FUNCTIONS WITH IDENTICAL $L_p$ NORMS

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Abstract. Suppose $P := (p_j)_{j=1}^\infty$ is a sequence of distinct real numbers $p_j > 0$. We prove that the equalities
\[ \|f\|_p = \|g\|_p, \quad p \in P, \]
imply
\[ \mu(\{x \in E : |f(x)| < \alpha\}) = \mu(\{x \in E : |g(x)| < \alpha\}), \quad \alpha \geq 0, \]
whenever $0 < \mu(E) < \infty$ and $f, g \in L_\infty(E)$ if and only if
\[ \sum_{j=1}^\infty \frac{p_j}{p_j^2 + 1} = \infty. \]

1. Introduction

Associated with a measure space $(E, \mathcal{A}, \mu)$ let
\[ \|f\|_p := \left( \int_E |f(x)|^p \, d\mu(x) \right)^{1/p}, \quad p > 0, \]
and
\[ \|f\|_\infty := \inf \{\alpha \in \mathbb{R} : \mu(\{x \in E : |f(x)| > \alpha\}) = 0\}. \]

Using the “Full Müntz Theorem in $C[0,1]$” [1,2] G. Klun [6] proved the following result.

Theorem 1.1. Suppose $f, g \in L_p(E)$ for all $p \geq 1$, $f, g \in L_\infty(E)$, and $P := (p_j)_{j=1}^\infty$ is a sequence of distinct real numbers $p_j \geq 1$ such that
\[ \sum_{j=1}^\infty \frac{p_j - 1}{(p_j - 1)^2 + 1} = \infty. \]
The equalities
\[ \|f\|_p = \|g\|_p, \quad p \in P, \]
imply
\[ \mu(\{x \in E : |f(x)| < \alpha\}) = \mu(\{x \in E : |g(x)| < \alpha\}), \quad \alpha \geq 0. \]

It is quite remarkable that Klun does not assume $\mu(E) < \infty$ in the above theorem as he applies it elegantly when $E := \mathbb{N}$ and $\mu(A)$ is the number of elements in $A \subset \mathbb{N}$.

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2. New Result

In this note we prove the following result.

**Theorem 2.1.** Suppose $P := (p_j)_{j=1}^\infty$ is a sequence of distinct real numbers $p_j > 0$. The equalities

$$\|f\|_p = \|g\|_p, \quad p \in P,$$

imply

$$\mu(\{x \in E : |f(x)| < \alpha\}) = \mu(\{x \in E : |g(x)| < \alpha\}), \quad \alpha \geq 0,$$

whenever $0 < \mu(E) < \infty$ and $f, g \in L_\infty(E)$ if and only if

$$\sum_{j=1}^\infty \frac{p_j}{p_j^2 + 1} = \infty.$$

Note that under the assumption $0 < \mu(E) < \infty$ the above theorem allows arbitrary sequences $P := (p_j)_{j=1}^\infty$ of distinct real numbers $p_j > 0$ rather than only $p_j \geq 1$, and it is an “if and only if” extension of Theorem 1.1.

3. Proof

The proof of the “if” part of Theorem 1 is based on the following result called “Full M"untz Theorem” in $L_p(A)$ for $p \in (0, \infty)$ and for compact sets $A \subset [0,1]$ with positive lower density at 0.

**Theorem 3.1.** Let $A \subset [0,1]$ be a compact set with positive lower density at 0. Let $p \in (0, \infty)$. Suppose $(\lambda_j)_{j=1}^\infty$ is a sequence of distinct real numbers greater than $-(1/p)$. Then $\text{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ is dense in $L_p(A)$ if and only if

$$\sum_{j=1}^\infty \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} = \infty.$$

Theorem 3.1 is proved in [5] by Erdélyi and Johnson, and it improves and extends earlier results of Müntz [7], Szász [9], Clarkson and Erdős [3], P. Borwein and Erdélyi [1,2], and Operstein [8]. Another proof of Theorem 3.1 is given in [4]. In fact, to prove the “if” part of Theorem 2.1 we need only the case where $p = 1$ and $A = [0,1]$ proved first in [2]. To prove the “only if” part of Theorem 2.1 we also need the following result.

**Theorem 3.2 (Full Müntz Theorem in $C[0,1]$).** Suppose $(\lambda_j)_{j=1}^\infty$ is a sequence of distinct positive real numbers. Then $\text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ is dense in $C[0,1]$ if and only if

$$\sum_{j=1}^\infty \frac{\lambda_j}{\lambda_j^2 + 1} = \infty.$$
Theorem 3.2 is proved in Borwein and Erdélyi [1,2].

Proof of Theorem 2.1. First we prove the “if” part of the theorem. Assume that (2.2) holds. Let \( 0 < \mu(E) < \infty \) and \( f, g \in L_{\infty}(E) \). Multiplying by constants, without loss of generality, we may assume that \( \mu(E) = 1 \) and

\[
\|f\|_\infty \leq \|g\|_\infty = 1.
\]

We define

\[
F(t) := \mu(\{x \in E : |f(x)| < t\}) \quad \text{and} \quad G(t) := \mu(\{x \in E : |g(x)| < t\}).
\]

Then \( h(t) := G(t) - F(t) \) is well defined for all \( t \in \mathbb{R} \), and \( h(t) = 0 \) for all \( t \in \mathbb{R} \setminus [0,1] \), and \( |h(t)| \leq 1 \) for all \( t \in [0,1] \). Hence (2.1) implies that

\[
0 = \int_E (|g(x)|^p - |f(x)|^p) \, d\mu(x) = \int_0^1 (G(y^{1/p}) - F(y^{1/p})) \, dy, \quad p \in P.
\]

Substituting \( t = y^{1/p} \) we obtain

\[
(3.1) \quad 0 = p \int_0^1 (G(t) - F(t)) t^{p-1} \, dt = p \int_0^1 h(t) t^{p-1} \, dt, \quad p \in P,
\]

where \( h(t) := G(t) - F(t) \in L_{\infty}[0,1] \). In the light of the case \( p = 1 \) and \( A = [0,1] \) of Theorem 3.1 (2.2) implies that \( \text{span}\{x^{p-1} : p \in P\} \) is dense in \( L_1[0,1] \). Using (3.1) we obtain

\[
\int_0^1 h(t) u(t) \, dt = 0
\]

for every \( u \in L_1[0,1] \), and hence, choosing \( u = h \in L_\infty[0,1] \subset L_1[0,1] \), we have

\[
\int_0^1 h(t)^2 \, dt = 0,
\]

and \( h(t) = 0 \) for almost every \( t \in [0,1] \) follows. We conclude that \( G(t) = F(t) \) for almost every \( t \in [0,1] \). However, as both \( F \) and \( G \) are continuous from left on \( \mathbb{R} \), we have \( G(t) = F(t) \) for every \( t \in \mathbb{R} \).

Assume now that (2.2) does not holds. Combining Theorem 3.2, the Hahn-Banach Theorem, and the Riesz Representation Theorem, we can deduce that there is a finite signed Borel measure \( \mu \) on \([0,1]\) and a function \( h \in C[0,1] \) such that

\[
(3.2) \quad \int_0^1 t^p \, d\mu(t) = 0, \quad p \in P,
\]

and

\[
(3.3) \quad \int_0^1 h(t) \, d\mu(t) = 1.
\]
Let \( \{E_1, E_2\} \) be the Hahn decomposition of the measure \( \mu \) on \([0, 1]\), that is, \( E_1 \) and \( E_2 \) are Borel measurable, \([0, 1] = E_1 \cup E_2, E_1 \cap E_2 = \emptyset \), and \( \mu(A) \geq 0 \) for any Borel set \( A \subset E_1 \) and \( \mu(A) \leq 0 \) for any Borel measurable set \( A \subset E_2 \). We define the Borel measurable functions

\[
  f(t) := \begin{cases} 
    t, & t \in E_1 \\
    0, & t \in E_2
  \end{cases}
  \quad \text{and} \quad
  g(t) := \begin{cases} 
    t, & t \in E_2 \\
    0, & t \in E_1
  \end{cases}
\]

We have \( \|f\|_\infty \leq 1 \) and \( \|g\|_\infty \leq 1 \). We define the finite nonnegative measure

\[
  |\mu|(A) := \mu(A \cap E_1) - \mu(A \cap E_2)
\]

for Borel sets \( A \subset [0, 1] \). It follows from (3.2) and the definitions of \( f \) and \( g \) that

\[
  0 = \int_0^1 (|f(t)|^p - |g(t)|^p) \, d|\mu|(t), \quad p \in P,
\]

that is,

\[
  \int_0^1 |f(t)|^p \, d|\mu|(t) = \int_0^1 |g(t)|^p \, d|\mu|(t), \quad p \in P.
\]

Now we show that there is an \( \alpha \geq 0 \) such that

\[
  |\mu|(\{x \in E : |f(x)| < \alpha\}) \neq |\mu|(\{x \in E : |g(x)| < \alpha\}).
\]

Suppose to the contrary that

\[
  \mu(\{x \in E : |f(x)| < \alpha\}) = \mu(\{x \in E : |g(x)| < \alpha\})
\]

for every \( \alpha \geq 0 \). Then

\[
  \int_0^1 (|f(t)|^p - |g(t)|^p) \, d|\mu|(t) = 0, \quad p > 0,
\]

and hence

\[
  \int_0^1 t^p \, d\mu(t) = 0, \quad p > 0.
\]

Hence it follows from the Weierstrass Theorem that

\[
  \int_0^1 u(t) \, d\mu(t) = 0
\]

for every \( u \in C[0, 1] \), which contradicts (3.3). This completes the proof of the “only if” part of the theorem. \( \Box \)
References


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