FUNCTIONS WITH IDENTICAL $L_p$ NORMS

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Abstract. Suppose $P := (p_j)_{j=1}^\infty$ is a sequence of distinct real numbers $p_j > 0$. We prove that the equalities

$$\|f\|_p = \|g\|_p, \quad p \in P,$$

imply

$$\mu(\{x \in E : |f(x)| > \alpha\}) = \mu(\{x \in E : |g(x)| > \alpha\}), \quad \alpha \geq 0,$$

whenever $0 < \mu(E) < \infty$ and $f, g \in L_\infty(E)$ if and only if

$$\sum_{j=1}^\infty \frac{p_j}{p_j^2 + 1} = \infty.$$

1. Introduction

Associated with a measure space $(E, A, \mu)$ let

$$\|f\|_p := \left( \int_E |f(x)|^p \, d\mu(x) \right)^{1/p}, \quad p > 0,$$

and

$$\|f\|_\infty := \inf\{\alpha \in \mathbb{R} : \mu(\{x \in E : |f(x)| > \alpha\}) = 0\}.$$

Using the “Full Müntz Theorem in $C[0,1]$” [1,2] G. Klun [6] proved the following result.

Theorem 1.1. Suppose $f, g \in L_p(E)$ for all $p \geq 1$, $f, g \in L_\infty(E)$, and $P := (p_j)_{j=1}^\infty$ is a sequence of distinct real numbers $p_j \geq 1$ such that

$$\sum_{j=1}^\infty \frac{p_j - 1}{(p_j - 1)^2 + 1} = \infty.$$

The equalities

$$\|f\|_p = \|g\|_p, \quad p \in P,$$

hold for all $p \geq 1$. Furthermore, $f, g \in L_p(E)$ for all $p \geq 1$ if and only if

$$\sum_{j=1}^\infty \frac{p_j}{p_j^2 + 1} = \infty.$$

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imply
\[ \mu(\{x \in E : |f(x)| < \alpha\}) = \mu(\{x \in E : |g(x)| < \alpha\}), \quad \alpha \geq 0. \]

It is quite remarkable that Klun does not assume \( \mu(E) < \infty \) in the above theorem as he applies it elegantly when \( E := \mathbb{N} \) and \( \mu(A) \) is the number of elements in \( A \subset \mathbb{N} \).

2. New Result

In this note we prove the following result.

**Theorem 2.1.** Suppose \( P := (p_j)_{j=1}^{\infty} \) is a sequence of distinct real numbers \( p_j > 0 \). The equalities
\[ (2.1) \quad \|f\|_p = \|g\|_p, \quad p \in P, \]
imply
\[ \mu(\{x \in E : |f(x)| < \alpha\}) = \mu(\{x \in E : |g(x)| < \alpha\}), \quad \alpha \geq 0, \]
whenever \( 0 < \mu(E) < \infty \) and \( f, g \in L_\infty(E) \) if and only if
\[ (2.2) \quad \sum_{j=1}^{\infty} \frac{p_j}{p_j^2 + 1} = \infty. \]

Note that under the assumption \( 0 < \mu(E) < \infty \) the above theorem allows arbitrary sequences \( P := (p_j)_{j=1}^{\infty} \) of distinct real numbers \( p_j > 0 \) rather than only \( p_j \geq 1 \), and it is an “if and only if” extension of Theorem 1.1.

Note also that a careful reading of the “only if” part of Theorem 2.1 means that if (2.2) does not hold, then there is a measure space \((E, A, \mu)\) with \( 0 < \mu(E) < \infty \) and there are two functions \( f, g \in L_\infty(E) \) such that (2.1) holds but
\[ \mu(\{x \in E : |f(x)| < \alpha\}) \neq \mu(\{x \in E : |g(x)| < \alpha\}) \]
for at least one value of \( \alpha \geq 0 \).

3. Proof

The proof of the “if” part of Theorem 2.1 is based on the following result called “Full Müntz Theorem” in \( L_p(A) \) for \( p \in (0, \infty) \) and for compact sets \( A \subset [0, 1] \) with positive lower density at 0. In Theorem 3.1 below \( L_p(A) \) is considered with respect to the Lebesgue measure.

**Theorem 3.1.** Let \( A \subset [0, 1] \) be a compact set with positive lower density at 0. Let \( p \in (0, \infty) \). Suppose \((\lambda_j)_{j=1}^{\infty}\) is a sequence of distinct real numbers greater than \(-1/p\). Then span\{\( x^{\lambda_1}, x^{\lambda_2}, \ldots \)\} is dense in \( L_p(A) \) if and only if
\[ \sum_{j=1}^{\infty} \frac{\lambda_j + (1/p)}{(\lambda_j + (1/p))^2 + 1} = \infty. \]

Theorem 3.1 is proved in [5] by Erdélyi and Johnson, and it improves and extends earlier results of Müntz [7], Szász [10], Clarkson and Erdős [3], P. Borwein and Erdélyi [1,2], and Operstein [8]. Another proof of Theorem 3.1 is given in [4]. In fact, to prove the “if” part of Theorem 2.1 we need only the case where \( p = 1 \) and \( A = [0, 1] \) proved first in [2]. To prove the “only if” part of Theorem 2.1 we also need the following result.
**Theorem 3.2 (Full M"untz Theorem in $C[0,1]$).** Suppose $(\lambda_j)_{j=1}^\infty$ is a sequence of distinct positive real numbers. Then $\text{span}\{1, x^{\lambda_1}, x^{\lambda_2}, \ldots\}$ is dense in $C[0,1]$ if and only if

$$\sum_{j=1}^\infty \frac{\lambda_j}{\lambda_j^2 + 1} = \infty.$$

Theorem 3.2 is proved by Borwein and Erdélyi [1,2].

**Proof of Theorem 2.1.** First we prove the “if” part of the theorem. Assume that (2.2) holds. Let $0 < \mu(E) < \infty$ and $f, g \in L_\infty(E)$. Multiplying by constants, without loss of generality, we may assume that $\mu(E) = 1$ and

$$\|f\|_\infty \leq \|g\|_\infty = 1.$$

We define

$$F(t) := \mu(\{x \in E : |f(x)| < t\}) \quad \text{and} \quad G(t) := \mu(\{x \in E : |g(x)| < t\}).$$

Then $h(t) := G(t) - F(t)$ is well defined for all $t \in \mathbb{R}$, and $h(t) = 0$ for all $t \in \mathbb{R} \setminus [0,1]$, and $|h(t)| \leq 1$ for all $t \in [0,1]$. Hence (2.1) implies that

$$0 = \int_E (|g(x)|^p - |f(x)|^p) \, d\mu(x) = \int_0^1 (G(y^{1/p}) - F(y^{1/p})) \, dy, \quad p \in P.$$

Substituting $t = y^{1/p}$ we obtain

$$0 = \int_0^1 (G(t) - F(t)) \, t^{p-1} \, dt = p \int_0^1 h(t) t^{p-1} \, dt, \quad p \in P,$$

where $h(t) := G(t) - F(t) \in L_\infty[0,1]$. In the light of the case $p = 1$ and $A = [0,1]$ of Theorem 3.1, (2.2) implies that $\text{span}\{x^{p-1} : p \in P\}$ is dense in $L_1[0,1]$. Using (3.1) we obtain

$$\int_0^1 h(t) u(t) \, dt = 0$$

for every $u \in L_1[0,1]$, and hence, choosing $u = h \in L_\infty[0,1] \subset L_1[0,1]$, we have

$$\int_0^1 h(t)^2 \, dt = 0,$$

and $h(t) = 0$ for almost every $t \in [0,1]$ follows. We conclude that $G(t) = F(t)$ for almost every $t \in [0,1]$. However, as both $F$ and $G$ are continuous from left on $\mathbb{R}$, we have $G(t) = F(t)$ for every $t \in \mathbb{R}$.

To prove the “only if” part of the theorem assume now that (2.2) does not holds. We show that there is a finite Borel measure $\mu$ on $E := [0,1]$ with $0 < \mu(E) < \infty$ and there are two functions $f, g \in L_\infty(E)$ such that (2.1) holds but

$$\mu(\{x \in E : |f(x)| < \alpha\}) \neq \mu(\{x \in E : |g(x)| < \alpha\}).$$
for at least one value of $\alpha \geq 0$.

Combining Theorem 3.2, the Hahn-Banach Theorem (see [9, page 107]), and the Riesz Representation Theorem (see [9, page 40]) we can deduce that there is a finite signed Borel measure $\nu$ on $[0, 1]$ and a function $h \in C[0, 1]$ such that

\[
\int_0^1 t^p \, d\nu(t) = 0, \quad p \in P,
\]

and

\[
\int_0^1 h(t) \, d\nu(t) = 1.
\]

Indeed, let $M$ be the closure of the subspace spanned by $\{t^p : p \in P\}$ in $C[0, 1]$, where $C[0, 1]$ is the linear space of continuous functions on $[0, 1]$ equipped with the uniform norm. If (2.2) does not hold then Theorem 3.2 implies that $M \neq C[0, 1]$, and hence there is an $h \in C[0, 1] \setminus M$. By the Hahn-Banach Theorem there is a bounded linear functional $f$ on $C[0, 1]$ such that $f(u) = 0$ for all $u \in M$ and $f(h) = 1$. By the Riesz representation Theorem this bounded linear functional $f$ on $C[0, 1]$ can be represented by a finite signed Borel measure $\nu$ on $[0, 1]$ such that

\[
f(u) = \int_0^1 u(t) \, d\nu(t),
\]

and the proof of (3.2) and (3.3) is finished.

Let $\{E_1, E_2\}$ be the Hahn decomposition of the measure $\nu$ on $[0, 1]$, that is, $E_1$ and $E_2$ are Borel measurable, $[0, 1] = E_1 \cup E_2$, $E_1 \cap E_2 = \emptyset$, and $\nu(A) \geq 0$ for any Borel set $A \subset E_1$ and $\nu(A) \leq 0$ for any Borel measurable set $A \subset E_2$. We define the Borel measurable functions

\[
f(t) := \begin{cases} t, & t \in E_1 \\ 0, & t \in E_2 \end{cases}, \quad \text{and} \quad g(t) := \begin{cases} t, & t \in E_2 \\ 0, & t \in E_1 \end{cases}.
\]

We have $\|f\|_{\infty} \leq 1$ and $\|g\|_{\infty} \leq 1$. We define the finite nonnegative measure

\[
|\nu|(A) := \nu(A \cap E_1) - \nu(A \cap E_2)
\]

for Borel sets $A \subset [0, 1]$. It follows from (3.2) and the definitions of $f$ and $g$ that

\[
0 = \int_0^1 (|f(t)|^p - |g(t)|^p) \, d|\nu|(t), \quad p \in P,
\]

that is,

\[
\int_0^1 |f(t)|^p \, d|\nu|(t) = \int_0^1 |g(t)|^p \, d|\nu|(t), \quad p \in P.
\]
Now we show that there is an \( \alpha \geq 0 \) such that
\[
|\nu|\{x \in E : |f(x)| < \alpha\} \neq |\nu|\{x \in E : |g(x)| < \alpha\}.
\]
Suppose to the contrary that
\[
|\nu|\{x \in E : |f(x)| < \alpha\} = |\nu|\{x \in E : |g(x)| < \alpha\}
\]
for every \( \alpha \geq 0 \). Then
\[
\int_0^1 (|f(t)|^p - |g(t)|^p) \, d|\nu|(t) = 0, \quad p > 0,
\]
and hence
\[
\int_0^1 t^p \, d\nu(t) = 0, \quad p > 0.
\]
Hence it follows from the Weierstrass Theorem that
\[
\int_0^1 u(t) \, d\nu(t) = 0
\]
for every \( u \in C[0,1] \), which contradicts (3.3). This completes the proof of the “only if” part of the theorem. \( \square \)

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References