Markov-Bernstein Type Inequalities for Classes of Polynomials with Restricted Zeros

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Abstract. We prove that there exists an absolute constant $c > 0$ such that

$$|p'(y)| \le c \min\left\{n(k+1), \left(\frac{n(k+1)}{1-y^2}\right)^{1/2}\right\} \max_{-1\le x\le 1} |p(x)|, \quad -1 \le y \le 1$$

for every real algebraic polynomial of degree at most n having at most k zeros in the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. This inequality, which has been conjectured for at least a decade, improves and generalizes several earlier results. Up to the multiplicative absolute constant c, it is a sharp generalization of both Markov's and Bernstein's inequalities.

1. Introduction, Notations

Bernstein's inequality [LO1, pp. 39-41] asserts that

$$\max_{-\pi\le t\le\pi} |p'(t)| \le n \max_{-\pi\le t\le\pi} |p(t)| \tag{1.1}$$

for every $p \in \mathcal{T}_n$, where $\mathcal{T}_n$ denotes the set of all trigonometric polynomials of degree at most n with real coefficients. The corresponding algebraic result [LO1, pp. 39-41], known as Markov's inequality, states that

$$\max_{-1\le x\le 1} |p'(x)| \le n^2 \max_{-1\le x\le 1} |p(x)| \tag{1.2}$$

for all $p \in \mathcal{P}_n$, where $\mathcal{P}_n$ denotes the set of all algebraic polynomials of degree at most n with real coefficients. The Chebyshev polynomials $Q_n \in \mathcal{T}_n$ and $T_n \in \mathcal{P}_n$ defined by

$$Q_n(t) := \cos(nt + \alpha), \qquad \alpha \in \mathbb{R} \tag{1.3}$$

and

$$T_n(x) := \cos(n \arccos x), \qquad -1 \le x \le 1 \tag{1.4}$$

show that inequalities (1.1) and (1.2) are sharp. The substitution $x = \cos t$ in (1.1), together with (1.2), yields

$$|p'(y)| \le \min\left\{n^2, \frac{n}{\sqrt{1-y^2}}\right\} \max_{-1\le x\le 1} |p(x)|, \qquad -1 \le y \le 1 \tag{1.5}$$

for every $p \in \mathcal{P}_n$. Markov-Bernstein type inequalities in weighted spaces and in L_p norms play a key role in proving inverse theorems of approximation and of course have their own intrinsic interest.

Denote by $\mathcal{P}(n,k)$ the set of all $p \in \mathcal{P}_n$ having at most k zeros (by counting multiplicities) in the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Markov-Bernstein type inequalities for constrained polynomials have been the subject of many research papers and the classes $\mathcal{P}(n,k)$, $0 \le k \le n$, have been of special interest. One might correctly suspect that the restrictions on the zeros of a polynomial imply an improvement in inequality (1.5). In 1940 Erdős [ERDŐS] proved that there is an absolute constant $c > 0$ such that

$$(1.6) \qquad |p'(y)| \le \min\left\{\frac{en}{2}, \frac{c\sqrt{n}}{(1-y^2)^2}\right\} \max_{-1 \le x \le 1} |p(x)|, \qquad -1 \le y \le 1$$

for every $p \in \mathcal{P}(n,0)$ having only real zeros. By taking the polynomials $p_n \in \mathcal{P}(n,0)$ defined by $p_n(x) := (1+x)^{n-1}(1-x)$, it is easy to see that the constant $e/2$ in (1.6) is asymptotically sharp. In 1963 G. G. Lorentz [LO2] showed that there is an absolute constant $c > 0$ such that

$$(1.7) \qquad |p'(y)| \le c \min\left\{n, \frac{\sqrt{n}}{\sqrt{1-y^2}}\right\} \max_{-1 \le x \le 1} |p(x)|, \quad -1 \le y \le 1$$

for every $p \in \mathcal{P}_n$ of the form

$$(1.8) \qquad p(x) = \sum_{j=0}^{n} a_j (1+x)^j (1-x)^{n-j} \text{ with all } a_j \ge 0 \text{ or all } a_j \le 0.$$

By an observation of G. G. Lorentz [SCH], every $p \in \mathcal{P}(n,0)$ is of the form (1.8), therefore (1.7) holds for every $p \in \mathcal{P}(n,0)$. Inequality (1.7) is sharp up to the multiplicative absolute constant $c > 0$; namely it was shown in [ER1] that there is an absolute constant $c > 0$ such that

$$(1.9) \qquad \sup_{p \in \mathcal{P}(n,0)} \frac{|p'(y)|}{\max_{-1 \le x \le 1} |p(x)|} \ge c \min\left\{n, \frac{\sqrt{n}}{\sqrt{1-y^2}}\right\}$$

for every $n \in \mathbb{N}$ and $y \in [-1,1]$. In 1972 Scheick [SCH] found the best possible constant in Lorentz's Markov-type inequality. Extending Erdős's Markov-type inequality, he proved that

$$(1.10) \qquad \max |p'(x)| < \frac{en}{2} \max |p(x)|$$

for every $p \in \mathcal{P}_n$ of the form (1.8), hence for every $p \in \mathcal{P}(n,0)$. In 1980 Szabados and Varma showed that there is a constant $c(k) > 0$ depending only on k such that

$$\max_{-1\le x\le 1} |p'(x)| \le c(k)n \max_{-1\le x\le 1} |p(x)| \tag{1.11}$$

for every $p \in \mathcal{P}(n,k)$, $0 \le k \le n$, having only real zeros. Subsequently Máté proved that

$$\max_{-1\le x\le 1} |p'(x)| \le 6n\exp(\pi\sqrt{k}) \max_{-1\le x\le 1} |p(x)| \tag{1.12}$$

for every $p \in \mathcal{P}(n,k)$, $1 \le k \le n$, having $n-k$ zeros in $\mathbb{R}\backslash(-1,1)$. Szabados' conjecture, proved by P. Borwein, [BO] establishes the Markov-type inequality

$$\max_{-1\le x\le 1} |p'(x)| \le 9n(k+1) \max_{-1\le x\le 1} |p(x)| \tag{1.13}$$

for every $p \in \mathcal{P}(n,k)$, $0 \le k \le n$, having $n-k$ zeros in $\mathbb{R}\backslash(-1,1)$. Inequality (1.13) was extended in [ER2] to all $p \in \mathcal{P}(n,k)$, $0 \le k \le n$. Another proof of (1.13) for all $p \in \mathcal{P}(n,k)$, $0 \le k \le n$, is obtained in [ER3] with the constant 11 instead of 9. The fact that (1.13) is sharp up to the multiplicative constant was shown by Szabados [SZA, Example 1]. While (1.13) is essentially sharp, it is only a good estimate for $|p'(y)|$ with $|y|$ close to 1.

It was proved in [ER-SZA] that there is an absolute constant $c > 0$ such that

$$|p'(y)| \le \frac{c\sqrt{n}(k+1)^2}{\sqrt{1-y^2}} \max_{-1\le x\le 1} |p(x)|, \qquad -1 < y < 1 \tag{1.14}$$

for every $p \in \mathcal{P}(n,k)$, $0 \le k \le n$. Subsequently it was shown in [ER3] that there is an absolute constant $c > 0$ such that

$$|p'(y)| \le \frac{c\sqrt{n(k+1)}}{1-y^2} \max_{-1\le x\le 1} |p(x)|, \qquad -1 < y < 1 \tag{1.15}$$

for every $p \in \mathcal{P}(n,k)$, $0 \le k \le n$. When $y = 0$, inequality (1.15) is sharp up to the multiplicative constant $c > 0$; namely it was verified in [ER2] that there is an absolute constant $c > 0$ such that

$$\sup_{p\in\mathcal{P}(n,k)} \frac{|p'(0)|}{\max_{-1\le x\le 1} |p(x)|} \ge c\sqrt{n(k+1)} \tag{1.16}$$

for every $0 \le k \le n$. The unpleasant thing about the Berstein-type inequalities (1.14) and (1.15) is the fact that none of them matches inequality (1.5) in the unrestricted case $k = n$ (note that $\mathcal{P}(n,n) = \mathcal{P}_n$). To formulate a Markov-Bersnstein type inequality for $\mathcal{P}(n,k)$, $0 \le k \le n$, which contains all of the earlier results as special cases, the following result was conjectured in [ER-SZA].

Theorem. *There is an absolute constant $c > 0$ such that*

$$|p'(y)| \le c \min\left\{ n(k+1), \left(\frac{n(k+1)}{1-y^2}\right)^{1/2} \right\} \max_{-1 \le x \le 1} |p(x)|, \qquad -1 \le y \le 1$$

for every $p \in \mathcal{P}(n,k)$, $0 \le k \le n$.

The purpose of this paper is to prove the above theorem, which seems to be the "right" Markov-Bernstein type inequality for the classes $\mathcal{P}(n,k)$, $0 \le k \le n$. The proof relies on a series of lemmas, some of which are interesting for their own merit.

2. Proof of the Theorem

First we prove the following Bernstein-type inequality for trigonometric polynomials $p \in \mathcal{T}_n$ having $2n-2k$ zeros at 0.

Theorem 1. *Let $0 \le k \le n$, $n \ge 1$ be integers. There is an absolute constant $c_1 > 0$ such that*

$$\max_{t \in \mathbb{R}} |p'(t)| \le c_1 \sqrt{n(k+1)} \max_{t \in \mathbb{R}} |p(t)| \tag{2.1}$$

for every $p \in \mathcal{T}_n$ of the form

$$p(t) = (\sin(t/2))^{2n-2k} q(t), \qquad q \in \mathcal{T}_k. \tag{2.2}$$

Theorem 1 implies immediately the following result for higher derivatives by induction on s.

Corollary 2. *Let $0 \le k \le n$, $n \ge 1$ and $s \ge 1$ be integers. Then*

$$\max_{t \in \mathbb{R}} |p^{(s)}(t)| \le \prod_{j=1}^{s} (c_1 \sqrt{n(k+j)}) \max_{t \in \mathbb{R}} |p(t)| \tag{2.3}$$

for every $p \in \mathcal{T}_n$ of the form (2.2).

The proof of Theorem 1 rests on the following.

Theorem 3. *Let $0 \le k \le n$ and $n \ge 1$ be integers. There is an absolute constant $c_2 > 0$ such that*

$$|p(z)| \le c_2 \max_{t \in \mathbb{R}} |p(t)| \tag{2.4}$$

for every $p \in \mathcal{T}_n$ of the form (2.2), and for every $z \in \mathbb{C}$ such that $|\mathrm{Im}(z)| \le (n(k+1))^{-1/2}$.

To prove Theorem 3 we need four lemmas.

Lemma 4. *Let $a < b$ and $r > 0$ be arbitrary real numbers. Assume that $P \in \mathcal{P}_{2n}$ and*

$$|P(b)| = \max_{a \le x \le b} |P(x)|. \tag{2.5}$$

Then P has at most $c_3 n r^{1/2}(b-a)^{-1/2}$ zeros (by counting multiplicities) in the interval $[b-r, b]$ with some $0 < c_3 \le 4$.

Lemma 4 is proved in [ER4, Corollary 1] with a slightly larger constant. The fact that $c_3 \le 4$ was pointed out by Lorentz and von Golitschek. Our next lemma is a special case of [BO-ER, Theorem 3.2].

Lemma 5. *Let $0 \le k \le n$ and $n \ge 1$ be integers. For every $c_4 > 0$ there is a constant $c_5 = c_5(c_4)$ depending only on c_4 such that*

$$\max_{a \le x \le b + c_4(b-a)(n(k+1))^{-1}} |P(x)| \le c_5 \max_{a \le x \le b} |P(x)| \tag{2.6}$$

holds for every $P \in \mathcal{P}_{2n}$ of the form

$$P(x) = (x-a)^{2n-2k} Q(x), \qquad Q \in \mathcal{P}_{2k}. \tag{2.7}$$

¿From Lemmas 4 and 5 we deduce

Lemma 6. *Let $0 \le k \le n$, $n \ge 1$ be integers. There is an absolute constant $c_6 > 0$ such that*

$$\max_{-1 \le x \le 1 + (n(k+1))^{-1}} |P(x)| \le c_6 \max_{-1 \le x \le 1} |P(x)| \tag{2.8}$$

for every polynomial of the form

$$P(x) = (x-a)^{2n-2k} Q(x), \quad Q \in \mathcal{P}_{2k}, \quad a \in [-1, 1]. \tag{2.9}$$

Proof. If $n/2 \le k \le n$, then inequality (2.8) holds for every $P \in \mathcal{P}_{2n}$ by Bernstein's inequality [LO1, pp. 42-43]. Therefore, in the sequel we assume that $0 \le k \le n/2$. Let $b \ge 1$ be the smallest real number for which

$$|P(b)| = \max |P(x)| = \max |P(x)|. \tag{2.10}$$

Since $0 \le k \le n/2$, P has $2n-2k \ge n$ zeros at a. On the other hand, Lemma 4 implies that P has at most $c_3 n(b-a)^{1/2}(b+1)^{-1/2}$ zeros (by counting multiplicities) in $[a,b]$, hence

$$c_3 n(b-a)^{1/2}(b+1)^{-1/2} \ge n,$$

thus

$$(b-a)^{-1} \le \frac{c_3^2}{2} =: c_7. \tag{2.11}$$

Now Lemma 5 and inequality (2.11) yield that

$$\begin{aligned} \max_{-1\le x\le 1+(n(k+1))^{-1}} |P(x)| &\le \max_{a\le x\le b+(n(k+1))^{-1}} |P(x)| \\ &\le \max_{a\le x\le b+c_7(b-a)(n(k+1))^{-1}} P(x) \le c_6 \max_{a\le x\le b} |P(x)| = c_6 \max_{-1\le x\le 1} |P(x)|, \end{aligned} \tag{2.12}$$

where $c_6 = c_5(c_7) > 0$ is an absolute constant, and the lemma is proved. $\square$

¿From Lemma 6 we easily obtain.

Lemma 7. *Let $0 \le k \le n$, $n \ge 1$ be integers. There is an absolute constant $c_8 > 0$ such that*

$$|p(i\delta)| \le c_8 \max_{t\in\mathbb{R}} |p(t)| \tag{2.13}$$

(i denotes the imaginary unit) for every $p \in \mathcal{T}_n$ of the form

$$p(t) = (\sin((t-\alpha)/2))^{2n-2k} q(t), \quad q \in \mathcal{T}_k, \quad \alpha \in \mathbb{R} \tag{2.14}$$

and for every $\delta \in \mathbb{R}$ such that

$$|\delta| \le (n(k+1))^{-1/2}. \tag{2.15}$$

Proof. Let

$$\tilde{p}(t) := p(t)p(-t) = \frac{1}{4^{n-k}}(\cos t - \cos\alpha)^{2n-2k} q(t)q(-t). \tag{2.16}$$

Since $q(t)q(-t) \in \mathcal{T}_{2k}$ is an even trigonometric polynomial, by the subsitutions $x = \cos t$ and $a = \cos\alpha$, we obtain that

$$\tilde{p}(t) = (x-a)^{2n-2k} Q(x), \quad Q \in \mathcal{P}_{2k}, \quad a \in [-1,1] \tag{2.17}$$

For the sake of brevity let

$$P(x) := (x-a)^{2n-2k} Q(x). \tag{2.18}$$

Now (2.15) – (2.18) and Lemma 6 yield

$$\begin{aligned} |p(i\delta)|^2 &= |p(i\delta)p(-i\delta)| = |\tilde{p}(i\delta)| \\ &= |(\cosh\delta - a)^{2n-2k} Q(\cosh\delta)| \leq \max_{-1\leq x\leq 1+\delta^2} |P(x)| \\ &\leq c_6 \max_{-1\leq x\leq 1} |P(x)| = c_6 \max_{t\in\mathbb{R}} |\tilde{p}(t)| \\ &= c_6 \max_{t\in\mathbb{R}} |p(t)p(-t)| \leq c_6 \max_{t\in\mathbb{R}} |p(t)|^2, \end{aligned} \tag{2.19}$$

and the lemma is proved. □

Theorem 3 follows from Lemma 7 immediately, while Theorem 1 can be obtained from Theorem 3 by the Cauchy integral formula. Corollary 2, which follows from Theorem 1 by induction on s, plays a key role in the proof of our next lemma.

Lemma 8. *Let $0 \leq k \leq n$, $n \geq 1$ be integers, let $p \in \mathcal{T}_n$ be of the form*

$$p(t) = (\sin(t/2))^{2n-2k} q(t), \qquad q \in \mathcal{T}_k, \tag{2.20}$$

let $t_0 \in \mathbb{R}$ be such that

$$p(t_0) = \max_{t\in\mathbb{R}} |p(t)| = 1. \tag{2.21}$$

Then p has at most $2ec_1\sqrt{n(k+1)}h$ zeros (by counting multiplicities) in the interval

$[t_0-h, t_0+h]$ for any $0 < h < (2ec_1)^{-1}(k+1)^{1/2}n^{-1/2}$, where $c_1 > 0$ is the same as in Theorem 1 (and Corollary 2).

Proof. For the sake of brevity let

$$s := [2ec_1\sqrt{n(k+1)}h] + 1, \tag{2.22}$$

where $c_1 > 0$ is the same as in Theorem 1. Assume that we can find $t_1 < t_2 < \cdots < t_\nu$ in $[t_0-h, t_0+h]$ such that p has μ_i repeated zeros at t_i, $1 \leq i \leq \nu$, and $\sum_{i=1}^{\nu} \mu_i = s$. Now let

$$\Omega(x) := \prod_{i=1}^{\nu} (x-t_i)^{\mu_i}. \tag{2.23}$$

Then, by a well-known relation for the remainder of the Hermite interpolation polynomial, there exists a $\xi \in [t_1, t_\nu] \subset [t_0 - h, t_0 + h]$ such that

$$p(t_0) - H(t_0) = \frac{1}{s!} p^{(s)}(\xi)\Omega(t_0), \tag{2.24}$$

where $H \in \mathcal{P}_{s-1}$ and $H^{(j)}(t_i) = p^{(j)}(t_i) = 0$ for every $1 \le i \le \nu$ and $0 \le j \le \mu_i - 1$, so $H \equiv 0$. Hence, by $s! > (s/e)^s$ and from (2.21) – (2.24), we can deduce that

$$\begin{aligned} |p^{(s)}(\xi)| &\ge s!\, h^{-s} > \left(\frac{s}{e}\right)^s \left(\frac{2ec_1\sqrt{n(k+1)}}{s}\right)^s \\ &\ge (2c_1)^s (n(k+1))^{s/2} \ge \prod_{j=1}^{s} (c_1\sqrt{n(k+j)}) \\ &= \prod_{j=1}^{s} (c_1\sqrt{n(k+j)}) \max_{t \in \mathbb{R}} |p(t)| \end{aligned} \tag{2.25}$$

for any $0 < h < (2ec_1)^{-1}(k+1)^{1/2} n^{-1/2}$ (these bounds for h imply $1 \le s \le k+1$ by (2.22), which is used in the last inequality). Since (2.25) constradicts Corollary 2, our assumption is false, and the lemma is proved. $\square$

By the substitution $x + 1 = 2\sin^2(t/2)$ (i.e. $x = -\cos t$), from Lemma 8 we obtain

Corollary 9. *Let $0 \le k \le n$ and $n \ge 1$ be integers, let $p \in \mathcal{P}_n$ be of the form*

$$p(x) = (x+1)^{n-k} q(x), \qquad q \in \mathcal{P}_k, \tag{2.26}$$

and let

$$|p(1)| = \max_{-1 \le x \le 1} |p(x)|. \tag{2.27}$$

Then p has at most $4ec_1\sqrt{n(k+1)h}$ zeros in $[1-h, 1]$ for any

$$0 < h < (4ec_1)^{-2}(k+1)n^{-1}$$

.

Our next lemma constructs a polynomial with some nice properties, which will be exploited in the proof of Lemma 11.

Lemma 10. *Let $1 \le k \le m \le n$, $m+k \le n$ be integers, let $0 < c_9 \le 1$ be real and let*

$$y := 1 - \frac{c_9 k}{4n} \quad \textit{and} \quad 0 < z \le 1 - \frac{c_9 m}{n}. \tag{2.28}$$

Let Q_{2k} be the Chebyshev polynomial $T_{2k} = \cos(2k \arccos x)$ transformed linearly from $[-1,1]$ to $[y,1]$. Denote the zeros of Q_{2k} by

$$(1 >)\, x_1 > x_2 > \dots > x_k > \tilde{x}_k > \tilde{x}_{k-1} > \dots \tilde{x}_1 (> y)$$

and define

$$P(x) := (x+1)^{n-m-k}(x-z)^m \prod_{j=1}^{k} (x - x_j). \tag{2.29}$$

Let $x_0 := 1$ and $x_{k+1} := y$. Let $\xi_j \in (x_j, x_{j-1}]$, $j = 1, 2, \dots, k+1$, be the (only) point for which

$$|P(\xi_j)| = \max_{x_j \le x \le x_{j-1}} |P(x)|. \tag{2.30}$$

Then

$$|P(\xi_j)| < |P(\xi_{j+1})|, \qquad j = 1, 2, \dots, k. \tag{2.31}$$

Proof. We have

$$\begin{aligned} P(x) &= (x+1)^{n-m-k}(x-z)^m \prod_{j=1}^{k} (x - x_j) \\ &= (x+1)^{n-m-k}(x-z)^m \prod_{j=1}^{k} (x - \tilde{x}_j)^{-1} Q_{2k}(x) \\ &= (x+1)^{n-m-k}(x-z)^m (x-y)^{-k} \prod_{j=1}^{k} \frac{x-y}{x-\tilde{x}_j} Q_{2k}(x). \end{aligned} \tag{2.32}$$

For the sake of brevity let

$$F(x) := \prod_{j=1}^{k} \frac{x-y}{x-\tilde{x}_j} \tag{2.33}$$

and

$$G(x) := (x+1)^{n-m-k}(x-z)^m(x-y)^{-k} \tag{2.34}$$

Then $P = FGQ_{2k}$. Since Q_{2k} equioscillates $k+1$ times on $[(1+y)/2, 1]$, it is sufficient to prove that both F and G are decreasing on $[(1+y)/2, 1]$. Since each of its factors is decreasing on $[(1+y)/2, \infty)$, so is F. To show that G is decreasing on $[(1+y)/2, 1]$ we write

$$G = G_1 G_2, \tag{2.35}$$

where

$$G_1(x) := (x+1)^{n-m-k}(x-y)^{-k/2} \tag{2.36}$$

and

$$G_2(x) := (x-z)^m (x-y)^{-k/2}. \tag{2.37}$$

We have

$$\begin{aligned} G_1'(x) &= \frac{(x+1)^{n-m-k-1}(x-y)^{k/2-1}((n-m-k)(x-y) - \frac{k}{2}(x+1))}{(x-y)^k} \\ &< (x+1)^{n-m-k-1}(x-y)^{-k/2-1}\left(n\frac{c_9 k}{2n} - \frac{k}{2}\right) \le 0 \end{aligned} \tag{2.38}$$

and

$$\begin{aligned} G_2'(x) &= \frac{(x-z)^{m-1}(x-y)^{k/2-1}(m(x-y) - \frac{k}{2}(x-z))}{(x-y)^k} \\ &< (x-z)^{m-1}(x-y)^{-k/2-1}\left(m\frac{c_9 k}{4n} - \frac{k}{2}\frac{c_9 m}{2n}\right) = 0 \end{aligned} \tag{2.39}$$

for every $x \in [(1+y)/2, 1]$, and the lemma is proved. □

Lemma 11. *Let k, m, n, c_9, y, z, x_j $(j = 1, 2, \dots, k)$ and P be the same as in Lemma 10. Let*

$$P^*(x) = (x+1)^{n-m-k}(x-z)^m Q^*(x), \qquad Q^* \in \mathcal{P}_k \tag{2.40}$$

be the constrained Chebyshev polynomials of degree n on $[y, 1]$, which equioscillates $k+1$ times on $[y, 1]$[1]*. Denote the zeros of P^* on $(y, 1)$ by $1 > y_1 > y_2 > \cdots > y_k (> y)$. Then $y_j \le x_j$ for every $j = 1, 2, \dots, k$.*

[1]It is well-known from the theory of Chebyshev approximation that there is a "constrained Chebyshev polynomial" $P^* \in \mathcal{P}_n$ of the form (2.40) such that $P^*(\gamma_j) = (-1)^{k+1-j} \max_{y \le x \le 1} |P^*(x)|$

Proof. This follows from Lemma 10, by proceeding exactly in the same way as in the proof of [BO, Lemma 4]. Suppose that the statement of the lemma is false. Choose the smallest j for which $y_j > x_j$. Then pick η so that

$$\max_{x_j \le x \le 1} |\eta P^*(x)| = \max_{x_j \le x \le 1} |P(x)|. \tag{2.41}$$

(We will specify the sign of η later.) We can deduce from the equioscillation of P^* that $\eta P^* - P$ has at least $j-1$ zeros on $[\beta, 1]$, where β is the smallest number greater than y_j, where $|\eta P^*|$ achieves its maximum on $[y, 1]$. From Lemma 10 we deduce that $\eta P^* - P$ has at least $k - j$ zeros on $[y, \alpha)$, where $\alpha := \xi_{j+1}$ (see (2.30)). We need only to observe that if we choose the sign of η so that

$$\operatorname{sign}(\eta P^*(\beta)) = -\operatorname{sign}(P(\alpha)), \tag{2.42}$$

then $\eta P^* - P$ must have at least two zeros in (α, β). Thus, together with the $n - m - k$ zeros at -1 and m zeros at z, $\eta P^* - P$ has at least

$$(n - m - k) + m + (j - 1) + (k - j) + 2 = n + 1$$

zeros, a contradiction. Therefore $y_j \le x_j$ for every $j = 1, 2, \ldots, k$, and the lemma is proved. □

¿From Lemma 11 we prove the following

Lemma 12. *Let k, m, n, c_9, y and z be the same as in Lemma 10 and 11. There is a constant $c_{10} = c_{10}(c_9) > 0$ depending only on c_9 such that*

$$|p(\alpha)| \le c_{10} \max_{y \le x \le 1} |p(x)| \tag{2.43}$$

for every p of the form

$$p(x) = (x+1)^{n-m-k}(x-z)^m q(x), \qquad q \in \mathcal{P}_k, \tag{2.44}$$

and for every

$$\alpha \in [1, 1 + (nk)^{-1}] \tag{2.45}$$

Proof. By a compactness argument and a variational method it is routine to show that it is sufficient to prove (2.43) for $p = P^*$, where P^* is the constrained Chebyshev polynomial of degree n on $[y,1]$ defined in Lemma 11. Recalling the definition of y and z, and using Lemma 10, (2.45) and the well-known formula

$$T_{2k}(x) = \frac{1}{2}((x+\sqrt{x^2-1})^{2k} + (x-\sqrt{x^2-1})^{2k}), \qquad x \in \mathbb{R}\backslash(-1,1)$$

for the Chebyshev polynomial of degree $2k$ on $[-1,1]$, we obtain

$$\begin{aligned}(2.46) \qquad \frac{|P^*(\alpha)|}{\max\limits_{y\le x\le 1}|P^*(x)|} &= \frac{|P^*(\alpha)|}{|P^*(1)|} \\ &= \left(\frac{\alpha+1}{2}\right)^{n-m-k}\left(\frac{\alpha-z}{1-z}\right)^m \prod_{j=1}^{k}\frac{\alpha-y_j}{1-y_j} \\ &\le \left(\frac{\alpha+1}{2}\right)^{n-m-k}\left(\frac{\alpha-z}{1-z}\right)^m \prod_{j=1}^{k}\frac{\alpha-x_j}{1-x_j} \\ &\le \left(\frac{\alpha+1}{2}\right)^{n-m-k}\left(\frac{\alpha-z}{1-z}\right)^m \frac{Q_{2k}(\alpha)}{Q_{2k}(1)} \\ &\le \left(1+\frac{1}{2nk}\right)^{n-m-k}\left(1+\frac{1}{c_9 mk}\right)^m \frac{T_{2k}(1+8c_9^{-1}k^{-2})}{T_{2k}(1)} \le c_{10},\end{aligned}$$

where Q_{2k} is the Chebyshev polynomial $T_{2k} = \cos(2k\ \arccos x)$ transformed linearly from $[-1,1]$ to $[y,1]$, and c_{10} depends only on c_9. The lemma is now proved. □

Lemma 12 and Corollary 9 allow us to prove the following.

Lemma 13. *Let $1 \le k \le n$, $0 \le m \le n-k$ be integers, let $-1 \le a \le 1$. There is an absolute constant $c_{11} > 0$ such that*

$$(2.47) \qquad |p(\alpha)| \le c_{11} \max_{-1\le x\le 1}|p(x)|$$

for every $p \in \mathcal{P}_n$ of the form

$$(2.48) \qquad p(x) = (x+1)^{n-m-k}(x-a)^m q(x), \qquad q \in \mathcal{P}_k$$

and for every

$$(2.49) \qquad \alpha \in [1, 1+(nk)^{-1}].$$

Proof. Without loss of generality we may assume that

$$(2.50) \qquad |p(1)| = \max |p(x)|.$$

Any other case can be reduced to this by a linear transformation. We distinguish two cases.

Case 1: $1 \le k \le m$. We show that there is an absolute constant $c_9 \in (0,1]$ such that

$$(2.51) \qquad a < 1 - \frac{c_9 m}{n}.$$

Indeed, let $0 < c_9 < (4ec_1)^{-2}$. By Corollary 9 p has at most

$$4ec_1\sqrt{n(k+m+1)c_9 mn^{-1}} < 8ec_1\sqrt{c_9}m$$

zeros (by counting multiplicities) in $[1 - c_9 mn^{-1}, 1]$. Therefore (2.51) holds with $c_9 = (8ec_1)^{-2}$, since p has a zero at a with multiplicity m. Since (2.51) holds, Lemma 12 can be applied to yield the desired result.

Case 2: $0 \le m < k$. Now $p \in \mathcal{P}_n$ has all but $m + k < 2k$ zeros at -1, hence [BO-ER, Theorem 3.2] (cf. Lemma 5) gives the conclusion of the lemma. $\square$

Transforming the result of Lemma 13 linearly from $[-1,1]$ to $[-|b|,1]$ $(-1 \le b \le 1)$, we immediately obtain

Corollary 14. *Let $1 \le k \le n, 0 \le m \le n-k$ be integers and let $-1 \le b \le 1$. There is an absolute constant $c_{12} > 0$ such that*

$$(2.52) \qquad |p(\alpha)| \le c_{12} \max_{-|b| \le x \le 1} |p(x)| \le c_{12} \max_{-1 \le x \le 1} |p(x)|$$

for every $p \in \mathcal{P}_n$ of the form

$$(2.53) \qquad p(x) = (x+b)^{n-m-k}(x-b)^m q(x), \qquad q \in \mathcal{P}_k,$$

and for every

$$(2.54) \qquad \alpha \in [1, 1 + (2nk)^{-1}].$$

Corollary 14 plays a key role in the proof of our next lemma.

Lemma 15. *Let $1 \le k \le n$, $0 \le m \le 2n-2k$ be integers. We have*

$$|p(i\delta)| \le \sqrt{c_{12}} \max_{t \in \mathbb{R}} |p(t| \tag{2.55}$$

(i denotes the imaginary unit) for every $p \in \mathcal{T}_n$ of the form

$$p(t) = (\sin((t+\gamma)/2)))^{2n-m-2k}(\sin((t+\gamma-\pi)/2))^m q(t), \tag{2.56}$$

with $q \in \mathcal{T}_k$ and $\gamma \in \mathbb{R}$, provided

$$\delta \in (-(8kn)^{-1/2}, (8kn)^{-1/2}). \tag{2.57}$$

Proof. The proof is quite similar to that of Lemma 7. Let

$$\begin{aligned} \tilde{p}(t) &:= p(t)p(-t) \\ &= 4^{k-n}(\cos t - \cos\gamma)^{2n-m-2k}(\cos t + \cos\gamma)^m q(t)q(-t). \end{aligned} \tag{2.58}$$

Since $q(t)q(-t) \in \mathcal{T}_{2k}$ is an even trigonometric polynomimal, by the substitutions $x = \cos t$ and $b = \cos\gamma$, we obtain that

$$\tilde{p}(t) = (x-b)^{2n-m-2k}(x+b)^m Q(x), \qquad Q \in \mathcal{P}_{2k}, \tag{2.59}$$

where $Q \in \mathcal{P}_{2k}$. Let

$$P(x) := (x-b)^{2n-m-2k}(x+b)^m Q(x). \tag{2.60}$$

Now (2.57) – (2.60) and Corollary 14 yield

$$\begin{aligned} |p(i\delta)|^2 &= |p(i\delta)p(-i\delta)| = |\tilde{p}(i\delta)| \\ &= |(\cosh\delta + b)^{2n-m-2k}(\cosh\delta - b)^m Q(\cosh\delta)| \\ &\le \max_{-1 \le x \le 1+\delta^2} |P(x)| \le c_{12} \max_{t \in \mathbb{R}} |\tilde{p}(t)| \\ &= c_{12} \max_{t \in \mathbb{R}} |p(t)p(-t)| \le c_{12} \max_{t \in \mathbb{R}} |p(t)|^2, \end{aligned} \tag{2.61}$$

and the lemma is proved. □

We will use a slightly different version of Lemma 15, namely the following.

Corollary 16. *Let $1 \le k \le n$, $0 \le m \le n$, $m + 2k \le 2n$ be integers. We have*

$$|p(z)| \le \sqrt{c_{12}} \max_{t \in \mathbb{R}} |p(t)| \tag{2.62}$$

for every $p \in \mathcal{T}_n$ of the form

$$p(t) = (\sin(t/2))^{2n-m-2k} (\sin((t-\pi)/2))^m q(t), \tag{2.63}$$

with $q \in \mathcal{T}_k$, provided

$$|\mathrm{Im}(z)| \le (8kn)^{-1/2}. \tag{2.64}$$

Proof. Let $\tilde{p}(t) := p(t - \gamma)$, $\gamma \in \mathbb{R}$, and apply Lemma 15. $\square$

Corollary 17. *Let $1 \le k \le n$, $0 \le m \le 2n - 2k$ be integers. There is an absolute constant $c_{13} > 0$ such that*

$$\max_{t \in \mathbb{R}} |p'(t)| \le c_{13} \sqrt{nk} \max_{t \in \mathbb{R}} |p(t)| \tag{2.68}$$

for every $p \in \mathcal{T}_n$ of the form

$$p(t) = \left(\sin \frac{t + \pi/2}{2} \right)^{2n-m-2k} \left(\sin \frac{t - \pi/2}{2} \right)^m q(t), \qquad q \in \mathcal{T}_k. \tag{2.69}$$

By the substitution $x = \sin t$, from Corollary 17 we deduce

Corollary 18. *Let $1 \le k \le n$, $0 \le m \le n - k$ be integers. We have*

$$|p'(y)| \le c_{13} \frac{\sqrt{nk}}{\sqrt{1 - y^2}} \max_{-1 \le x \le 1} |p(x)|, \qquad -1 < y < 1 \tag{2.70}$$

for every $p \in \mathcal{P}_n$ of the form

$$p(x) = (x+1)^{n-m-k} (x-1)^m q(x), \qquad q \in \mathcal{P}_k, \tag{2.71}$$

where $c_{13} > 0$ is the same as in Corollary 17.

Now we are ready to prove the Theorem stated in Section 1.

Proof of the Theorem. The case $k = 0$ follows from the case $k = 1$, hence, without loss of generality, we may assume that $1 \le k$. A simple compactness argument shows that for every $-1 < y < 1$ there is a $\tilde{p} \in \mathcal{P}(n,k)$ such that

$$\frac{|\tilde{p}'(y)|}{\max |\tilde{p}(x)|} = \sup_{p \in \mathcal{P}(n,k)} \frac{|p'(y)|}{\max |p(x)|}. \tag{2.72}$$

Observe that $\tilde{p}$ cannot attain its maximum modulus on $[-1,1]$ at y, otherwise $\tilde{p}'(y) = 0$, a contradiction. We will show that $\tilde{p}$ has at most $k+1$ zeros (by counting multiplicities) different from ± 1. First we prove that $\tilde{p}$ has only real zeros. Indeed, if $\tilde{p}(a) = 0$ for an $a \in \mathbb{C} \setminus \mathbb{R}$, then $\tilde{p}(\overline{a}) = 0$, and a simple calculation shows that, with a sufficiently small $\epsilon > 0$,

$$p_\varepsilon(x) := \tilde{p}(x) \left(1 - \varepsilon \frac{(x-y)^2}{(x-a)(x-\overline{a})}\right) \tag{2.73}$$

is in $\mathcal{P}(n,k)$ and contradicts the extremality of $\tilde{p} \in \mathcal{P}(n,k)$.

Next we show that $\tilde{p}$ can not have more than one zero (by counting multiplicities) in $\mathbb{R} \setminus [-1,1]$. Indeed, if there are $a \neq b \in \mathbb{R} \setminus [-1,1]$ such that $\tilde{p}(a) = \tilde{p}(b) = 0$, or there are $a = b \in \mathbb{R} \setminus [-1,1]$ such that $\tilde{p}(a) = \tilde{p}'(a) = 0$, then the polynomial

$$p_\varepsilon(x) := \tilde{p}(x) \left(1 - \varepsilon \, \mathrm{sign}(ab) \frac{(x-y)^2}{(x-a)(x-b)}\right) \tag{2.74}$$

with sufficiently small $\varepsilon > 0$ is in $\mathcal{P}(n,k)$, and it contradicts the extremal property of $\tilde{p}$. Since $\tilde{p} \in \mathcal{P}(n,k)$, $\tilde{p}$ has at most $k+1$ zeros (by counting multiplicities) different from ± 1, hence Corollary 18 and the extremal property of $\tilde{p} \in \mathcal{P}(n,k)$ yield

$$|p'(y)| \leq c_{13} \frac{\sqrt{n(k+1)}}{\sqrt{1-y^2}} \max_{-1 \leq x \leq 1} |p(x)| \tag{2.75}$$

for every $p \in \mathcal{P}(n,k)$. The uniform bound

$$\max_{-1 \leq x \leq 1} |p'(x)| \leq 9n(k+1) \max_{-1 \leq x \leq 1} |p(x)| \tag{2.76}$$

for every $p \in \mathcal{P}(n,k)$ is proved in [ER2, Corollary 1.3] extending [BO]. Other proofs of (2.76) are given in [ER3, Theorem 1] and [BO-ER, Theorem 3.4] with slightly larger multiplicative constants than 9. By (2.75) and (2.76) the theorem is proved. $\square$

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