THE SIZE OF EXPONENTIAL SUMS ON INTERVALS OF THE REAL LINE

Tamás Erdélyi, Kaveh Khodjasteh, and Lorenza Viola

Abstract. We prove that there is a constant $c > 0$ depending only on $M \geq 1$ and $\mu \geq 0$ such that

$$\int_y^{y+a} |g(t)|\, dt \geq \exp(-c/(a\delta))\,, \qquad a\delta \in (0, \pi]\,,$$

for every g of the form

$$g(t) = \sum_{j=0}^{n} a_j e^{i\lambda_j t}, \qquad a_j \in \mathbb{C},\ |a_j| \leq M j^{\mu}\,,\ |a_0| = 1\,,\ n \in \mathbb{N}\,,$$

where the exponents $\lambda_j \in \mathbb{R}$ satisfy

$$\lambda_0 = 0\,, \qquad \lambda_j \geq j\delta > 0\,, \qquad j = 1, 2, \dots\,,$$

and for every subinterval $[y, y+a]$ of the real line. Establishing inequalities of this variety is motivated by problems in physics.

0. Introduction

The well known Littlewood Conjecture was solved by Konyagin [8] and independently by McGehee, Pigno, and B. Smith [10]. Based on these Lorentz [5] worked out a textbook proof of the conjecture.

Theorem 0.1. *There is an abcolute constant $c > 0$ such that*

$$\int_0^{2\pi} \Big| \sum_{j=1}^{n} e^{i\lambda_j t} \Big|\, dt \geq c \log n$$

whenever $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct integers.

This is an obvious consequence of the following result.

2000 Mathematics Subject Classifications. 41A17

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Theorem 0.2. *Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ be integers. Let $a_1, a_2, \ldots, a_k$ be arbitrary complex numbers. We have*

$$\int_0^{2\pi} \Big| \sum_{j=1}^{n} a_j e^{i\lambda_j t} \Big| \, dt \geq \frac{1}{30} \sum_{j=1}^{n} \frac{|a_j|}{j} .$$

Pichorides, who contributed essentially to the proof of the Littlewood conjecture, observed in [10] that the original Littlewood conjecture (when all the coefficients are from $\{0, 1\}$) would follow from a result on the L_1 norm of such polynomials on sets $E \subset \partial D$ of measure π. Namely if

$$\int_E \Big| \sum_{j=0}^{n} z^{\lambda_j} \Big| \, |dz| \geq c$$

for any subset $E \subset \partial D$ of measure π and for any nonnegative integers $\lambda_0 < \lambda_1 < \cdots < \lambda_n$ with an absolute constant $c > 0$, then the original Littlewood conjecture holds. Here ∂D denotes the unit circle of the complex plane and the measure of a set $E \subset \partial D$ is the linear Lebesgue measure of the set

$$\{t \in [-\pi, \pi) : e^{it} \in E\} .$$

Konyagin [9] gives a lovely probabilistic proof showing that this hypothesis fails. He does however conjecture the following: for any *fixed* set $E \subset \partial D$ of positive measure there exists a constant $c = c(E) > 0$ depending only on E such that

$$\int_E \Big| \sum_{j=0}^{n} z^{\lambda_j} \Big| \, |dz| \geq c(E) ,$$

for any nonnegative integers $\lambda_0 < \lambda_1 < \cdots < \lambda_n$. In other words, the sets $E_\varepsilon \subset \partial D$ of measure π in his example where

$$\int_{E_\varepsilon} \Big| \sum_{j=0}^{n} z^{\lambda_j} \Big| \, |dz| < \varepsilon$$

must vary with $\varepsilon > 0$.

In [2] we show, among other things, that Konyagin's conjecture holds on subarcs of the unit circle ∂D.

In [7] S. Güntürk constructs certain types of near-optimal approximations of a class of analytic functions in the unit disk by power series with two distinct coefficients. More precisely, it is shown that if all the coefficients of the power series $f(z)$ are real and lie in $[-\mu, \mu]$, where $\mu < 1$, then there exists a power series $Q(z)$ with coefficients in $\{-1, +1\}$ such that $|f(z) - Q(z)| \to 0$ at the rate $\exp(C|1 - z|^{-1})$ as $z \to 1$ non-tangentially inside the unit disk. Güntürk refers to P. Borwein, Erdélyi, and Kós in [5] to see that this type of decay rate is best possible. The special case $f \equiv 0$ yields a near-optimal solution to the "fair duel problem" of Konyagin, as it is described in the Introduction of [7].

In this paper we extend the polynomial inequalities of [2] to exponential sums of the form

$$g(t) = \sum_{j=0}^{n} a_j e^{i\lambda_j t} , \qquad a_j \in \mathbb{C}, \;\; |a_j| \leq M j^{\mu}, \;\; |a_0| = 1, \;\; n \in \mathbb{N},$$

where $\lambda_0 = 0$ and the real exponents satisfy the "minimum growth condition"

$$\lambda_j \geq j\delta > 0, \quad j = 1, 2, \ldots$$

In addition to being interesting on its own, this extension is motivated by physical applications in the context of decoherence control in open quantum systems using dynamical decoupling methods [6]. In the paradigmatic case of a single two-level quantum system undergoing pure dephasing due to coupling to a quantum bosonic environment, for instance, the residual decoherence error at a time t after the application of n ideal "spin-flip" pulses at times $0 < t_1 < t_2 < \ldots < t_n < T$, is quantified by a decay factor of the form

$$\chi_{\{t_j\}} = \int_0^\infty \Lambda(\omega) |f_{\{t_j\}}(\omega)|^2 d\omega \,, \quad f_{\{t_j\}}(\omega) = \sum_{j=0}^{n} (-1)^j (e^{it_j\omega} - e^{it_{j+1}\omega}) \,,$$

where $\Lambda(\omega)$ is a real function whose details depend on both the temperature and the density of modes at frequency ω in the environment and, in addition, $t_0 := 0$ and $t_{n+1} := T$. Thus, the decoherence error corresponds directly to the size of the exponential sum $f_{\{t_j\}}(\omega)$.

Decoupling methods aim to design the "filter function" $f_{\{t_j\}}(\omega)$ in such a way that the error $\chi_{\{t_j\}}$ is minimized [13]. Under the assumption that the spectral density of the environment (hence $\Lambda(\omega)$) vanishes for frequencies higher than an "ultraviolet cut-off frequency" ω_c, the decay factor $\chi_{\{t_j\}}$ may be made small by requiring the exponential sum to vanish perturbatively, i.e., to start its Taylor series at a sufficiently high order $(\omega_c T)^m$. In particular, it has been recently shown [12] that if the pulse timings are chosen according to $t_j = T \sin^2(j\pi/(2n+2))$, cancellation of $\chi_{\{t_j\}}$ is achieved to order $m = n$ by using n pulses, the so-called Uhrig decoupling. Physically, however, a minimum growth condition is always imposed by the fact that the separation between any two consecutive pulses cannot be made arbitrarily small due to finite timing resources, thus $t_{j+1} - t_j > \tau > 0$ for all j. As shown in [6], the results established here may then be used to obtain a *non-perturbative* lower bound on $\chi_{\{t_j\}}$, determined solely in terms the parameter $\omega_c \tau$. As an additional implication of our analysis, we find that Uhrig decoupling arises naturally as a consequence of representing certain polynomials of degree at most $2n+1$ in terms of Lagrange interpolation at the extreme points of the Chebyshev polynomials U_{2n+1}.

1. Notation

For $N > 0$ and $\mu \geq 0$, let $\mathcal{S}_N^\mu$ denote the collection of all analytic functions f on the open unit half-disk $D^+ := \{z \in \mathbb{C} : |z| < 1, \ \mathrm{Re}(z) > 0\}$ that satisfy

$$|f(z)| \leq \frac{N}{(1-|z|)^\mu} \,, \qquad z \in D^+ \,.$$

In this note the value of μ will always assumed to be a nonnegative integer. We define the following subsets of $\mathcal{S}_1^1$. Let

$$\mathcal{F}_n := \left\{ f : f(x) = \sum_{j=0}^{n} a_j x^j \,, \ \ a_j \in \{-1, 0, 1\} \right\}$$

and denote the set of all polynomials with coefficients from the set $\{-1,0,1\}$ by

$$\mathcal{F} := \bigcup_{n=0}^{\infty} \mathcal{F}_n \,.$$

More generally we define the following classes of Müntz polynomials. For $M > 0$, $\mu \geq 0$, and a sequence $\Lambda := (\lambda_j)_{j=0}^{\infty}$ of real numbers let

$$\mathcal{K}_M^{\mu}(\Lambda) := \left\{ f : f(x) = \sum_{j=0}^{n} a_j x^{\lambda_j}\,,\ \ a_j \in \mathbb{C}\,,\ \ |a_j| \leq M j^{\mu}\,,\ \ |a_0| = 1\,,\ \ n \in \mathbb{N} \right\}.$$

Here we define the analytic function $z^{\lambda_j} := \exp(\lambda_j \log z)$ by taking the principal analytic branch of $\log z$ in $\mathbb{C} \setminus (-\infty, 0]$.

2. New Results

Theorem 2.1. *There is a constant $c > 0$ depending only on $M \geq 1$ and $\mu \geq 0$ such that*

$$\int_y^{y+a} |g(t)|\,dt \geq \exp(-c/(a\delta))\,, \qquad a\delta \in (0,\pi]\,,$$

for every g of the form

$$g(t) = \sum_{j=0}^{n} a_j e^{i\lambda_j t}, \qquad a_j \in \mathbb{C},\ \ |a_j| \leq M j^{\mu}\,,\ \ |a_0| = 1\,,\ \ n \in \mathbb{N}\,,$$

where the exponents $\lambda_j \in \mathbb{R}$ satisfy

$$\lambda_0 = 0\,, \qquad \lambda_j \geq j\delta > 0\,, \qquad j = 1,2,\ldots\,,$$

and for every subinterval $[y, y+a]$ of the real line.

Using the substitution $u = t/\delta - y - a/2$ we need to prove Theorem 2.1 only in the case when $[y, y+a] = [-a/2, a/2]$ and $\delta = 1$. Hence, using the substitution $z = e^{it}$, we need to prove only the following result.

Theorem 2.2. *There is a constant $c > 0$ depending only on $M \geq 1$ and $\mu \geq 0$ such that*

$$\int_A |f(z)|\,|dz| \geq \exp(-c/a)\,, \qquad a \in (0,\pi]\,,$$

for every f of the form

$$f(z) = \sum_{j=0}^{n} a_j z^{\lambda_j}, \qquad a_j \in \mathbb{C},\ \ |a_j| \leq M j^{\mu}\,,\ \ |a_0| = 1\,,\ \ n \in \mathbb{N}\,,$$

where the exponents $\lambda_j \in \mathbb{R}$ satisfy

$$\lambda_0 = 0\,, \qquad \lambda_j \geq j > 0\,, \qquad j = 1, 2, \dots\,,$$

and for every subarc $A := \{e^{it} : t \in [-a/2, a/2]\}$ of the unit circle.

Remark 2.3. Without a growth condition on the exponents λ_j a sequence (g_k) of exponential sums

$$g_k(t) = \sum_{j=0}^{n_k} a_{j,k} e^{i\lambda_j t}, \qquad a_{j,k} \in \mathbb{C},\ \ |a_{j,k}| \leq 1\,,\ \ |a_{0,k}| = 1\,,$$

can certainly converge to 0 uniformly on any fixed interval $[-a, a]$. This can easily be seen by taking

$$\lambda_0 := 0\,, \qquad \lambda_j := 2/j\,, \quad j = 1, 2, \dots\,,$$

and the two term exponential sums

$$g_k(t) := 1 - \exp(i\lambda_k t)\,.$$

Clearly,

$$\lim_{k\to\infty} \max_{[-a,a]} |g_k(t)| = \lim_{k\to\infty} 2\sin(a/k) = 0\,.$$

Remark 2.4. The lower bounds in Theorems 2.1 and 2.2 cannot be essentially improved even if we assume that the exponents

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

are integers and $a_j \in \{-1, 1\}$ for each j. Namely, in [2] the authors proved that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$\inf_{0 \neq f \in \mathcal{F}} \max_{z\in A} |f(z)| \leq \exp(-c_1/a)$$

whenever A is a subarc of the unit circle with arclength $\ell(A) = a \leq c_2$.

Remark 2.5. An explicit construction showing that Theorems 2.1 and 2.2 cannot be essentially improved can be given by utilizing the fact that if n is even then

$$(2.1) \qquad f_n(z) := -z + 1 + 2\sum_{k=1}^{n} (-1)^k z^{d_k}\,, \qquad d_k := \sin^2\left(\frac{k\pi}{2n+2}\right),$$

has a zero at 1 with multiplicity at least $n+1$. Namely we prove that there is an absolute constant $c > 0$ such that

$$\inf_g \max_{t\in[-a,a]} |g(t)| \leq 12\,\exp(-c/a)\,,$$

for all $a \in (0, 1/3]$, where the infimum is taken for all exponential sums g of the form

$$g(t) = \sum_{k=0}^{n+1} a_k e^{i\lambda_k t}$$

with $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n+1}$ satisfying the gap condition

$$\lambda_{k+1} - \lambda_k \geq 1\,, \qquad k = 0, 1, \dots, n\,,$$

($n = 2, 4, \dots$ can be arbitrary) and with

$$a_0 = 1, \quad a_{n+1} = -1, \qquad a_k = (-1)^k 2\,, \qquad k = 1, 2, \dots, n\,.$$

Note that in the context of dynamical decoupling theory, the exponents d_k are the relative timings of the Uhrig protocol with n pulses [12]. To see that g_n has a zero at 1 with multiplicity at least $n+1$ observe that the Lagrange interpolation formula associated with $2n+1$ distinct points (see [3, p. 8]) reproduces any polynomial of degree at most $2n$. In particular, choosing the nodes to be the zeros

$$\alpha_k := \cos\left(\frac{k\pi}{2n+2}\right)\,, \qquad 1 \leq k \leq 2n+1\,,$$

of the Chebyshev polynomial U_{2n+1} (see [3, Section 2.1] about some of the basic facts about Chebyshev polynomials including the symmetry of the their zeros

$$\alpha_{2n+2-k} = -\alpha_k\,, \qquad k = 1, 2, \dots, n\,,$$

and $\alpha_{m+1} = 0$), we deduce that

$$\begin{aligned} Q(x) &= \sum_{k=1}^{2n+1} Q(\alpha_k) \frac{U_{2n+1}(x)}{U'_{2n+1}(\alpha_k)(x-\alpha_k)} \\ &= \sum_{k=1}^{2n+1} Q(\alpha_k) \frac{(-1)^{k+1}(1-\alpha_k^2)U_{2n+1}(x)}{(2n+2)(x-\alpha_k)} \\ &= Q(0)\frac{-U_{2n+1}(x)}{(2n+2)x} + \sum_{k=1}^{n} Q(\alpha_k) \frac{(-1)^{k+1}(1-\alpha_k^2)2x\,U_{2n+1}(x)}{(2n+2)(x^2-\alpha_k^2)}\,, \end{aligned}$$

hence

$$Q(1) = Q(0) + 2\sum_{k=1}^{n} Q(\alpha_k) \frac{(-1)^{k+1}U_{2n+1}(1)(1-\alpha_k^2)}{(2n+2)(1-\alpha_k^2)} = 1 + 2\sum_{k=1}^{n} (-1)^{k+1} Q(\alpha_k)$$

for every polynomial Q of degree at most $2n+1$. Here we used that

$$\alpha_{2n+2-k} = -\alpha_k\,, \qquad k = 1, 2, \dots, n\,,$$

and $\alpha_{n+1} = 0$. Choosing

$$Q(x) := \left(1 - x^2\right)^m \,, \qquad m = 1, 2, \dots, n \,,$$

we obtain

$$0 = 1 + 2 \sum_{k=1}^{n} (-1)^{k+1} \left(1 - \alpha_k^2\right)^m \,, \qquad m = 1, 2, \dots, n \,.$$

That is,

$$0 = -1 + 2 \sum_{k=1}^{n} (-1)^k d_k^m \,, \qquad m = 1, 2, \dots, n \,,$$

while the assumption that n is even yields

$$0 = \sum_{k=1}^{n} (-1)^k d_k^m \,, \qquad m = 0 \,.$$

Thus,

$$f_n^{(m)}(1) = 0, \qquad m = 0, 1, \dots, n \,, \tag{2.2}$$

indeed. Now let

$$g_n(t) := f_n(e^{it}) \,.$$

Then

$$g_n^{(m)}(0) = 0 \,, \qquad m = 0, 1, \dots, n \,,$$

and

$$|g_n^{(m)}(u)| = \left| -i^m e^{iu} + 2 \sum_{k=1}^{n} (-1)^k (id_k)^m e^{id_k u} \right| \le 2n + 1 \,, \qquad u \in \mathbb{R} \,, \quad m = 1, 2, 3, \dots \,.$$

Let $b = \dfrac{3}{n+1}$. By using the integral form of the well known Taylor's Remainder Theorem we have

$$\begin{aligned} |g_n(t)| &= \left| \frac{1}{n!} \int_0^t g_n^{(n+1)}(u)(t-u)^n \, du \right| \\ &\le \frac{1}{n!} \int_0^t \left| g_n^{(n+1)}(u)(t-u)^n \right| \, du \\ &\le \frac{1}{(n+1)!} \max_{u \in [-|t|, |t|]} |g_n^{(n+1)}(u)| \, |t|^{n+1} \le \left(\frac{et}{n+1} \right)^{n+1} (2n+1) \\ &\le \frac{6}{b} \left(\frac{e}{3} \right)^{3/b} \end{aligned} \tag{2.3}$$

whenever $|t| \le 1/b$. (Note that the integral form of Taylor's Remainder Theorem is valid for complex-valued functions, so we do not need to separate the real and imaginary parts of the function g_n to apply it.) Now with $\lambda_0 := 0$, $\lambda_{n+1} := 9/b^2$, and $\lambda_k := (9/b^2)d_k$, $k = 1, 2, \dots, n$, let

$$G_b(t) := g_n(9t/b^2) = -e^{i\lambda_{n+1}t} + e^{i\lambda_0 t} + 2\sum_{k=1}^{n}(-1)^{k+1}e^{i\lambda_k t}\,.$$

Elementary calculus shows that the exponents λ_k satisfy the gap condition

$$\begin{aligned}\lambda_{k+1} - \lambda_k \ge \lambda_1 - \lambda_0 = (9/b^2)(d_1 - d_0) = (9/b^2)\sin^2\left(\frac{\pi}{2n+2}\right)\\ \ge (9/b^2)\left(\frac{1}{n+1}\right)^2 \ge (9/b^2)(b^2/9) \ge 1\,, \qquad k = 0, 1, \dots, n\,,\end{aligned}$$

and it follows from (2.3) that

$$\max_{-b/9\le t\le b/9}|G_b(t)| \le \frac{6}{b}\left(\frac{e}{3}\right)^{3/b} \le 12\exp(-c/b)$$

with an absolute constant $c > 0$. If $b \le 3$ is not of the form $b = \dfrac{3}{n+1}$ with an even non-negative integer n, then we choose the largest even integer n such that $b < \beta := \dfrac{3}{n+1}$ and the example in the already studied case shows that

$$\max_{-b/9\le t\le b/9}|G_\beta(t)| \le \max_{-\beta/9\le t\le \beta/9}|G_\beta(t)| \le 12\,\exp(-c/\beta) \le 12\,\exp(-c^*/b)$$

with an absolute constant $c^* := c/3 > 0$. Choosing $b = 9a \le 3$ we obtain our claim for all $a \in (0, 1/3]$.

Remark 2.6. Using a slightly better lower bound for $n!$, by a straightforward modification of Remark 2.5 one can see that

$$\inf_{g}\max_{t\in[-a,a]}|g(t)| \le \exp\left(\frac{-1}{e^2 a}\right)\left(\frac{2}{e} + ea\right)\sqrt{\frac{e^2 + 1/a}{2\pi}}$$

for all $a \in (0, 1/(2e^2)]$, where the infimum is taken for all exponential sums g of the form

$$g(t) = \sum_{k=0}^{n+1} a_k e^{i\lambda_k t},$$

with $0 = \lambda_0 < \lambda_1 < \dots < \lambda_{n+1}$ satisfying the gap condition

$$\lambda_{k+1} - \lambda_k \ge 1, \qquad k = 0, 1, \dots, n,$$

($n = 2, 4, \ldots$ can be arbitrary) and with a_k as in Remark 2.5.

To see this let n be even and consider

$$\widetilde{g}_n(z) := 1 - z^{\widetilde{\lambda}_{n+1}} + 2\sum_{k=1}^{n}(-1)^k z^{\widetilde{\lambda}_k}\,,$$

where $\widetilde{\lambda}_k$ are defined as

$$\widetilde{\lambda}_k := \csc^2\left(\frac{\pi}{2n+2}\right)\sin^2\left(\frac{k\pi}{2n+2}\right), \quad k = 1, \ldots, n,$$

and satisfy the gap condition $\widetilde{\lambda}_{k+1} - \widetilde{\lambda}_k \geq 1$ for $k = 1, 2, \ldots, n$. Notice that the rescaled timings satisfy $\widetilde{\lambda}_k = d_k/d_1$, where the numbers d_k are defined in Remark 2.5. Let $a := e^{-2}/n$ and define $F_a(t) := \widetilde{g}_n(e^{it})$. Following the argument given in Remark 2.5, we can show that

$$|F_a(t)| \leq \exp\left(\frac{-1}{e^2 a}\right)\left(\frac{2}{e} + ea\right)\sqrt{\frac{e^2 + 1/a}{2\pi}}$$

for all $|t| \leq a$. Notice that the right hand side of the above inequality is an increasing function of a. If a cannot be written as e^{-2}/n for an even integer n, we may simply use the smallest even integer n such that $a > \alpha := e^{-2}/n$ whenever $0 < a < 1/(2e^2)$. Thus

$$\max_{-a\leq t\leq a}|F_\alpha(t)| \leq \max_{-\alpha\leq t\leq \alpha}|F_\alpha(t)| \leq \exp\left(\frac{-1}{e^2 a}\right)\left(\frac{2}{e} + ea\right)\sqrt{\frac{e^2 + 1/a}{2\pi}}\,.$$

Remark 2.7 Finding a polynomial $\sum_{j=1}^{n} a_j z^{\lambda_j}$ with $a_j \in \{-1, 1\}$, *integer* exponents $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{2n}$, and with a zeros at 1 of multiplicity at least n is closely related to Wright's conjecture (1934) on ideal solutions of the Prouhet-Tarry-Escott Problem. This seems extremely difficult to settle. See [1, Chapter 11] about the history of this problem. However, as P. Borwein writes it in [1, p. 87], heuristic arguments suggest that Wright's conjecture should be false.

3. Lemmas

To prove Theorem 2.2 we modify the proof given in [2] in the case where $\lambda_j = j$ for each j. We need the lemmas.

Lemma 3.1. *Let $0 < a \leq \pi$ and $N \geq 1$. For every $g \in \mathcal{S}_N^\mu$ with $|g(\frac{1}{4Ne^\mu})| \geq 4^{-(\mu+1)}$ there is a value $b \in [\frac{1}{2}, \frac{3}{4}]$ such that $|g(b)| \geq c_2 > 0$, where c_2 depends only on $N \geq 1$ and $\mu \geq 0$.*

Proof of Lemma 3.1. The proof is a standard normal family argument. Suppose the lemma is not true for some $N \geq 1$ and $\mu \geq 0$. Then there is a sequence (g_n) such that $g_n \in \mathcal{S}_N^\mu$, $|g_n(\frac{1}{4Ne^\mu})| = 4^{-(\mu+1)}$ and

$$\lim_{n\to\infty} K_n = 0\,, \qquad K_n := \max_{z\in[\frac{1}{2},\frac{3}{4}]}|g_n(z)|\,, \qquad n = 1, 2, \ldots\,.$$

Then there is a subsequence of (g_n), without loss of generality we may assume that this is (g_n) itself, that converges to a function $g \in \mathcal{S}_N^\mu$ locally uniformly on every compact subset of

$$H := \{z \in \mathbb{C} : |z| < 1, \ \mathrm{Re}(z) > 0\} .$$

Now $g(z) = 0$ for all $z \in [\frac{1}{2}, \frac{3}{4}]$ while $g(\frac{1}{4Ne^\mu}) = 4^{-(\mu+1)}$. This contradicts the Unicity Theorem. □

Lemma 3.2. *Let* $0 < a \le \pi$. $b \in [\frac{1}{2}, \frac{3}{4}]$, *and* $N \ge 1$. *Let* $\Gamma_{a,b}$ *be the circle centered at* b *with diameter* $[2b - \cos(a/8), \cos(a/8)]$. *Let* I *be the subarc of* $\Gamma_{a,b}$ *with length* $\ell(I) \ge c_3 a$ *with midpoint* $\cos(a/8)$ *on the real line, where* $c_3 \in (0,1]$. *Then there is a constant* $c_4 > 0$ *depending only on* N, μ, c_2, *and* c_3 *such that*

$$\max_{z \in I} |g(z)| \ge \exp(-c_4/a)$$

for every $g \in \mathcal{S}_N^\mu$ *with* $|g(b)| \ge c_2$.

Proof of Lemma 3.2. Let $2m \ge 4$ be the smallest even integer not less than $4\pi/(c_3 a)$. Let

$$\xi := \exp\left(\frac{\pi i}{m}\right)$$

be the first $(2m)$-th root of unity. We define $2m$ equally spaced points on $\Gamma_{a,b}$ by

$$\eta_k := b + (\cos(a/8) - b)\xi^k , \qquad k = 0, 1, \dots , 2m - 1 .$$

Then there is a constant $c_5 > 0$ depending only on c_3 such that

$$1 - |z| \ge c_5 (ka)^2 , \qquad k = 1, 2, \dots m - 1 ,$$

whenever z is on the smaller subarc of the circle $\Gamma_{a,b}$ with endpoints η_k and η_{k+1} or with endpoints η_{2m-k} and η_{2m-k-1}, respectively. We define the function

$$h(z) := \prod_{j=0}^{2m-1} g(b + (\cos(a/8) - b)\xi^j (z - b)) .$$

If $g \in \mathcal{S}_N^\mu$, then

$$\begin{aligned}
\max_{z \in \Gamma_{a,b}} |h(z)| &\le \left(\prod_{k=1}^{m-1} \left(N \left(\frac{1}{c_5 (ka)^2} \right)^\mu \right) \right)^2 \left(\max_{z \in I} |g(z)| \right)^2 \\
&\le \left(\frac{1}{c_5 a} \right)^{(4m-4)\mu} \frac{N^{2m-2}}{((m-1)!)^{4\mu}} \left(\max_{z \in I} |g(z)| \right)^2 \\
&\le \left(\frac{m}{2\pi c_3 c_5} \right)^{(4m-4)\mu} \left(\frac{e}{m-1} \right)^{(4m-4)\mu} N^{2m-2} \left(\max_{z \in I} |g(z)| \right)^2 \\
&\le e^{4\mu} N^{2m-2} \left(\frac{e}{2\pi c_3 c_5} \right)^{(4m-4)\mu} \left(\max_{z \in I} |g(z)| \right)^2 \\
&\le \exp(c_6/a) \left(\max_{z \in I} |g(z)| \right)^2
\end{aligned}$$

with a constant $c_6 > 0$ depending only on N, μ, and c_3. Now the Maximum Principle yields that

$$|g(b)|^{2m} = |h(b)| \le \max_{z \in \Gamma_{a,b}} |h(z)| \le \exp(c_6/a) \left(\max_{z \in I} |g(z)| \right)^2 .$$

Since $2m \le 2 + 4\pi/(c_3 a)$ and $|g(b)| \ge c_2$, we obtain

$$\left(\max_{z \in I} |g(z)| \right)^2 \ge \exp(-c_6/a)\, |g(b)|^{2m} \ge \exp(-c_6/a)\, (c_2)^{2m} \ge \exp(-2c_4/a)$$

with a constant $c_4 > 0$ depending only on N, μ, c_2, and c_3. $\square$

Lemma 3.3. *Let $0 < a < \pi$, $N \ge 1$, and $\mu = 1, 2, \dots$. Let $A := \{e^{it} : t \in [-a/2, a/2]\}$. There is a constant $c_7 > 0$ depending only on N and μ such that*

$$\int_A |f(z)|\, |dz| \ge \exp(-c_7/a)$$

for every $f \in \mathcal{S}_N^\mu$ that is analytic on the arc A and satisfies $|f(\frac{1}{4Ne^\mu})| \ge \frac{1}{2}$.

To prove Lemma 3.3 we need the following.

Lemma 3.4. *Let $w_1 \ne w_2 \in \mathbb{C}$ and let $z_0 := \frac{1}{2}(w_1 + w_2)$. Assume that J_1 is an arc that connects w_1 and w_2. Let J_2 be the arc that is the symmetric image of J_1 with respect to the z_0. Let $J := J_1 \cup J_2$ be positively oriented. Suppose that g is an analytic function inside and on J. Suppose that the region inside J contains the disk centered at z_0 with radius $\gamma > 0$. Let $|g(z)| \le K$ for $z \in J_2$. Then*

$$|g(z_0)|^2 \le (\pi\gamma)^{-1} K \int_{J_1} |g(z)|\, |dz| .$$

Proof of Lemma 3.4. Applying Cauchy's integral formula with

$$G(z) := g(z_0 + (z - z_0))g(z_0 - (z - z_0))$$

on J, we obtain

$$\begin{aligned} |g(z_0)|^2 = |G(z_0)| &= \left| \frac{1}{2\pi i} \int_J \frac{G(z)\, dz}{z - z_0} \right| \\ &= \frac{2}{2\pi} \left| \int_{J_1} \frac{G(z)\, dz}{z - z_0} \right| \le \frac{1}{\pi} \int_{J_1} \frac{|G(z)|\, |dz|}{|z - z_0|} \\ &= \frac{1}{\pi} \int_{J_1} \frac{|g(z_0 + (z - z_0))g(z_0 - (z - z_0))|\, |dz|}{|z - z_0|} \\ &\le (\pi\gamma)^{-1} K \int_{J_1} |g(z)|\, |dz| . \end{aligned}$$

□

Proof of Lemma 3.3. Without loss of generality we may assume that $\ell(A) = a \le \pi/2$. Suppose $f \in \mathcal{S}_N^\mu$ and $|f(\frac{1}{4Ne^\mu})| \ge \frac{1}{2}$. Let the region H_a be defined by

$$H_a := \left\{ z = re^{i\theta} : \ \cos(a/4) < r < \cos(a/8)\,, \ -a/4 < \theta < a/4 \right\}.$$

Associated with $a \in (0,1]$ and $b \in [\frac{1}{2}, \frac{3}{4}]$ (the choice of b will be specified later), let $\Gamma_{a,b}$ be the circle as in Lemma 3.2. It is easy to see that the arc $I := \Gamma_{a,b} \cap H_a$ has length greater than $c_3 a$ with an absolute constant $c_3 > 0$. Let $f \in \mathcal{S}_N^\mu$. Let $z_0 \in I \subset H_a$ be chosen so that

$$|f(z_0)| = \max_{z \in I} |f(z)|.$$

Also, we can choose $w_1 \in A$ and $w_2 \in A$ such that $z_0 = \frac{1}{2}(w_1 + w_2)$. Let J_1 be the arc connecting w_1 and w_2 on the unit circle. Note that J_1 is a subarc of A of length at least $a/4$. Let J_2 be the arc which is the symmetric image of J_1 with respect to the line segment connecting w_1 and w_2. Let

$$g(z) := 4^{-\mu}((z - w_1)(z - w_2))^\mu f(z).$$

It is elementary geometry again to show that

$$|g(z)| \le \frac{4^{-\mu} N |(z - w_1)(z - w_2)|^\mu}{(1 - |z|)^\mu} \le \frac{4^{-\mu} N 2^\mu}{\sin^\mu(a/8)} = \frac{2^{-\mu} N}{\sin^\mu(a/8)}\,, \qquad z \in J_2.$$

By Lemma 3.4 we obtain

$$|g(z_0)|^2 \le (\pi(1 - \cos(a/8))^{-1} \frac{N 2^{-\mu}}{\sin^\mu(a/2)} \int_{J_1} |g(z)|\,|dz|. \tag{3.1}$$

Observe that $f \in \mathcal{S}_N^\mu$ implies $g \in \mathcal{S}_N^\mu$. Also, since $N \ge 1$, $\mu \ge 1$, and $|f(\frac{1}{4Ne^\mu})| \ge \frac{1}{2}$, we have

$$\begin{aligned} \left|g\left(\tfrac{1}{4Ne^\mu}\right)\right| \ge & 4^{-\mu}\left(1 - \tfrac{1}{4Ne^\mu}\right)^{2\mu} \left|f\left(\tfrac{1}{4Ne^\mu}\right)\right| \ge 4^{-\mu}\left(1 - \frac{1}{8\mu}\right)^{2\mu} \frac{1}{2} \ge 4^{-\mu}(\tfrac{7}{8})^2 \tfrac{1}{2} \\ \ge & 4^{-(\mu+1)}. \end{aligned}$$

Hence, by Lemma 3.1, we can pick $b \in [\frac{1}{2}, \frac{3}{4}]$ so that $|g(b)| \ge c_2$ with an absolute constant $c_2 > 0$ depending only on N and μ. Now we can deduce from Lemma 3.2 that

$$|g(z_0)| \ge \exp(-c_4/a). \tag{3.2}$$

Combining (3.2) with (3.1) and $J_1 \subset A$ gives

$$\begin{aligned} \int_A |f(z)|\,|dz| \ \ge & \int_A |f(z)|\,|dz| \ge \int_{J_1} |g(z)|\,|dz| \\ \ge & \ \pi(1 - \cos(a/8)) \frac{2^\mu \sin^\mu(a/8)}{N} |g(z_0)|^2 \\ \ge & \exp(-c_1/a) \end{aligned}$$

with a constant $c_1 > 0$ depending only N and μ. □

4. Proof of the theorems

Proof of Theorem 2.2. Let $f \in \mathcal{K}_M^\mu(\Lambda)$, where the exponents $\lambda_j \in \mathbb{R}$ satisfy

$$\lambda_0 = 0\,, \qquad \lambda_j \geq j > 0\,, \qquad j = 1, 2, \dots\,.$$

Then $f \in \mathcal{S}_{M\mu!}^{\mu+1}$ and f is analityc on the arc A. Also, if

$$|z_0| \leq \frac{1}{4M(\mu+1)!e^{\mu+1}}\,,$$

then

$$|f(z_0)| \geq 1 - M\sum_{j=1}^{\infty} \lambda_j^\mu \left(\frac{1}{4M(\mu+1)!e^{\mu+1}}\right)^{\lambda_j} \geq 1 - \frac{M}{4M(\mu+1)!}\sum_{j=1}^{\infty}\left(\frac{j}{e^j}\right)^{\mu+1}$$
$$\geq 1 - \frac{1}{4}\sum_{j=1}^{\infty}\left(\frac{j}{e^j}\right) \geq 1 - \frac{1}{4}\sum_{j=1}^{\infty}\left(\frac{j}{2^j}\right) 1 - \frac{2}{4} \geq \frac{1}{2}\,.$$

So the assumptions of Theorem 3.3 are satisfied with N replaced by $M\mu!$, and the theorem follows from Lemma 3.3. □

Theorem 2.1 is an obvious consequence of Theorem 2.2.

Acknowledgments

Work at Dartmouth was supported by the National Science Foundation under grant number PHY-0903727.

References

1. P. Borwein, *Computational Excursions in Analysis and Number Theory*, Springer, New York, 2002.
2. P. Borwein and T. Erdélyi, *Littlewood-type problems on subarcs of the unit circle*, Indiana Univ. Math. J. **46** (1997), 1323–1346.
3. P. Borwein and T. Erdélyi, *Polynomials and Polynomial Inequalities*, Springer-Verlag, New York, N.Y., 1995.
4. P. Borwein, T. Erdélyi, and G. Kós, *Littlewood-type problems on* $[0,1]$, Proc. London Math. Soc. **(3) 79** (1999), 22–46.
5. R.A. DeVore and G.G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, 1993.
6. K. Khodjasteh, T. Erdélyi, and L. Viola, *Limits on preserving quantum coherence using multi-pulse control* (to appear).
7. S. Güntürk, *Approximation by power series with* ± 1 *coefficients.*, Int. Math. Res. Not. **26** (2005), 1601–1610.
8. S.V. Konyagin, *On a problem of Littlewood*, Mathematics of the USSR, Izvestia **18** (1981), 205–225.

9. S. Konyagin, *On a question of Pichorides*, C.R. Acad. Sci. Paris Sér I Math. **324** (1997), 385–388.

10. O.C. McGehee, L. Pigno, and B. Smith, *Hardy's inequality and the L_1 norm of exponential sums*, Ann. of Math. **113** (1981), 613–618.

11. S.K. Pichorides, *Notes on trigonometric polynomials.*, in: Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I (Chicago, Illinois, 1981), 84–94, Wadsworth Math. Ser., Wadsworth, Belmont, Calif., 1983..

12. G.S. Uhrig, Phys. Rev. Lett. **98**, 100504 (2007).

13. L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999).

Department of Mathematics, Texas A&M University, College Station, Texas 77843
E-mail address: terdelyi@math.tamu.edu

Department of Physics and Astronomy, Dartmouth College, 6127 Wilder Laboratory, Hanover, NH 03755
E-mail address: kaveh.khodjasteh@dartmouth.edu

Department of Physics and Astronomy, Dartmouth College, 6127 Wilder Laboratory, Hanover, NH 03755
E-mail address: Lorenza.Viola@Dartmouth.edu