ON THE OSCILLATION OF THE MODULUS OF THE RUDIN-SHAPIRO POLYNOMIALS AROUND THE MIDDLE OF THEIR RANGES

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Abstract. Let either \( R_k(t) := |P_k(e^{it})|^2 \) or \( R_k(t) := |Q_k(e^{it})|^2 \), where \( P_k \) and \( Q_k \) are the usual Rudin-Shapiro polynomials of degree \( n - 1 \) with \( n = 2^k \). The graphs of \( R_k \) on the period suggest many zeros of the equation \( R_k(t) = n \) in a dense fashion. Let \( N(I, R_k - n) \) denote the number of zeros of \( R_k - n \) in an interval \( I := [\alpha, \beta] \subset [0, 2\pi] \). Improving earlier results stated only for \( I := [0, 2\pi] \), in this paper we show that

\[
\frac{n|I|}{8\pi} - \frac{4}{\pi}(n \log n)^{1/2} - 1 \leq N(I, R_k - n) \leq \frac{n|I|}{\pi} + \frac{16}{\pi}(n \log n)^{1/2}, \quad k \geq 2,
\]

for every \( I := [\alpha, \beta] \subset [0, 2\pi] \), where \( |I| = \beta - \alpha \) denotes the length of the interval \( I \).

1. Introduction

Let \( D := \{ z \in \mathbb{C} : |z| < 1 \} \) denote the open unit disk of the complex plane. Let \( \partial D := \{ z \in \mathbb{C} : |z| = 1 \} \) denote the unit circle of the complex plane. Littlewood polynomials are polynomials with each of their coefficients in \( \{-1, 1\} \). A special sequence of Littlewood polynomials, the Rudin-Shapiro polynomials appear in Harold Shapiro’s 1951 thesis [S-51] at MIT and are sometimes called just Shapiro polynomials. They also arise independently in Golay’s paper [G-51]. They are remarkably simple to construct and are a rich source of counterexamples to possible conjectures. The Rudin-Shapiro polynomials are defined recursively as follows:

\[
P_0(z) := 1, \quad Q_0(z) := 1, \\
P_{k+1}(z) := P_k(z) + z^{2^k} Q_k(z), \\
Q_{k+1}(z) := P_k(z) - z^{2^k} Q_k(z),
\]

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for $k = 0, 1, 2, \ldots$. Note that both $P_k$ and $Q_k$ are polynomials of degree $n - 1$ with $n := 2^k$ having each of their coefficients in \{-1, 1\}. In signal processing, the Rudin-Shapiro polynomials have good autocorrelation properties and their values on the unit circle are small. Binary sequences with low autocorrelation coefficients are of interest in radar, sonar, and communication systems.

It is well known and easy to check by using the parallelogram law that

$$|P_{k+1}(z)|^2 + |Q_{k+1}(z)|^2 = 2(|P_k(z)|^2 + |Q_k(z)|^2), \quad z \in \partial D.$$ 

Hence

(1.1) $$|P_k(z)|^2 + |Q_k(z)|^2 = 2^{k+1} = 2n, \quad z \in \partial D.$$ 

It is also well known (see Section 4 of [B-02], for instance), that

$$Q_k(-z) = P_k^*(z) = z^{n-1}P_k(1/z), \quad z \in \partial D,$$

and hence

(1.2) $$|Q_k(-z)| = |P_k(z)|, \quad z \in \partial D.$$ 

Despite the simplicity of their definition not much is known about the Rudin-Shapiro polynomials. Various properties of the Rudin-Shapiro polynomials are discussed in [B-73] by Brillhart and in [BL-76] by Brillhart, Lemont, and Morton. As for $k \geq 1$ both $P_k$ and $Q_k$ have odd degree, both $P_k$ and $Q_k$ have at least one real zero. The fact that for $k \geq 1$ both $P_k$ and $Q_k$ have exactly one real zero was proved in [B-73]. It has been shown in [E-16c] fairly recently that the Mahler measure (geometric mean) and the maximum modulus of the Rudin-Shapiro polynomials $P_k$ and $Q_k$ of degree $n - 1$ with $n := 2^k$ on the unit circle of the complex plane have the same size, that is, the Mahler measure of the Rudin-Shapiro polynomials of degree $n - 1$ with $n := 2^k$ is bounded from below by $cn^{1/2}$, where $c > 0$ is an absolute constant. In [E-20a] various results on the zeros of Rudin-Shapiro polynomials are proved and some open problems are raised. In [E-20b] a conjecture of Saffari on the asymptotic value of the Mahler measure of the Rudin-Shapiro polynomials is proved.

More on Littlewood polynomials may be found in [B-02], [E-02], [M-17], and [O-18], for example. There are many other publications on the zeros of constrained polynomials. Some of them are [BE-05], [BP-32], [BE-97], [BE-01], [BE-07], [BE-08a], [BE-08b], [BE-99], [BE-13], [B-97], [BN-73], [CG-00], [D-08], [E-08a], [E-08b], [E-16a], [E-16b], [E-20c], [HJ-20], [L-61], [L-64], [L-66a], [L-66b], [L-68], [M-06a], [M-06b], [R-17], [S-19], [Sch-32], [Sch-33], [Sz-34], and [TV-07].

For a monic polynomial

(1.3) $$P(z) = \prod_{j=1}^n (z - \alpha_j) = z^n + \sum_{j=0}^{n-1} a_j z^j, \quad a_j \in \mathbb{C},$$
let
\[ H(P) := \frac{1}{|a_0|^{1/2}} \max_{z \in \partial D} |P(z)|. \]

Obviously, with \( a_n := 1 \), we have
\[ H(P) \leq \frac{1}{|a_0|^{1/2}} \sum_{j=0}^{n} |a_j|. \]

Let
\[ \alpha_j = \rho_j e^{i\theta_j}, \quad \rho_j \geq 0, \quad \theta_j \in [0, 2\pi). \]

For \( I := [\alpha, \beta] \subset [0, 2\pi] \) let \( N(I, P) \) denote the number of the values \( j \in \{1, 2, \ldots, n\} \) for which \( \theta_j \in I \). In 1950 Erdős and Turán [ET-50] proved the following result.

**Theorem 1.3.** We have
\[ |N(I, P) - \frac{n|I|}{2\pi}| \leq 16\sqrt{n \log H(P)} \]
for every monic polynomial of the form (1.3) and for every \( I := [\alpha, \beta] \subset [0, 2\pi) \), where \( |I| = \beta - \alpha \) denotes the length of the interval \( I \).

In [S-19] K. Soundararajan proved that the constant 16 in the above result may be replaced by \( 8/\pi \).

### 2. New Results

Let either \( R_k(t) := |P_k(e^{it})|^2 \) or \( R_k(t) := |Q_k(e^{it})|^2 \), and \( n := 2^k \). In [AC-18] we combined close to sharp upper bounds for the modulus of the autocorrelation coefficients of the Rudin-Shapiro polynomials with a deep theorem of Littlewood (see Theorem 1 in [L-66]) to prove that there is an absolute constant \( A > 0 \) such that the equation \( R_k(t) = (1 + \eta)n \) with \( n := 2^k \) has at least \( An^{0.5394282} \) distinct zeros in \([0, 2\pi)\) whenever \( \eta \) is real, \( |\eta| \leq 2^{-11} \), and \( n \) is sufficiently large. In this paper we improve this result substantially.

**Theorem 2.1.** Let \( n := 2^k \) and let \( N(I, R_k - n) \) denote the number of zeros of the equation \( R_k - n \) in an interval \( I := [\alpha, \beta] \subset [0, 2\pi] \). We have
\[ \frac{n|I|}{8\pi} - \frac{4}{\pi} (n \log n)^{1/2} - 1 \leq N(I, R_k - n) \leq \frac{n|I|}{\pi} + \frac{16}{\pi} (n \log n)^{1/2}, \quad k \geq 2, \]
for every \( I := [\alpha, \beta] \subset [0, 2\pi] \), where \( |I| = \beta - \alpha \) denotes the length of the interval \( I \).

This extends the main result in [E-21] from the case \( I := [0, 2\pi) \) to the case \( I = [\alpha, \beta] \subset [0, 2\pi] \). In our proof of Theorem 2.1 we combine subtle ideas used in [E-21] and and old and classical result of Erdős and Turán [ET-50] with a constant improved recently by Soundararajan [S-19].
In the proof of Theorem 2.1 we need the lemma below stated and proved as Lemma 3.1 in [E-16].

**Lemma 3.1.** Let \( n \geq 2 \) be an integer, \( n := 2^k \), and let

\[
z_j := e^{it_j}, \quad t_j := \frac{2\pi j}{n}, \quad j \in \mathbb{Z}.
\]

We have

\[
P_k(z_j) = 2P_{k-2}(z_j), \quad j = 2u, \quad u \in \mathbb{Z},
\]

\[
P_k(z_j) = (-1)^{(j-1)/2}2iQ_{k-2}(z_j), \quad j = 2u + 1, \quad u \in \mathbb{Z},
\]

where \( i \) is the imaginary unit.

For a trigonometric polynomial of the form

\[
T(\theta) = \pm 2 \cos(m\theta) + \sum_{j=-m+1}^{m-1} a_j e^{ij\theta}, \quad a_j \in \mathbb{C},
\]

let

\[
H(T) := \max_{\theta \in \mathbb{R}} |T(\theta)|.
\]

Obviously, with \( a_m = a_{-m} := 1 \), we have

\[
H(T) \leq \sum_{j=-m}^{m} |a_j|.
\]

For \( I := [\alpha, \beta] \subset [0, 2\pi] \) let \( N(I, T) \) denote the number of zeros of \( T \) in \( I \) counted with multiplicities.

**Lemma 3.2.** We have

\[
\left| N(I, T) - \frac{m|I|}{\pi} \right| \leq \frac{8}{\pi} \sqrt{(2m+1) \log H(T)}
\]

for every trigonometric polynomial of the form (3.1) and for every \( I := [\alpha, \beta] \subset [0, 2\pi] \), where \( |I| := \beta - \alpha \).

**Proof.** This follows from the Erdős-Turán inequality (Theorem 1.3) with Soundararajan’s constant. □
Lemma 3.3. With the notation \( m = n/4 - 1 \) we have
\[
|N(I, R_{k-2}) - \frac{m|I|}{\pi}| \leq \frac{8}{\pi} \sqrt{n \log n}
\]
for every monic polynomial of the form (1.3) and for every \( I := [\alpha, \beta] \subset [0, 2\pi] \), where \(|I| := \beta - \alpha|.

Proof. Observe that \( R_{k-2} - n/4 \) is of the form
\[
R_{k-2}(t) - n/4 = \pm 2 \cos(m\theta) + \sum_{j=-m+1}^{m-1} a_j e^{ij\theta}, \quad a_0 = 0, \quad |a_j| \leq |n - j + 1|.
\]
(In fact, each \( a_j \) above is an integer.) Hence,
\[
H(T) \leq -\sum_{j=-m}^{-1} |m - j + 1| + \sum_{j=1}^{m} |m - j + 1| = m(m + 1) \leq \frac{n^2}{16}.
\]
and the lemma follows from Lemma 3.2. \( \Box \)

4. Proof of Theorem 2.1

Proof of Theorem 2.1. Let \( k \geq 2 \) be an integer and let \( I := [\alpha, \beta] \subset [0, 2\pi] \). Assume that \( R_k(t) = |P_k(e^{it})|^2 \), the proof in the case \( R_k(t) = |Q_k(e^{it})|^2 \) is the same.

The upper bound of the theorem follows from Lemma 3.2 applied to \( T := R_k \). Indeed, observe that
\[
R_k(t) - n = \pm 2 \cos(m\theta) + \sum_{j=-m+1}^{m-1} a_j e^{ij\theta}, \quad a_0 = 0, \quad |a_j| \leq |n - j + 1|,
\]
where \( m = n - 1 = 2^k - 1 \), so \( 2m + 1 = 2n - 1 \leq 2n \) and
\[
H(T) \leq -\sum_{j=-m}^{-1} |m - j + 1| + \sum_{j=1}^{m} |m - j + 1| = m(m + 1) \leq n^2.
\]

Now we prove the lower bound of the theorem. Without loss of generality we may assume that
\[
|I| \geq \frac{4\pi}{n},
\]
otherwise there is nothing to prove. For the sake of brevity let
\[
A_j := R_{k-2}(t_j) - n/4, \quad j \in \mathbb{Z}.
\]
Let
\[
t_h < \alpha \leq t_{h+1} < t_{h+M+1} \leq \beta < t_{h+M+2}.
\]
Observe that
\[ M \geq \frac{n|I|}{2\pi} - 2. \]

We study the \( N \)-tuple \( \langle A_{h+1}, A_{h+2}, \ldots, A_{h+M} \rangle \). Observe that \( R_{k-2} \) is a real trigonometric polynomial of degree \( n/4 - 1 = 2^k/4 - 1 \), and hence Lemma 3.3 implies that \( R_{k-2}(t) - n/4 \) has at most
\[ \frac{n}{4} \left| \frac{I}{\pi} \right| + \frac{8}{\pi} \sqrt{n \log n} \]
zeros in \( I \). Therefore the Intermediate Value Theorem yields that the number of sign changes in the \( M \)-tuple \( \langle A_{h+1}, A_{h+2}, \ldots, A_{h+M} \rangle \) is at most as large as the value in (4.2). Thus there are integers
\[ h + 1 \leq j_1 < j_2 < \cdots < j_N \leq h + M \]
with
\[ N \geq \frac{n|I|}{2\pi} - 2 - \frac{n}{4} \left| \frac{I}{\pi} \right| - \frac{8}{\pi} \sqrt{n \log n} \geq \frac{n|I|}{4\pi} - \frac{8}{\pi} \sqrt{n \log n} - 2 \]
such that
\[ A_{j_\nu}A_{j_\nu+1} \geq 0, \quad \nu = 1, 2, \ldots, N. \]

Using Lemma 3.1 we have either
\[ 16A_{j_\nu}A_{j_\nu+1} = (4(R_{k-2}(t_{j_\nu}) - n/4))(4(R_{k-2}(t_{j_\nu+1}) - n/4)) = (4|P_{k-2}(e^{it_{j_\nu}})|^2 - n)(4|P_{k-2}(e^{it_{j_\nu+1}})|^2 - n) = (|P_k(e^{it_{j_\nu}})|^2 - n)(|Q_k(e^{it_{j_\nu+1}})|^2 - n) = (|P_k(e^{it_{j_\nu}})|^2 - n)(n - |P_k(e^{it_{j_\nu+1}})|^2), \]
or
\[ 16A_{j_\nu}A_{j_\nu+1} = (4(R_{k-2}(t_{j_\nu}) - n/4))(4(R_{k-2}(t_{j_\nu+1}) - n/4)) = (4|P_{k-2}(e^{it_{j_\nu}})|^2 - n)(4|P_{k-2}(e^{it_{j_\nu+1}})|^2 - n) = (|Q_k(e^{it_{j_\nu}})|^2 - n)(|P_k(e^{it_{j_\nu+1}})|^2 - n) = (n - |P_k(e^{it_{j_\nu}})|^2)(|P_k(e^{it_{j_\nu+1}})|^2 - n). \]

Combining (4.4), (4.5), and (4.6), we can deduce that
\[ (|P_k(e^{it_{j_\nu}})|^2 - n)(|P_k(e^{it_{j_\nu+1}})|^2 - n) = -16A_{j_\nu}A_{j_\nu+1} \leq 0, \quad \nu = 1, 2, \ldots, N. \]
Hence the the Intermediate Value Theorem implies that \( R_k(t) - n = |P_k(e^{it})|^2 - n \) has at least one zero in each of the intervals
\[ [t_{j_\nu}, t_{j_\nu+1}], \quad \nu = 1, 2, \ldots, N. \]
Recalling (4.3) we conclude that \( R_k(t) - n = |P_k(e^{it})|^2 - n \) has at least
\[ N/2 \geq \frac{n|I|}{8\pi} - \frac{4}{\pi} \sqrt{n \log n} - 1 \]
distinct zeros in \( I \). \( \square \)
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References


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