CHAPTER 13 – FAIR DIVISION

A fair division procedure is ***equitable*** if each player believes he or she received the same fractional part of the total value.

A fair division procedure is ***envy-free*** if each player has a strategy that can guarantee him or her a share of whatever is being divided that is, in the eyes of that player, at least as large as that received by any other player, no matter what the other players do.

A fair division procedure is said to be ***Pareto-optimal*** if it produces an allocation of the property that no other allocation can make one player better off without making some other player worse off.

Adjusted Winner Procedure:

Step 1. Each party distributes 100 points over the items in a way that reflects their relative worth to that party.

Step 2. Each item is initially given to the party that assigns it more points. If there is a tie, the item is not assigned.

Step 3. Each party totals up the number of points it has received and the party that has received the fewest number of points is now given the item that that had a tie.

Step 4. If the number of points each party has is tied, the procedure is complete. If one party has more points, it is named party A and the party with fewer points in named party B.

Step 5. Items are now transferred from party A to party B until the point totals are equal. Fractional transfers are allowed. Transfers are determined using ***point ratios***. Transfer item with lowest ratio.

To determine an item's ***point ratio***, find the fraction

$$\frac{\text{A's point value of the item}}{\text{B's point value of the item}}$$

This procedure is well suited to dividing several items between two people or parties.

Example 1

Rand and Mat will split 4 items using the adjusted winner procedure with the point values listed below. How are the items distributed?

Item	Rand	Mat
Gold coin	25	5
Saddle Bag	25	20
Cape	25	35
Hat	25	40
Total	100	100

Rand Mat

50

75

Mat is party A

Rand is party B

Cape $\frac{35}{25} = 1.4 \leftarrow$ smaller

Hat $\frac{40}{25} = 1.6$

$x =$ portion of Cape Rand gets

Rand Mat

Gold coin + Saddle bag + part of Cape = Hat and part of Cape

$$25 + 25 + x(25) = 40 + (1-x)35$$

$$50 + 25x = 40 + 35 - 35x$$

$$50 + 25x = 75 - 35x$$

$$+35x \qquad +35x$$

$$50 + 60x = 75$$

$$\begin{array}{rcl} -50 & & -50 \\ \hline \frac{60x}{60} & = & \frac{25}{60} \end{array}$$

$x = \frac{25}{60} = \frac{5 \cdot 5}{5 \cdot 12} = \frac{5}{12}$ Rand gets $\frac{5}{12}$ of Cape

Mat gets $1 - \frac{5}{12} = \frac{12}{12} - \frac{5}{12} = \frac{7}{12}$ of cape

Rand gets: Gold coin
Saddle bag
$\frac{5}{12}$ of cape

Mat gets Hat
$\frac{7}{12}$ of cape

Check:
Rand

$$25 + 25 + \frac{5}{12}(25) = 40 + \frac{7}{12}(35)$$
$$60.41\overline{6} \approx 60.41\overline{6}$$

Example 2

Katniss and Peeta will split 5 items using the adjusted winner procedure using the point values listed below. How are the items distributed?

Item	Katniss	Peeta
Bow and Arrows	50	0
Water bottle	20	20
Knife	15	35
Food	5	40
Blanket	10	5
	100	100

Katniss Peeta
60 75

so
Katniss gets water bottle initially

now
Katniss Peeta
80 75
party A party B

Bow & Arrows $\frac{50}{0}$ Big # ~~Hd~~ Heads toward

Water bottle $\frac{20}{20} = 1$

Blanket $\frac{10}{5} = 2$

$$\boxed{\begin{array}{l}\text{Katniss gets Bow, Blanket } \frac{7}{8} \text{ of WB} \\ \text{Peeta gets Knife, Food \& } \frac{1}{8} \text{ of WB}\end{array}}$$

Katniss Peeta

Bow + Blanket + part of WB = Knife + Food + part of WB

$$\underbrace{50 + 10}_{} + 20(x) = \underbrace{35 + 40}_{} + 20(1-x)$$

$$60 + 20x = \underbrace{75 + 20}_{} - 20x$$

$$60 + 20x = 95 - 20x$$

$$+20x \qquad\qquad +20x$$

$$60 + 40x = 95$$

$$-60 \qquad\qquad -60$$

$$\frac{40x}{40} = \frac{35}{40}$$

$$x = \frac{35}{40} = \frac{5\cdot 7}{5\cdot 8} = \frac{7}{8}$$

Katniss gets $\frac{7}{8}$ of WB.

Peeta gets $1-\frac{7}{8} = \frac{8}{8}-\frac{7}{8} = \frac{1}{8}$ of WB

Example 3

Ozma and Dorothy will split some jewelry using the adjusted winner procedure using the point values listed below. How are the items distributed?

Item	Ozma	Dorothy
Gold Crown	10	5
Silver Crown	10	20
Diamond Bracelet	15	20
Sapphire Bracelet	11	~~14~~
Emerald Bracelet	20	30
Ruby Bracelet	22	8
Gold Earrings	12	3
	100	100

Ozma 44 Dorothy 84

Silver Crown $\frac{20}{10} = 2$

Diamond $\frac{20}{15} = 1.\overline{3}$ ←

Sapphire $\frac{14}{11} = 1.273$ ← Can transfer completely

Emerald $\frac{30}{20} = 1.5$

Oz 55 Dorothy 70

Oz

Gold C + Sapphire + Ruby + Gold E + part of Diamond = Silver + Emerald + part of Diamond (Dorothy)

$$\underbrace{10 + 11 + 22 + 12}_{55} + 15(x) = \underbrace{20 + 30}_{50} + 20(1-x)$$

$$55 + 15x = \underbrace{50 + 20}_{70} - 20x$$

$$55 + 15x = 70 - 20x$$

$$+20x \qquad +20x$$

$$55 + 35x = 70$$

$$-55 \qquad -55$$

$$\frac{35x}{35} = \frac{15}{35}$$

$$x = \frac{15}{35} = \frac{5 \cdot 3}{5 \cdot 7} = \frac{3}{7}$$

Oz gets $\frac{3}{7}$ of Diamond

Dorothy gets $1 - \frac{3}{7} = \frac{4}{7}$ of Diamond

The Knaster Inheritance Procedure

Step 1. The heirs – independently and simultaneously – submit monetary bids for the object.

Step 2. The high bidder is awarded the object and he or she places all but $1/n$ of his or her bid in a kitty.

Step 3. Each of the other heirs withdraws from the kitty $1/n$ of his or her bid.

Step 4. The remaining money in the kitty is divided equally

This procedure is well suited to dividing a few items between two people or more people.

Example 4

Janice, Cindy and Teri receive a coat. To decide who gets the coat they use the Knaster Inheritance Procedure. Janice bids \$90, Cindy bids \$75 and Teri bids \$60. What are the results of the division? $1-\frac{1}{3}$

Janice gets coat and puts $\frac{2}{3}(90) = \$60$ in kitty

Cindy gets $\frac{1}{3}(\$75) = \$25$

Teri gets $\frac{1}{3}(\$60) = 20$

Kitty $= 60 - 25 - 20 = \$15$ left to divide equally among all 3

Janice gets coat $- \$60 + \$5 =$ coat $- \$55$

Cindy gets $\$25 + \$5 = \$30$

Teri gets $\$20 + \$5 = \$25$

Example 5

John, Paul, George, and Ringo receive a piano and a drum set . To decide who gets these items they use the Knaster Inheritance Procedure.
John bids $800 on the piano and $500 on the drums,
Paul bids $720 on the piano and $440 on the drums,
George bids $600 on the piano and $620 on the drums.
Ringo bids $400 on the piano and $400 on the drums.
What are the results of the division?

Piano

Kitty $= \frac{3}{4}(800) = \$600 - 180 - 150 - 100 = 170$ to split 4 ways

$\frac{170}{4} = \$42.50$

John $=$ piano $- \frac{3}{4}(800)$ $+42.50$

Paul $= \frac{1}{4}(720) = 180$ $+42.50$

George $= \frac{1}{4}(600) = 150$ $+42.50$

Ringo $= \frac{1}{4}(400) = 100$ $+42.50$

Drums

Kitty $= \frac{3}{4}(620) = \$465 - 125 - 110 - 100 = \$130$ to split 4 ways

$\frac{130}{4} = 32.50$

John $= \frac{1}{4}(500) = 125$ $+32.50$

Paul $= \frac{1}{4}(440) = 110$ $+32.50$

George $=$ Drums $- \frac{3}{4}(620)$ $+32.50$

Ringo $= \frac{1}{4}(400) = 100$ $+32.50$

John $=$ piano $- \frac{3}{4}(800) + 42.50 + 125 + 32.50$ $=$ piano $- \$400$

Paul $= 180 + 42.50 + 110 + 32.50$ $= \$365$

George $= 150 + 42.50 +$ Drums $- \frac{3}{4}(620) + 32.50 =$ Drums $- \$240$

Ringo $= 100 + 42.50 + 100 + 32.50 = \$275$

Divide and Choose Procedures

When an item is to be divided between two players, one player will divide the item into two pieces that in the dividers opinion are of equal value.

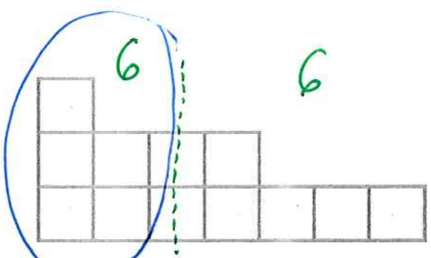
We will divide a "cake" by making a single vertical cut in the cake to create two pieces.

The chooser will pick the left or right piece to get, in the chooser's opinion, at least half of the cake.

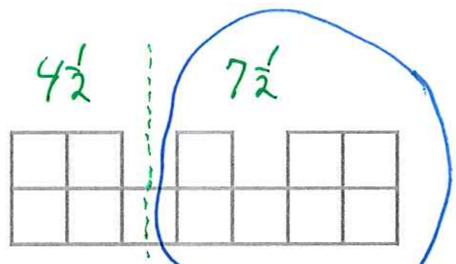
<u>*Example 7*</u>

Robert and Bob will divide a cake using divide and choose.

(a) With Robert dividing, which side will Bob choose? Right

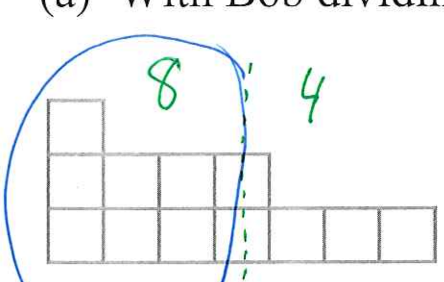


Robert's view

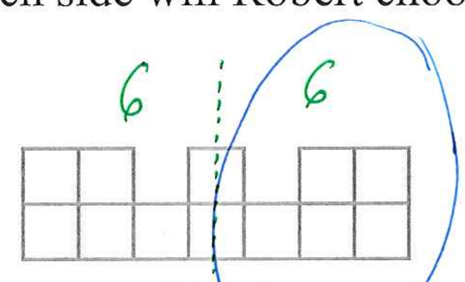


Bob's view

(a) With Bob dividing, which side will Robert choose? Left



Robert's view



Bob's view

A ***cake-division procedure*** for *n* players is a procedure that the players can use to allocate a cake among them so that each player has a strategy that will guarantee that player a piece with which he or she is "satisfied."

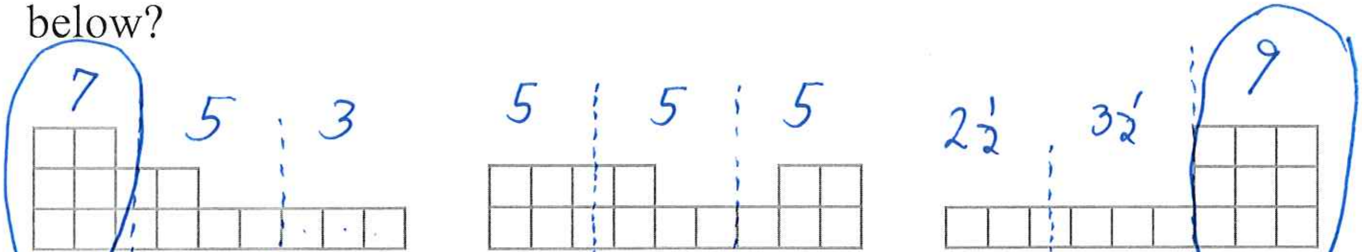
A cake-division procedure for *n* player is called ***proportional*** if each player's strategy guarantees that player a piece that is worth at least $1/n$ of the whole, in that player's estimation.

The Steinhaus Proportional Procedure (Lone Divider) for Three Players

Step 1. The players (A, B, and C) let player A be the divider.
Step 2. Player A divides the cake into three equal pieces, i, ii, and iii
Step 3. If players B and C each like different pieces, they get those pieces and A gets the remaining piece.
Step 4. If players B and C both want the same piece, they give a not wanted piece to player A. The remaining two pieces are combined and then B divides and C chooses.

<u>Example 8</u>

William, Bill and Jane will divide a cake using the Steinhaus proportional procedure. The divider will be Bill and William and Jane will choose. How will the cake be divided if the view of each person is as shown below?

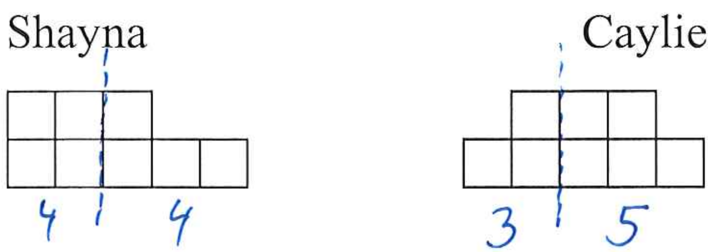


William's view Bill's view Jane's view

Leaves Middle for Bill

SAMPLE EXAM QUESTIONS FROM CHAPTER 13

Suppose that Shayna and Caylie view a cake as shown below. They agree to divide the cake using the divide-and-choose procedure.



1. If Shayna divides the cake, where will the cut be made?

A) 3 columns from the left B) 2 ½ columns from the left
C) 2 columns from the left D) 1 ½ columns from the left

2. If Shayna divides the cake, which side will Caylie choose?

A) The right side B) The left side

5. Match the terms with the correct definition:

(a) Equitable iii

(b) Pareto-optimal i

(i) A fair-division procedure that produces an allocation such that no other allocation can make one person better off without making another person worse off.

(ii) A fair-division procedure that can guarantee that each person has a share of whatever is being divided that is, in the eyes of that player, at least as large as that received by any other player. Envy Free

(iii) A fair-division procedure that results in a division that each person believes he or she received the same fractional part of the total value as the other people.

(iv) A fair-division process where people take turns choosing items.

(v) A fair-division process where one person divides and the other person chooses.

6. Lucy and Sandy must make a fair division of a printer, a microwave and a lamp. They place point values on the objects as shown below. Using the adjusted winner procedure, what do Lucy and Sandy receive?

Object	Lucy's points	Sandy's points
Printer	40	30
Microwave	10	50
Lamp	50	20
	100	100

Lucy 90 Sandy 50

Printer $\frac{40}{30} = 1.\overline{3}$

Lamp $\frac{50}{20} = 2.5$

Lucy gets 3/7 of Printer + Lamp

Sandy gets $\frac{4}{7}$ printer + MW

Lucy Lamp + part of Printer = Sandy MW + part of printer

$50 + 40(x) = 50 + 30(1-x)$

$50 + 40x = 50 + 30 - 30x$

$50 + 40x = 80 - 30x$

$+30x \quad +30x$

$50 + 70x = 80$

$-50 \quad -50$

$\frac{70x}{70} = \frac{30}{70}$

$x = \frac{30}{70} = \frac{3}{7}$

7. Nancy, Elayne, and Teri must make a fair division of a boat left to them by their father using the Knaster inheritance procedure. The values they bid on the boat are Nancy - \$4200, Elayne - \$3600, and Teri - \$3000. What are the results of the division?

Kitty $= \frac{2}{3}(4200) - 1200 - 1000 = 600$ left to share 3 ways

Nancy gets Boat $- \frac{2}{3}(4200) + 200$

Elayne gets $\frac{1}{3}(3600) = 1200 \quad +200$

Teri get $\frac{1}{3}(3000) = 1000 \quad +200$

Nancy gets Boat $- 2800 + 200 =$ Boat $-\$2600$

Elayne $= \$1400$

Teri $= \$1200$