FUNCTIONS WITH IDENTICAL $L_p$ NORMS

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Abstract. Suppose $P := (p_j)_{j=1}^\infty$ is a sequence of distinct real numbers $p_j > 0$. We prove that the equalities
$$\|f\|_p = \|g\|_p, \quad p \in P,$$
imply
$$\mu(\{x \in E : |f(x)| < \alpha\}) = \mu(\{x \in E : |g(x)| < \alpha\}), \quad \alpha \geq 0,$$whenever $0 < \mu(E) < \infty$ and $f, g \in L_\infty(E)$ if and only if
$$\sum_{j=1}^\infty \frac{p_j}{p_j^2 + 1} = \infty.$$

1. Introduction

Associated with a measure space $(E, \mathcal{A}, \mu)$ let
$$\|f\|_p := \left(\int_E |f(x)|^p \, d\mu(x)\right)^{1/p}, \quad p > 0,$$and
$$\|f\|_\infty := \inf\{\alpha \in \mathbb{R} : \mu(\{x \in E : |f(x)| > \alpha\}) = 0\}.$$Using the “Full Müntz Theorem in $C[0,1]$” [1,2] G. Klun [6] proved the following result.

Theorem 1.1. Suppose $f, g \in L_p(E)$ for all $p \geq 1$, $f, g \in L_\infty(E)$, and $P := (p_j)_{j=1}^\infty$ is a sequence of distinct real numbers $p_j \geq 1$ such that
$$\sum_{j=1}^\infty \frac{p_j - 1}{(p_j - 1)^2 + 1} = \infty.$$The equalities
$$\|f\|_p = \|g\|_p, \quad p \in P,$$imply
$$\mu(\{x \in E : |f(x)| < \alpha\}) = \mu(\{x \in E : |g(x)| < \alpha\}), \quad \alpha \geq 0.$$It is quite remarkable that Klun does not assume $\mu(E) < \infty$ in the above theorem as he applies it elegantly when $E := \mathbb{N}$ and $\mu(A)$ is the number of elements in $A \subset \mathbb{N}$.

Key words and phrases. “Full Müntz Theorem”, denseness in $L_p[0,1]$, functions with identical $L_p$ norms.

2010 Mathematics Subject Classifications. 41A17
2. New Result

In this note we prove the following result.

**Theorem 2.1.** Suppose \( P := (p_j)_{j=1}^{\infty} \) is a sequence of distinct real numbers \( p_j > 0 \). The equalities

\[
\|f\|_p = \|g\|_p, \quad p \in P,
\]

imply

\[
\mu(\{x \in E : |f(x)| < \alpha\}) = \mu(\{x \in E : |g(x)| < \alpha\}), \quad \alpha \geq 0,
\]

whenever \( 0 < \mu(E) < \infty \) and \( f, g \in L_\infty(E) \) if and only if

\[
\sum_{j=1}^{\infty} \frac{p_j}{p_j^2 + 1} = \infty.
\]

Note that under the assumption \( 0 < \mu(E) < \infty \) the above theorem allows arbitrary sequences \( P := (p_j)_{j=1}^{\infty} \) of distinct real numbers \( p_j > 0 \) rather than only \( p_j \geq 1 \), and it is an “if and only if” extension of Theorem 1.1.

3. Proof

The proof of the “if” part of Theorem 1 is based on the following result called “Full Müntz Theorem” in \( L_p(A) \) for \( p \in (0, \infty) \) and for compact sets \( A \subset [0,1] \) with positive lower density at 0.

**Theorem 3.1.** Let \( A \subset [0,1] \) be a compact set with positive lower density at 0. Let \( p \in (0, \infty) \). Suppose \( (\lambda_j)_{j=1}^{\infty} \) is a sequence of distinct real numbers greater than \(-1/p\). Then \( \text{span}\{x^{\lambda_1}, x^{\lambda_2}, \ldots\} \) is dense in \( L_p(A) \) if and only if

\[
\sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} = \infty.
\]

Theorem 3.1 is proved in [5] by Erdélyi and Johnson, and it improves and extends earlier results of Müntz [7], Szász [9], Clarkson and Erdős [3], P. Borwein and Erdélyi [1,2], and Operstein [8]. Another proof of Theorem 3.1 is given in [4]. In fact, to prove the “if” part of Theorem 3.1 we need only the case where \( p = 1 \) and \( A = [0,1] \) proved first in [2]. To prove the “only if” part of Theorem 3.1 we also need the following result.

**Theorem 3.2 (Full Müntz Theorem in \( C[0,1] \)).** Suppose \( (\lambda_j)_{j=1}^{\infty} \) is a sequence of distinct positive real numbers. Then \( \text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\} \) is dense in \( C[0,1] \) if and only if

\[
\sum_{j=1}^{\infty} \frac{\lambda_j}{\lambda_j^2 + 1} = \infty.
\]
Theorem 3.2 is proved in Borwein and Erdélyi [1,2].

Proof of Theorem 2.1. First we prove the “if” part of the theorem. Assume that (2.2) holds. Let $0 < \mu(E) < \infty$ and $f, g \in L_\infty(E)$. Multiplying by constants, without loss of generality, we may assume that $\mu(E) = 1$ and
\[ \|f\|_\infty \leq \|g\|_\infty = 1. \]

We define
\[ F(t) := \mu(\{x \in E : |f(x)| < t\}) \quad \text{and} \quad G(t) := \mu(\{x \in E : |g(x)| < t\}). \]

Then $h(t) := G(t) - F(t)$ is well defined for all $t \in \mathbb{R}$, and $h(t) = 0$ for all $t \in \mathbb{R} \setminus [0, 1]$, and $|h(t)| \leq 1$ for all $t \in [0, 1]$. Hence (2.1) implies that
\[ 0 = \int_E (|g(x)|^p - |f(x)|^p) \, d\mu(x) = \int_0^1 (G(y^{1/p}) - F(y^{1/p})) \, dy, \quad p \in P. \]

Substituting $t = y^{1/p}$ we obtain
\[ (3.1) \quad 0 = p \int_0^1 (G(t) - F(t)) t^{p-1} \, dt = p \int_0^1 h(t) t^{p-1} \, dt, \quad p \in P, \]

where $h(t) := G(t) - F(t) \in L_\infty[0, 1]$. In the light of the case $p = 1$ and $A = [0, 1]$ of Theorem 3.1 (2.2) implies that span$\{x^{p-1} : p \in P\}$ is dense in $L_1[0, 1]$. Using (3.1) we obtain
\[ \int_0^1 h(t) u(t) \, dt = 0 \]

for every $u \in L_1[0, 1]$, and hence, choosing $u = h \in L_\infty[0, 1] \subset L_1[0, 1]$, we have
\[ \int_0^1 h(t)^2 \, dt = 0, \]

and $h(t) = 0$ for almost every $t \in [0, 1]$ follows. We conclude that $G(t) = F(t)$ for almost every $t \in [0, 1]$. However, as both $F$ and $G$ are continuous from left on $\mathbb{R}$, we have $G(t) = F(t)$ for every $t \in \mathbb{R}$.

Assume now that (2.2) does not holds. Combining Theorem 3.2, the Hahn-Banach Theorem, and the Riesz Representation Theorem, we can deduce that there is a finite signed Borel measure $\mu$ on $[0, 1]$ and a function $h \in C[0, 1]$ such that
\[ (3.2) \quad \int_0^1 t^p \, d\mu(t) = 0, \quad p \in P, \]

and
\[ (3.3) \quad \int_0^1 h(t) \, d\mu(t) = 1. \]
Let \( \{E_1, E_2\} \) be the Hahn decomposition of the measure \( \mu \) on \([0,1]\), that is, \( E_1 \) and \( E_2 \) are Borel measurable, \([0,1] = E_1 \cup E_2, E_1 \cap E_2 = \emptyset \), and \( \mu(A) \geq 0 \) for any Borel set \( A \subset E_1 \) and \( \mu(A) \leq 0 \) for any Borel measurable set \( A \subset E_2 \). We define the Borel measurable functions
\[
f(t) := \begin{cases} 
  t, & t \in E_1 \\
  0, & t \in E_2,
\end{cases}
\quad \text{and} \quad g(t) := \begin{cases} 
  t, & t \in E_2 \\
  0, & t \in E_1 .
\end{cases}
\]
We have \( \|f\|_\infty \leq 1 \) and \( \|g\|_\infty \leq 1 \). We define the finite nonnegative measure
\[
|\mu|(A) := \mu(A \cap E_1) - \mu(A \cap E_2)
\]
for Borel sets \( A \subset [0,1] \). It follows from (3.2) and the definitions of \( f \) and \( g \) that
\[
0 = \int_0^1 (|f(t)|^p - |g(t)|^p) \, d|\mu|(t) , \quad p \in P ,
\]
that is,
\[
\int_0^1 |f(t)|^p \, d|\mu|(t) = \int_0^1 |g(t)|^p \, d|\mu|(t) , \quad p \in P .
\]
Now we show that there is an \( \alpha \geq 0 \) such that
\[
|\mu|(\{x \in E : |f(x)| < \alpha\}) \neq |\mu|(\{x \in E : |g(x)| < \alpha\}) .
\]
Suppose to the contrary that
\[
\mu(\{x \in E : |f(x)| < \alpha\}) = \mu(\{x \in E : |g(x)| < \alpha\})
\]
for every \( \alpha \geq 0 \). Then
\[
\int_0^1 (|f(t)|^p - |g(t)|^p) \, d|\mu|(t) = 0 , \quad p > 0 ,
\]
and hence
\[
\int_0^1 t^p \, d\mu(t) = 0 , \quad p > 0 .
\]
Hence it follows from the Weierstrass Theorem that
\[
\int_0^1 u(t) \, d\mu(t) = 0
\]
for every \( u \in C[0,1] \), which contradicts (3.3). This completes the proof of the “only if” part of the theorem. \( \Box \)
References


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