and a truck. To decide who gets these items they use the Knaster Inheritance Procedure.

What are the results of the division?

	Boat	Truck
<b>Step 1</b>	Marcia bid \$1800, Jan bid \$1500, Cindy bid \$2100, and Bobby bid \$2400.	Marcia bid \$15,500, Jan bid \$14,400, Cindy bid \$16,000, and Bobby bid \$14,000.
<b>Step 2</b>	_____ gets the boat and places _____ in a holding account.	_____ gets the truck and places _____ in a holding account.
<b>Steps 3-4</b>	Marcia  Jan  Cindy  Bobby	Marcia  Jan  Cindy  Bobby
<b>Step 5</b>	Marcia  Jan  Cindy  Bobby	

### ***Divide and Choose Procedures***

With two “players”, one player divides the object into two parts then the second player chooses the part he or she wants.

With more players, we can use the Steinhaus Proportional Procedure. For three players, it looks like this.

- Step 1** The players (A, B, and C) let player A be the divider.
- Step 2** Player A divides the cake into three equal pieces: i, ii, and iii
- Step 3** If players B and C each like different pieces, they get those pieces and A gets the remaining piece.
- Step 4** If players B and C both want the same piece, they give the least desirable piece to player A. The remaining two pieces are combined. Player B divides the combined pieces and C chooses.

### ***Vickrey Auctions***

In a Vickrey auction, bidders independently submit sealed bids for the object being sold. The winner is the high bidder, but he or she pays only the amount of the second-highest bid. For our examples, we will assume that ties do not occur.

#### **Example**

Four people were bidding on a new computer. Janie bid \$700, Shawn bid \$600, Stacy bid \$650, and Ronnie bid \$800.

- (a) Who wins the computer?
- (b) How much does he/she pay for the computer?

eBay uses a variation on Vickrey Auctions for online bidding. An eBay auction has a minimum bid and a bid increment set by the seller before bidding starts. A bidder is free to enter the highest price that he/she is willing to pay for the item, because he/she will only have to pay the amount of the second-highest bid plus the bid increment if he/she wins. Each time a higher bid is placed, the “current eBay bid” is updated to be the second-highest bid plus one bid increment. Bidding continues until time expires.

Example

Sherry, Monica, and Alyssa are bidding on a copy of the 1895 Olio (TAMU’s first yearbook) on eBay. The minimum bid was set at \$600, and the bid increment is \$100. Complete the following chart to show the progress of the auction before time ran out.

(a)

Bidder	Bid	Current Winner	Current eBay bid
Sherry	\$600	Sherry	\$600
Monica	\$2000		
Sherry	\$800		
Alyssa	\$1200		
Sherry	\$1500		
Alyssa	\$1800		

(b) Who won the auction?

(c) How much did he/she pay for the yearbook?



**SAMPLE EXAM QUESTIONS FROM CHAPTER 13**

1. Five people were bidding on a new camping trailer using a Vickery auction. Josh bid \$30,000, Jimmy bid \$25,000, Donnie bid \$28,000, Jennifer bid \$32,000, and Lloyd bid \$40,000.

(a) Who wins the trailer?

(b) How much does he/she pay for the trailer?

2. Lucy and Sandy must make a fair division of a printer, a microwave and a lamp. They place point values on the objects as shown below. Using the adjusted winner procedure, what do Lucy and Sandy receive?

Object	Lucy's points	Sandy's points
Printer	30	30
Microwave	25	50
Lamp	45	20

**3.** Nancy, Elayne, and Teri must make a fair division of a boat left to them by their father using the Knaster inheritance procedure. The values they bid on the boat are Nancy - \$4200, Elayne - \$5700, and Teri - \$3900. What are the results of the division?