CHAPTER 13 – FAIR DIVISION

We have three goals for "fairness".

- A fair division procedure is ***equitable*** if each player believes he or she received the same fractional part of the total value.
- A fair division procedure is ***envy-free*** if each player has a strategy that can guarantee him or her a share of whatever is being divided that is, in the eyes of that player, at least as large as that received by any other player, no matter what the other players do.
- A fair division procedure is said to be ***Pareto-optimal*** if it produces an allocation of the property that no other allocation can make one player better off without making some other player worse off.

Adjusted Winner Procedure:

The adjusted winner procedure can be used to divide items between two parties. It achieves all three goals for fairness.

Step 1   Each party distributes 100 points over the items in a way that reflects their relative worth to that party.

Step 2   Each item is initially given to the party that assigns it more points. Each party totals up the number of points it has received. If there is a tie, the item goes to the party with fewer points.

Step 3   If the number of points for each party is equal, the procedure is complete. Otherwise, the party with more points is called the "initial winner" and the other party is called the "initial loser".

Step 4   Calculate the ***point ratio*** for each item that belongs to the initial winner. The point ratio is $\dfrac{\text{the initial winner's point value for the item}}{\text{the initial loser's point value for the item}}$.

Step 5   Move items from the initial winner to the initial loser in increasing order of point ratio. Stop when you get to an item whose move will cause the initial winner to have fewer points than the initial loser. This item will need to be shared.

Step 6   Let x represent the fractional part of the shared item that will be transferred from the initial winner to the initial loser. Set the initial winner's total points after the sharing of the item equal to the initial loser's total points after the sharing of the item.

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Step 7 Solve the equation and state the final division of items between the two parties. Notice that the parties now have an equal number of points of value.

Example

Kim and Andy will split 5 items using the adjusted winner procedure with the point values listed below. How are the items distributed?

Step 1

Item	Kim	Andy
Dresser	5	(35)
Necklace	(30)	10
Hat Box	(30)	5
Table	5	(10)
China Cabinet	30	(40) ← share

Step 3

The initial winner is *Andy*

The initial loser is *Kim*

Step 2

60   85
~~If the China Cab~~ +30 −40
~~90~~ ~~45~~ ← changes who is winning Share instead

Step 4

Item	
Dresser	$\frac{35}{5} = 7$
Table	$\frac{10}{5} = 2$
China Cab.	$\frac{40}{30} = 1.3$ ← smallest

Steps 5, 6, 7

Kim's valuation of China Cab

Let X = portion of China Cab. Andy gives Kim

$$Kim = Andy$$
$$60 + 30x \qquad 85 - 40x$$
(gain)   (give up) Andy's valuation of China Cab
$$+40x \qquad\qquad +40x$$
$$\overline{60 + 70x = 85}$$
$$-60 \qquad\qquad -60$$
$$\overline{\quad 70x = 25}$$
$$x = \frac{25}{70} = \frac{5 \cdot 5}{5 \cdot 14} = \frac{5}{14}$$

35.7%
↓

So Andy gives Kim $\frac{5}{14}$ of China Cab and keeps $1 - \frac{5}{14} = \frac{14}{14} - \frac{5}{14} = \frac{9}{14}$ of China Cab

So Kim gets Necklace, Hat Box and $\frac{5}{14}$ of China Cabinet.
Andy gets Dresser, Table and $\frac{9}{14}$ of China Cabinet
↑
64.3%

Example

Rock and Tina will split 4 items using the adjusted winner procedure using the point values listed below. How are the items distributed?

Step 1

Item	Rock	Tina
Bar Bells	25 Tie	(25) Share
Elliptical	(50)	5
Stair Master	10	(60)
Treadmill	(15)	10

Step 2

$$\begin{array}{cc} 65 & 60 \\ +0 & 25 \\ \hline 65 & 85 \end{array}$$ initially so Tina gets tied item

If give Bar Bells $\begin{array}{cc} +25 & -25 \\ \hline 90 & 60 \end{array}$ changes initial winner, so share instead

Step 4

Item	
Bar Bells	$\frac{25}{25} = 1$ ←
Stair Master	$\frac{60}{10} = 6$

Step 3

The initial winner is Tina

The initial loser is Rock

Steps 5, 6, 7

Let X = portion of Bar Bells Tina gives Rock

$$\begin{array}{ll} \text{Rock} & \text{Tina} \\ 65 + 25X & = 85 - 25X \\ +25X & +25X \\ \hline 65 + 50X & = 85 \\ -65 & -65 \\ \hline 50X & = 20 \\ X & = \frac{20}{50} = \frac{2}{5} = 40\% \end{array}$$

so Tina gives Rock 40% of Bar Bells and keeps 100% − 40% = 60% of Bar Bells

So Rock gets Elliptical, Treadmill, and 40% of Bar Bells.
Tina gets Stair Master and 60% of Bar Bells.

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Example

Onza and Clyde will split some TAMU yearbooks using the adjusted winner procedure using the point values listed below. How are the items distributed?

Step 1

Item	Onza	Clyde
Aggieland '76	11	(13)
Aggieland '92	2	(3)
Aggieland '64	15	(18)
Longhorn '48	(12)	8
Longhorn '42	(18	21) share
Aggieland '49	(35)	29
Aggieland '63	(7)	(8̶)
	47	63

Step 3

The initial winner is Clyde

The initial loser is Onza

Step 2

If give the '63

$$\begin{array}{cc} 47 & 63 \\ +7 & -8 \\ \hline 54 & 55 \end{array}$$ so give the '63 to Onza

If give '42 $\dfrac{\cancel{+18}}{\cancel{72}}\ \dfrac{\cancel{21}}{\cancel{34}}$ would change initial winner, so share '42.

Step 4

Item	
'76	$\frac{13}{11} \approx 1.18182$
'92	$\frac{3}{2} = 1.5$
'64	$\frac{18}{15} = 1.2$
'42	$\frac{21}{18} = 1.1\overline{6}$ ← Next smallest
'63	$\frac{8}{7} \approx 1.14286$ ← Smallest

Steps 5, 6, 7

Let x = portion of '42 that Clyde gives Onza.

$$\begin{array}{rcl} \text{Onza} & = & \text{Clyde} \\ 54 + 18x & = & 55 - 21x \\ +21x & & +21x \\ \hline 54 + 39x & = & 55 \\ -54 & & -54 \\ \hline 39x & = & 1 \\ x & = & \frac{1}{39} \end{array}$$

So Clyde gives Onza $\frac{1}{39}$ of the '42 and keeps $1 - \frac{1}{39} = \frac{38}{39}$ of the '42.

So Onza gets the '48, '49, '63 and $\frac{1}{39}$ of '42.

Clyde gets the '76, '92, '64, and $\frac{38}{39}$ of the '42.

The Knaster Inheritance Procedure

The Knaster inheritance procedure can be used to divide items among more than two parties. This procedure allocates the items one at a time but requires the parties to have a large amount of cash available.

Step 1   The n heirs – independently and simultaneously – submit monetary bids for the item.

Step 2   The high bidder is awarded the item and places $\left(\frac{n-1}{n}\right)$ (bid) in a holding account.

Step 3   Each of the other heirs withdraws $\frac{1}{n}$ of *his or her own bid* from the holding account.

Step 4   The money left in the holding account is divided equally among all n heirs.

Step 5   The final division of items and cash for the heirs is stated.

Example

Shelly, Toni, and Melissa receive a bookshelf. To decide who gets the bookshelf they use the Knaster Inheritance Procedure. Shelly bids $120, Toni bids $160, and Melissa bids $110. What are the results of the division?

Step 2   _Toni_ gets the bookshelf and places _$\frac{2}{3}(160) = \$106.67$_ in a holding account.

← because high bidder

Holding Acct:
106.67 Toni contributes
− 40 Shelly
− 36.67 Melissa
_____
30 Left to Share equally among ALL 3 siblings, so each gets 10

Steps 3-4 Shelly $\frac{1}{3}(120)$

Toni   Bookshelf − 106.67

Melissa $\frac{1}{3}(110)$

Step 5   Shelly   40   + 10   = 50
Toni   Bookshelf − 106.67   + 10   =   Bookshelf and pays   96.67
Melissa   36.67   + 10   =   46.67

Notice: The amount Toni pays equals the amount Shelly and Melissa receive.

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Example

Marcia, Jan, Cindy, and Bobby receive a boat and a truck. To decide who gets these items they use the Knaster Inheritance Procedure. What are the results of the division?

	Boat	Truck
Step 1	Marcia bid $1800, Jan bid $1500, Cindy bid $2100, and Bobby bid $2400.	Marcia bid $15,500, Jan bid $14,400, Cindy bid $16,000, and Bobby bid $14,000.

Step 2   _Bobby_ gets the boat and places $\frac{3}{4}(2400) = 1800$

Holding Acct:
1800 Bobby contributes
− 450 Marcia
− 375 Jan
− 525 Cindy

450 to share equally among all 4 siblings, so 112.50 each

in a holding account.

Cindy gets the truck and places $\frac{3}{4}(16,000) = 12000$

Holding Acct
12000 Cindy contributes
− 3875 Marcia
− 3600 Jan
− 3500 Bobby

1025 to share equally, so each gets 256.25

in a holding account.

Steps 3-4

Marcia $\frac{1}{4}(1800)$

Jan $\frac{1}{4}(1500)$

Cindy $\frac{1}{4}(2100)$

Bobby Boat −1800

Marcia $\frac{1}{4}(15,500)$

Jan $\frac{1}{4}(14,400)$

Cindy Truck −12000

Bobby $\frac{1}{4}(14,000)$

Step 5

Marcia 450 + 112.50   +3875   +256.25   = 4693.75

Jan 375 + 112.50   +3600   +256.25   = 4343.75

Cindy 525 + 112.50   +Truck −12000 +256.25 = Truck and pays 11,106.25

Bobby Boat −1800 +112.50   +3500   +256.25 = Boat +2068.75

Notice: The amount paid (by Cindy) equals the amount received (by the others)

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Divide and Choose Procedures
With two "players", one player divides the object into two parts then the second player chooses the part he or she wants.

With more players, we can use the Steinhaus Proportional Procedure. For three players, it looks like this.

Step 1   The players (A, B, and C) let player A be the divider.

Step 2   Player A divides the cake into three equal pieces: i, ii, and iii

Step 3   If players B and C each like different pieces, they get those pieces and A gets the remaining piece.

Step 4   If players B and C both want the same piece, they give the least desirable piece to player A. The remaining two pieces are combined. Player B divides the combined pieces and C chooses.
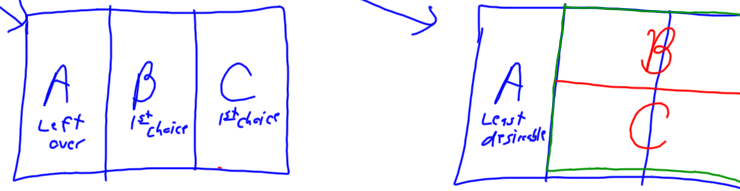
Vickrey Auctions
In a Vickrey auction, bidders independently submit sealed bids for the object being sold. The winner is the high bidder, but he or she pays only the amount of the second-highest bid. For our examples, we will assume that ties do not occur.

Example
Four people were bidding on a new computer. Janie bid $700, Shawn bid $600, Stacy bid $650, and Ronnie bid $800.

(a)   Who wins the computer? *Ronnie highest bidder*

(b)   How much does he/she pay for the computer? *700 2ⁿᵈ highest bid*

eBay uses a variation on Vickrey Auctions for online bidding. An eBay auction has a minimum bid and a bid increment set by the seller before bidding starts. A bidder is free to enter the highest price that he/she is willing to pay for the item, because he/she will only have to pay the amount of the second-highest bid plus the bid increment if he/she wins. Each time a higher bid is placed, the "current eBay bid" is updated to be the second-highest bid plus one bid increment. Bidding continues until time expires.

Example

Sherry, Monica, and Alyssa are bidding on a copy of the 1895 Olio (TAMU's first yearbook) on eBay. The minimum bid was set at $600, and the bid increment is $100. Complete the following chart to show the progress of the auction before time ran out.

(a)

Bidder	Bid	Current Winner	Current eBay bid
Sherry	$600	Sherry	$600
Monica	$2000	Monica	$600 + $100 = $700
Sherry	$800	Monica	$800 + $100 = $900
Alyssa	$1200	Monica	$1200 + $100 = $1300
Sherry	$1500	Monica	$1500 + $100 = $1600
Alyssa	$1800	Monica	$1800 + $100 = $1900

(b)   Who won the auction?   Monica

(c)   How much did he/she pay for the yearbook? $1900

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SAMPLE EXAM QUESTIONS FROM CHAPTER 13

1. Five people were bidding on a new camping trailer using a Vickery auction. Josh bid $30,000, Jimmy bid $25,000, Donnie bid $28,000, Jennifer bid $32,000, and Lloyd bid $40,000.

(a) Who wins the trailer?   *Lloyd*
highest bidder

(b) How much does he/she pay for the trailer?   *$32,000*
2ⁿᵈ highest bid

2. Lucy and Sandy must make a fair division of a printer, a microwave and a lamp. They place point values on the objects as shown below. Using the adjusted winner procedure, what do Lucy and Sandy receive?

Object	Lucy's points	Sandy's points
Printer	30 *Tie*	30 *share*
Microwave	25	50
Lamp	45	20

Initial winner: Lucy

$$45 \quad\quad 50$$

Lucy gets tie $\quad \dfrac{30}{75} \quad \dfrac{+0}{50}$

FF give Printer $\quad \dfrac{-30}{45} \quad \dfrac{+30}{80}$ *changes initial winner, so share instead*

Ratio

Printer $\dfrac{30}{30} = 1$ ← Give or share

Lamp $\dfrac{45}{20} = 2.25$

Let X = portion of Printer Lucy gives Sandy

$$\begin{array}{rcl}
\text{Lucy} & & \text{Sandy} \\
75 - 30X & = & 50 + 30X \\
+30X & & +30X \\
\hline
75 & = & 50 + 60X \\
-50 & & -50 \\
\hline
25 & = & 60X \\
\dfrac{25}{60} & & \dfrac{60X}{60}
\end{array}$$

$$\frac{5}{12} = \frac{2.5}{5 \cdot 12} = \frac{25}{60} = X$$

So Lucy gives Sandy $\frac{5}{12}$ of printer and
Keeps $1 - \frac{5}{12} = \frac{12}{12} - \frac{5}{12} = \frac{2}{12}$ of printer

Lucy gets Lamp and $\frac{2}{12}$ of printer

Sandy gets Microwave and $\frac{5}{12}$ of printer

3. Nancy, Elayne, and Teri must make a fair division of a boat left to them by their father using the Knaster inheritance procedure. The values they bid on the boat are Nancy - $4200, Elayne - $5700, and Teri - $3900. What are the results of the division?