11.1. Sequences

Definition. A sequence \( \{a_n\}_{n=1}^{\infty} \) is a list of real numbers in a definite order. We may think about sequences as functions defined on the set of natural numbers (positive integers), that is,

\[
a_n := f(n), \quad n = 1, 2, \ldots.
\]

The number \( a_n \) is called the \( n \)th element of the sequence. Often times the function \( f \) defining the \( n \)th element of a sequence is seen explicitly. If \( a_n := \frac{2^n - 1}{2^n} \), \( n = 1, 2, \ldots \), then

\[
f(n) = \frac{2^n - 1}{2^n}, \quad n = 1, 2, \ldots.
\]

We may write

\[
\{a_n\}_{n=1}^{\infty} = \left\{ \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots \right\}.
\]

However the function \( f \) defining the \( n \)th element of a sequence may be hidden, but often times even in such a case it can be found explicitly. An example is the Fibonacci sequence defined by the recursion

\[
a_0 = 0, \quad a_1 = 1, \quad a_n = a_{n-2} + a_{n-1}, \quad n = 2, 3, \ldots
\]

So \( \{a_n\}_{n=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \} \). We claim that

\[
a_n = f(n) = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right), \quad n = 0, 1, \ldots.
\]

Indeed if \( \alpha \) is a zero of the equation \( x^2 = x + 1 \), that is

\[
\alpha = \frac{1 + \sqrt{5}}{2} \quad \text{or} \quad \alpha = \frac{1 - \sqrt{5}}{2},
\]

then \( \alpha^2 = \alpha + 1 \). Multiplying by \( \alpha^n \) we have \( \alpha^{n+2} = \alpha^{n+1} + \alpha^n \), that is, the sequence \( \{\alpha^n\}_{n=0}^{\infty} \) satisfies the recursion (1). Hence if \( A \) and \( B \) are arbitrary real numbers then

\[
a_n := A \left( \frac{1 + \sqrt{5}}{2} \right)^n + B \left( \frac{1 - \sqrt{5}}{2} \right)^n, \quad n = 0, 1, 2, \ldots,
\]

satisfies the recursion (1). To arrange \( a_0 = 0 \) and \( a_1 = 1 \) we have to solve the simple linear system of equations

\[
A + B = 0, \quad A \left( \frac{1 + \sqrt{5}}{2} \right) + B \left( \frac{1 - \sqrt{5}}{2} \right) = 1
\]

to get \( A = 1/\sqrt{5} \) and \( B = -1/\sqrt{5} \).
Definition of the Limit of a Sequence

Recall the three definitions below as you have learned them from Math 151.

Definition 1.
\[ \lim_{x \to \infty} f(x) = L \]
if the values \( f(x) \) are as close to \( L \) as we wish for all sufficiently large \( x \).

Note that in advanced calculus the above definition is formulated more precisely.

Definition 1*.
\[ \lim_{x \to \infty} f(x) = L \]
if for every \( \varepsilon > 0 \) there is an \( N \) such that \( |f(x) - L| < \varepsilon \) for all \( x \geq N \).

Definition 2.
\[ \lim_{x \to \infty} f(x) = \infty \]
if the values \( f(x) \) are as large positive numbers as we wish for all sufficiently large \( x \).

Note that in advanced calculus the above definition is formulated more precisely.

Definition 2*.
\[ \lim_{x \to \infty} f(x) = \infty \]
if for every \( M > 0 \) there is an \( N > 0 \) such that \( |f(x)| > M \) for all \( x \geq N \).

Definition 3.
\[ \lim_{x \to \infty} f(x) = -\infty \]
if the values \( f(x) \) are as large negative numbers as we wish for all sufficiently large \( x \).

Note that in advanced calculus the above definition is formulated more precisely.

Definition 3*.
\[ \lim_{x \to \infty} f(x) = -\infty \]
if for every \( M < 0 \) there is an \( N > 0 \) such that \( f(x) < M \) for all \( x \geq N \).

The next three definitions of the limit of a sequence are analogous to the above definitions by thinking about a sequence \( \{a_n\}_{n=1}^{\infty} \) as a function \( a_n = f(n) \) defined on the positive integers.

Definition 1.
\[ \lim_{n \to \infty} a_n = L \]
if the values \( a_n \) are as close to \( L \) as we wish for all sufficiently large \( n \).

Note that in advanced calculus the above definition is formulated more precisely.

Definition 1*.
\[ \lim_{n \to \infty} a_n = L \]
if for every \( \varepsilon > 0 \) there is an \( N \) such that \( |a_n - L| < \varepsilon \) for all \( n \geq N \).

Definition 2.

\[
\lim_{n \to \infty} a_n = \infty
\]

if the values \( a_n \) are as large positive numbers as we wish for all sufficiently large \( n \).

Note that in advanced calculus the above definition is formulated more precisely.

Definition 2*.

\[
\lim_{x \to \infty} a_n = \infty
\]

if for every \( M > 0 \) there is an \( N > 0 \) such that \( a_n > M \) for all \( n \geq N \).

Definition 3.

\[
\lim_{n \to \infty} a_n = -\infty
\]

if the values \( a_n \) are as large negative numbers as we wish for all sufficiently large \( n \).

Note that in advanced calculus the above definition is formulated more precisely.

Definition 3*.

\[
\lim_{x \to \infty} a_n = -\infty
\]

if for every \( M < 0 \) there is an \( N > 0 \) such that \( a_n < M \) for all \( n \geq N \).

If

\[
\lim_{n \to \infty} a_n = L
\]

then we say the sequence \( \{a_n\}_{n=1}^\infty \) converges to \( L \). In any other cases we say that sequence \( \{a_n\}_{n=1}^\infty \) diverges. The sequence \( \{a_n\}_{n=1}^\infty \) may diverge “nicely” to \( \infty \) if

\[
\lim_{n \to \infty} a_n = \infty
\]

or to \( -\infty \) if

\[
\lim_{n \to \infty} a_n = -\infty .
\]

The sequence \( \{a_n\}_{n=1}^\infty \) may diverge due to oscillation. For example, the sequence \( \{a_n\}_{n=1}^\infty \) defined by \( a_n = (-1)^n \) diverges due to oscillation (roughly speaking). Convince yourself why there is no real number \( L, \infty \) or \( -\infty \) satisfying the definition.

**Tools to calculate the limit of a sequence**

To calculate limits of sequences we learn seven tools.

1. the Definition,
2. the “Theorem”,
3. SST (Subsequence Theorem)
4. LL (Limit Laws)
5. ST (Squeeze Theorem)
6. CT (Continuity Theorem)
7. MST (Monotone Sequence Theorem)

In this section we refer to the following simple statement as the “Theorem”.

\[
3
\]
Theorem. If \(a_n = f(n)\) for all sufficiently large positive integers, where \(f\) is a function defined for all sufficiently large real numbers \(x\), and

\[
\lim_{x \to \infty} f(x) = L,
\]

then

\[
\lim_{n \to \infty} a_n = L.
\]

In this statement the real number \(L\) may be replaced with \(\infty\) or \(-\infty\).

Often times by using this theorem we can reduce the calculation of the limit of a sequence to the calculation of the limit of a function at \(\infty\) as we learned it in Math 151.

If \(0 < k_1 < k_2 < \cdots < k_n < \cdots\) are integers then the sequence \(\{a_{k_n}\}_{n=1}^{\infty}\) is called a subsequence of \(\{a_n\}_{n=1}^{\infty}\).

SST (Subsequence Theorem). If the limit of a sequence exists as a real number \(L\), \(\infty\), or \(-\infty\), then its every subsequence has the same limit. That is, if

\[
\lim_{n \to \infty} a_n = L,
\]

and \(0 < k_1 < k_2 < \cdots < k_n < \cdots\) are integers then

\[
\lim_{n \to \infty} a_{k_n} = L,
\]

and in this statement the real number \(L\) may be replaced with \(\infty\) or \(-\infty\).

Limit Laws.

Sum/Difference Law: \[\lim_{n \to \infty} (a_n \pm b_n) = \left(\lim_{n \to \infty} a_n\right) \pm \left(\lim_{n \to \infty} b_n\right).\]

Constant Multiple Law: \[\lim_{n \to \infty} (ca_n) = c \left(\lim_{n \to \infty} a_n\right).\]

Product Law: \[\lim_{n \to \infty} (a_nb_n) = \left(\lim_{n \to \infty} a_n\right) \left(\lim_{n \to \infty} b_n\right).\]

Quotient Law: \[\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} \quad \text{if} \quad \lim_{n \to \infty} b_n \neq 0.\]

Often times when the “Theorem” is applicable we use the corresponding limit laws for functions of a real variable tending to \(\infty\). We list these as well as you learned it in Math 151.

Sum/Difference Law: \[\lim_{x \to \infty} (f(x) \pm g(x)) = \left(\lim_{x \to \infty} f(x)\right) \pm \left(\lim_{x \to \infty} g(x)\right).\]

Constant Multiple Law: \[\lim_{x \to \infty} (cf(x)) = c \left(\lim_{n \to \infty} f(x)\right).\]
Product Law: \( \lim_{x \to \infty} (f(x)g(x)) = \left( \lim_{x \to \infty} f(x) \right) \left( \lim_{x \to \infty} g(x) \right) \).

Quotient Law: \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \to \infty} f(x)}{\lim_{x \to \infty} g(x)} \) if \( \lim_{x \to \infty} g(x) \neq 0 \).

Recall that

\[ \infty + L = L + \infty = \infty, \quad -\infty + L = L - \infty = -\infty, \quad \infty + \infty = \infty, \quad -\infty - \infty = -\infty, \]

\[ L \cdot \infty = \infty \cdot L = \infty \quad \text{if} \quad L > 0, \quad L \cdot \infty = \infty \cdot L = -\infty \quad \text{if} \quad L < 0. \]

\[ \infty \cdot \infty = \infty, \quad (-\infty) \cdot \infty = -\infty, \quad (-\infty) \cdot (-\infty) = \infty, \]

\[ \frac{L}{\infty} = 0, \quad \frac{\infty}{L} = \infty \quad \text{if} \quad L > 0, \quad \frac{\infty}{L} = -\infty \quad \text{if} \quad L < 0, \]

\[ \frac{L}{0^+} = \infty, \quad \frac{\infty}{0^+} = \infty \quad \text{and} \quad \frac{L}{0^-} = \infty, \quad \frac{\infty}{0^-} = -\infty \quad \text{if} \quad L > 0, \]

\[ \infty - \infty, \quad \pm\infty, \quad 0, \quad 0 \cdot (\pm\infty), \quad (\pm\infty) \cdot 0 \]

are inconclusive. Review L’Hospital’s Rule for the inconclusive cases \(0/0\) and \(\infty/\infty\).

**ST (Squeeze Theorem).** If \( b_n \leq a_n \leq c_n \) for all sufficiently large \( n \geq n_0 \) and

\[ \lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n = L, \]

then

\[ \lim_{n \to \infty} a_n = L. \]

This statement remains valid if \( L \) is replaced by \( \infty \) or \(-\infty\).

**CT (Continuity Theorem).** If \( a_n = f(c_n) \), \( \lim_{n \to \infty} c_n = L \), and \( f \) is continuous at \( L \), then

\[ \lim_{n \to \infty} a_n = f(L). \]

Definition. A sequence \( \{a_n\}^\infty_{n=1} \) is called bounded above if there is an \( M \) such that \( a_n \leq M \) for each \( n = 1, 2, \ldots \).

Definition. A sequence \( \{a_n\}^\infty_{n=1} \) is called bounded below if there is an \( M \) such that \( a_n \geq M \) for each \( n = 1, 2, \ldots \).

Definition. A sequence \( \{a_n\}^\infty_{n=1} \) is called bounded if it is bounded above and bounded below, that is, there is an \( M \) such that \( |a_n| \leq M \) for each \( n = 1, 2, \ldots \).

Definition. A sequence \( \{a_n\}^\infty_{n=1} \) is called increasing if \( a_n \leq a_{n+1} \) for each \( n = 1, 2, \ldots \).

Definition. A sequence \( \{a_n\}^\infty_{n=1} \) is called decreasing if \( a_n \geq a_{n+1} \) for each \( n = 1, 2, \ldots \).

Definition. A sequence \( \{a_n\}^\infty_{n=1} \) is called monotonic if it is either increasing or decreasing.
MST (Monotonic Sequence Theorem). Every bounded monotonic sequence is convergent.

E.1. Find \( \lim_{n \to \infty} \frac{1}{n}. \)

The limit is 0 by the definition of limit as \( 1/n \) is as close to 0 as we wish for all sufficiently large \( n \).

E.2. Find \( \lim_{n \to \infty} \frac{\ln n}{n}. \)

Observe that \( a_n = f(n) \), where 
\[
f(x) := \frac{\ln x}{x}
\]
is defined for all \( x > 0 \). We have 
\[
\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \frac{1}{x} = \frac{1}{\infty} = 0.
\]
Here the first limit is an \( \infty/\infty \) type limit, which is inconclusive, but the assumptions of L'Hospital Rule are satisfied, and it can be used to evaluate the new limit by using the Quotient Law. We conclude by the “Theorem” that 
\[
\lim_{n \to \infty} \frac{\ln n}{n} = 0.
\]

E.3. Find \( \lim_{n \to \infty} \arctan(3n). \)

Observe that \( a_n = f(n) \), where 
\[
f(x) := \arctan(3x)
\]
is defined for all \( x > 0 \). We have 
\[
\lim_{x \to \infty} \arctan(3x) = \lim_{u \to \infty} \arctan u = \frac{\pi}{2}.
\]
We conclude by the “Theorem” that 
\[
\lim_{n \to \infty} \arctan(3n) = \frac{\pi}{2}.
\]

E.4. Find \( \lim_{n \to \infty} (-1)^n. \)

Note that \((-1)^{(2k+1)/2}\) is not defined. Also \((-1)^x\) is not defined for irrational values of \( x \). So the “Theorem” cannot be used.
Let \( a_n := (-1)^n \). Observe that

\[
\lim_{n \to \infty} a_{2n} = \lim_{n \to \infty} 1 = 1
\]

and

\[
\lim_{n \to \infty} a_{2n+1} = \lim_{n \to \infty} -1 = -1.
\]

Thus we have found two different subsequences of the given sequence with different limits. In the light of the Subsequence Theorem we conclude that

\[
\lim_{n \to \infty} (-1)^n
\]

does not exist.

E.5. Find \( \lim_{n \to \infty} \frac{(-1)^n}{n} \).

Let \( a_n := \frac{(-1)^n}{n} \). Observe that

\[
\frac{-1}{n} \leq a_n \leq \frac{1}{n}, \quad n = 1, 2, \ldots,
\]

where

\[
\lim_{n \to \infty} \frac{-1}{n} = \lim_{n \to \infty} \frac{1}{n} = 0,
\]

by the definition of limit. Hence the Squeeze Theorem gives

\[
\lim_{n \to \infty} \frac{(-1)^n}{n} = 0.
\]

E.6. Find \( \lim_{n \to \infty} \frac{(-1)^n \ln n}{n} \).

Let

\[
a_n := \frac{(-1)^n \ln n}{n}.
\]

Observe that

\[
\frac{-\ln n}{n} \leq a_n \leq \frac{\ln n}{n}, \quad n = 1, 2, \ldots,
\]

where

\[
\lim_{n \to \infty} \frac{-\ln n}{n} = \lim_{n \to \infty} \frac{\ln n}{n} = 0,
\]

by E.2 and the Constant Multiple Law. Hence the Squeeze Theorem gives

\[
\lim_{n \to \infty} \frac{(-1)^n \ln n}{n} = 0.
\]

E.7. Find \( \lim_{n \to \infty} (\ln(n+1) - \ln n) \).
If we try to use the Difference Law we get $\infty - \infty$, which is inconclusive. However, observe that

$$\lim_{n \to \infty} (\ln(n+1) - \ln n) = \lim_{n \to \infty} \ln \left( \frac{n+1}{n} \right) = \lim_{n \to \infty} \ln \left( 1 + \frac{1}{n} \right) = \lim_{x \to 1} \ln x = \ln 1 = 0,$$

where we used the rules of the logarithmic functions as well as the fact that the function $\ln x$ is continuous on its understood domain, and hence it is continuous at 1. We have concluded by using the Continuity Theorem.

E.8. Find $\lim_{n \to \infty} \frac{\ln(2 + e^n)}{3n}$.

First we handle this limit by the “Theorem”. Observe that

$$a_n = f(n) = \frac{\ln(2 + e^n)}{3n},$$

where

$$f(x) := \frac{\ln(2 + e^x)}{3x}.$$

We have

$$\lim_{x \to \infty} \frac{\ln(2 + e^x)}{3x} = \lim_{x \to \infty} \frac{1}{3} \frac{2 + e^x}{3} = \frac{1}{3} \lim_{x \to \infty} \left( 1 - \frac{2}{2 + e^x} \right) = \frac{1}{3} \left( 1 - \frac{2}{\infty} \right) = \frac{1}{3} (1 - 0) = \frac{1}{3}.$$

Here the first limit is an $\infty/\infty$ type limit, which is inconclusive, but the assumptions of L'Hospital Rule are satisfied, and it can be used to evaluate the new limit by using the Quotient Law. We conclude by the “Theorem” that

$$\lim_{n \to \infty} \frac{\ln(2 + e^n)}{3n} = \frac{1}{3}.$$

Another way to calculate the limit is to use the Squeeze Theorem as follows. Observe that

$$\frac{1}{3} = \frac{n}{3n} = \frac{\ln(e^n)}{3n} \leq \frac{\ln(2 + e^n)}{3n} \leq \frac{\ln(2e^n)}{3n} = \frac{\ln(e^n)}{3n} \leq \frac{\ln(e^{n+1})}{3n} = \frac{n + 1}{3n} = \frac{1}{3} + \frac{1}{3n}.$$

Here

$$\lim_{n \to \infty} \frac{1}{3} = \lim_{n \to \infty} \left( \frac{1}{3} + \frac{1}{3n} \right) = \frac{1}{3},$$
and hence the Squeeze Theorem yields that

\[ \lim_{n \to \infty} \frac{\ln(2 + e^n)}{3n} = \frac{1}{3}. \]

E.9. Find \( \lim_{n \to \infty} \left( \sqrt{n^2 + n + 1} - \sqrt{n^2 - n} \right) \).

Observe that

\[ a_n = f(n) = \sqrt{n^2 + n + 1} - \sqrt{n^2 - n}, \]

where

\[ f(x) := \sqrt{x^2 + x + 1} - \sqrt{x^2 - x}. \]

If we try to use the Difference Law to find \( \lim_{n \to \infty} f(x) = \infty \), we get \( \infty - \infty \), which is inconclusive. However, observe that

\[
\lim_{x \to \infty} \left( \sqrt{x^2 + x + 1} - \sqrt{x^2 - x} \right) = \lim_{x \to \infty} \frac{(x^2 + x + 1) - (x^2 - x)}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} = \lim_{x \to \infty} \frac{2x + 1}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} = \lim_{x \to \infty} \frac{2 + 1/x}{\sqrt{1 + 1/x + (1/x)^2 + 1/x - (1/x)}} = \frac{2 + 0}{\sqrt{1 + 0 + 0 + 1 - 0}} = \frac{2}{1 + 1} = 1.
\]

Here we multiplied and divided by the conjugate expression

\[ \sqrt{x^2 + x + 1} + \sqrt{x^2 - x} \]

and to avoid to have an \( \infty/\infty \) type limit we divided both the numerator and denominator by \( x \) to put ourself in the position to be able to use the Quotient Law. We also used the continuity of the \( \sqrt{u} \) function at \( u = 1 \). We conclude by the “Theorem” that

\[ \lim_{n \to \infty} \left( \sqrt{n^2 + n + 1} - \sqrt{n^2 - n} \right) = 1. \]

E.10. Let \( a_1 := \sqrt{2} \) and

\[ a_{n+1} = \sqrt{2 + a_n}, \quad n = 1, 2, \ldots. \]

Find \( \lim_{n \to \infty} a_n \).

We prove three simple claims first by the method of mathematical induction. Mathematical induction is a simple method to prove some simple statements for every positive integers.
Claim 1. $a_n \geq \sqrt{2}$, $n = 1, 2, \ldots$.

Claim 2. $a_n \leq 2$, $n = 1, 2, \ldots$.

Claim 3. $a_n \leq a_{n+1}$, $n = 1, 2, \ldots$.

Proof of Claim 1. We have $a_1 = \sqrt{2} \geq \sqrt{2}$, so Claim 1 is true for $n = 1$. Suppose now that Claim 1 is true for a positive integer $n$, that is, $a_n \geq \sqrt{2}$. Then

$$a_{n+1} = \sqrt{2} + a_n \geq \sqrt{2} + \sqrt{2} = \sqrt{2},$$

so Claim 1 is true for $n + 1$. Hence by the method of mathematical induction we see that Claim 1 is true for all $n = 1, 2, \ldots$.

Proof of Claim 2. We have $a_1 = \sqrt{2} \leq 2$, so Claim 2 is true for $n = 1$. Suppose now that Claim 2 is true for a positive integer $n$, that is, $a_n \leq 2$. Then

$$a_{n+1} = \sqrt{2} + a_n \leq \sqrt{4} = 2,$$

so Claim 2 is true for $n + 1$. Hence by the method of mathematical induction we see that Claim 2 is true for all $n = 1, 2, \ldots$.

Proof of Claim 3. We have $a_1 = \sqrt{2} \leq \sqrt{2 + \sqrt{2}} = a_2$, so Claim 3 is true for $n = 1$. Suppose now that Claim 3 is true for a positive integer $n$, that is, $a_n \leq a_{n+1}$. We already know from Claim 1 that $\sqrt{2} \leq a_n$ also holds. Since the function $\sqrt{u}$ is increasing on $[0, \infty)$ we can deduce that

$$a_{n+1} = \sqrt{2} + a_n \leq \sqrt{2 + 2} = a_{n+2},$$

so Claim 3 is true for $n + 1$. Hence by the method of mathematical induction we see that Claim 3 is true for all $n = 1, 2, \ldots$.

By Claims 1, 2, and 3 the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded and increasing. So by the MST we know that

$$\lim_{n \to \infty} = L$$

exists. Therefore the Limit Laws imply that

$$L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{2 + a_n} = \sqrt{\lim_{n \to \infty} (2 + a_n)} = \sqrt{2 + L},$$

so we have $L^2 = 2 + L$, and hence $(L - 2)(L + 1) = 0$. Thus we have only two candidates for the limit: $L = 2$ and $L = -1$. However, $L = -1$ is impossible by Claim 1. So $L = 2$ must be the case.

E.11. Let $a_1 := 1$ and

$$a_{n+1} = 3 - \frac{1}{a_n}, \quad n = 1, 2, \ldots.$$
Find \( \lim_{n \to \infty} a_n \).

We prove three simple claims first by the method of mathematical induction.

Claim 1. \( a_n \geq 1, \quad n = 1, 2, \ldots \)

Claim 2. \( a_n \leq 3, \quad n = 1, 2, \ldots \)

Claim 3. \( a_n \leq a_{n+1}, \quad n = 1, 2, \ldots \)

Proof of Claim 1. We have \( a_1 = 1 \geq 1 \), so Claim 1 is true for \( n = 1 \). Suppose now that Claim 1 is true for a positive integer \( n \), that is, \( a_n \geq 1 \). Then

\[
a_{n+1} = 3 - \frac{1}{a_n} \geq 3 - \frac{1}{1} = 2 \geq 1,
\]

so Claim 1 is true for \( n + 1 \). Hence by the method of mathematical induction we see that Claim 1 is true for all \( n = 1, 2, \ldots \).

Proof of Claim 2. We have \( a_1 = 1 \leq 3 \), so Claim 2 is true for \( n = 1 \). Suppose now that Claim 2 is true for a positive integer \( n \), that is, \( a_n \leq 3 \). Then, as \( a_n \geq 1 \) by Claim 1, we have

\[
a_{n+1} = 3 - \frac{1}{a_n} \leq 3,
\]

so Claim 2 is true for \( n + 1 \). Hence by the method of mathematical induction we see that Claim 2 is true for all \( n = 1, 2, \ldots \).

Proof of Claim 3. We have \( a_1 = 1 \leq 2 = 3 - 1/1 = a_2 \), so Claim 3 is true for \( n = 1 \). Suppose now that Claim 3 is true for a positive integer \( n \), that is, \( a_n \leq a_{n+1} \). By Claim 1 we already know that \( a_n \geq 1 \) also holds. Since the function \( 3 - 1/x \) is increasing on \((0, \infty)\) we can deduce that

\[
a_{n+1} = 3 - \frac{1}{a_n} \leq 3 - \frac{1}{a_{n+1}} = a_{n+2},
\]

so Claim 3 is true for \( n + 1 \). Hence by the method of mathematical induction we see that Claim 3 is true for all \( n = 1, 2, \ldots \).

By Claims 1, 2, and 3 the sequence \( \{a_n\} \) is bounded and increasing. So by the MST we know that

\[
\lim_{n \to \infty} a_n = L
\]

exists. Therefore the Limit Laws implies that

\[
L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \left( 3 - \frac{1}{a_n} \right) = 3 - \frac{1}{\lim_{n \to \infty} a_n} = 3 - \frac{1}{L},
\]

so we have \( L^2 = 3L - 1 \), and hence

\[
L = \frac{3 \pm \sqrt{5}}{2}.
\]
Thus we have only two candidates for the limit. However,

\[ L = \frac{3 - \sqrt{5}}{2} < 1 \]

is impossible by Claim 1. So

\[ L = \frac{3 + \sqrt{5}}{2} \]

must be the case.

E.12. Let \(a_1 := 1\) and

\[ a_{n+1} = \frac{1}{3 - a_n}, \quad n = 1, 2, \ldots \]

Find \(\lim_{n \to \infty} a_n\).

We prove two simple claims first by the method of mathematical induction.

Claim 1. \(a_n \in [0, 1], \quad n = 1, 2, \ldots \)

Claim 2. \(a_n \geq a_{n+1}, \quad n = 1, 2, \ldots \)

Proof of Claim 1. We have \(a_1 = 1 \in [0, 1]\), so Claim 1 is true for \(n = 1\). Suppose now that Claim 1 is true for a positive integer \(n\), that is, \(a_n \in [0, 1]\). As the function \(1/(3 - x)\) is increasing on \([0, 3)\) we have

\[ a_{n+1} = \frac{1}{3 - a_n} \in \left[ \frac{1}{3}, \frac{1}{2} \right] \subset [0, 1], \]

so Claim 1 is true for \(n + 1\). Hence by the method of mathematical induction we see that Claim 1 is true for all \(n = 1, 2, \ldots\).

Proof of Claim 2. We have \(a_1 = 1 \geq 1/2 = 1/(3 - 1) = a_2\), so Claim 3 is true for \(n = 1\). Suppose now that Claim 3 is true for a positive integer \(n\), that is, \(a_n \geq a_{n+1}\). Combining this with Claim 1 we have \(1 \geq a_n \geq a_{n+1} \geq 0\). Since the function \(1/(3 - x)\) is increasing on \([0, 3)\) we can deduce that

\[ a_{n+1} = \frac{1}{3 - a_n} \geq \frac{1}{3 - a_{n+1}} = a_{n+2}, \]

so Claim 2 is true for \(n + 1\). Hence by the method of mathematical induction we see that Claim 2 is true for all \(n = 1, 2, \ldots\).

By Claims 1 and 2 the sequence \(\{a_n\}_{n=1}^\infty\) is bounded and decreasing. So by the MST we know that

\[ \lim_{n \to \infty} a_n = L \]

exists. Therefore the Limit Laws implies that

\[ L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{1}{3 - a_n} = \frac{1}{\lim_{n \to \infty} (3 - a_n)} = \frac{1}{3 - L}, \]
so we have $L^2 - 3L + 1 = 0$, and hence

$$L = \frac{3 \pm \sqrt{5}}{2}.$$  

Thus we have only two candidates for the limit. However,

$$L = \frac{3 + \sqrt{5}}{2} > 1$$

is impossible by Claim 1. So

$$L = \frac{3 - \sqrt{5}}{2}$$

must be the case.

E.13. We show that if $q \in [0, 1)$, then $\lim_{n \to \infty} q^n = 0$. Let $a_n := q^n$. Observe that the sequence $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence of nonnegative numbers. Hence, the MST implies that $\lim_{n \to \infty} a_n = L$ exists as a real number. We have

$$a_{n+1} = qa_n, \quad n = 1, 2, \ldots$$

Taking the limit of both sides as $n \to \infty$ we get $L = qL$, that is $L(q - 1) = 0$. As $q - 1 \neq 0$, we conclude that $L = 0$.

E.14. We show that if $q \in (-1, 1)$, then $\lim_{n \to \infty} q^n = 0$.

Observe that

$$-|q|^n \leq q^n \leq |q|^n, \quad q \in (-1, 1),$$

where E.13 gives

$$\lim_{n \to \infty} -|q|^n = (-1) \lim_{n \to \infty} |q|^n = 0.$$  

Therefore the Squeeze Theorem implies that $\lim_{n \to \infty} q^n = 0$, indeed.

E.15. Let $a_n = n(\pi/2 - \arctan n), \quad n = 1, 2, \ldots$ Find $\lim_{n \to \infty} a_n$.

Observe that $a_n = f(n)$, where

$$f(x) = x(\pi/2 - \arctan x) = \frac{\pi/2 - \arctan x}{x^{-1}}.$$  

We have

$$\lim_{n \to \infty} f(x) = \lim_{x \to \infty} \frac{\pi/2 - \arctan x}{x^{-1}} = \lim_{x \to \infty} \frac{-1/(x^2 + 1)}{(-1)x^{-2}} = \lim_{x \to \infty} \left(1 - \frac{1}{x^2 + 1}\right) = 1 - \frac{1}{\infty} = 1 - 0 = 1.$$  

Here to avoid the inconclusive $0 \cdot \infty$ situation we write the product as a ratio to reach an inconclusive $0/0$ situation, which can be handled by L’Hospital’s Rule. We conclude by the “Theorem” that

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} n(\pi/2 - \arctan n) = 1.$$  

E.16. Let $a_n = \ln \left(\frac{2n^2 + n - 3}{n^2 - 3n + 5}\right)$. Find $\lim_{n \to \infty} a_n = 0$. 

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