11.2. Series, Infinite Sums

Definition of an infinite sum (series).

(1) \[ \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots := \lim_{n \to \infty} s_n , \]

whenever the above limit exists, where

\[ s_n := \sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_n \]

are the \( n \)th partial sums of the infinite sum (series). The real number \( a_n \) is called the \( n \)th term of the infinite sum (series). If the limit in (1) is a finite real number we say that the infinite sum \( \sum_{n=1}^{\infty} a_n \) converges. In any other case we say that the infinite sum diverges. It may diverges “nicely” to \( \infty \) or \( -\infty \) if the limit in (1) is \( \infty \) or \( -\infty \). However, an infinite sum may diverge due to the oscillation of the \( n \)th partial sums (roughly speaking) as well.

The following two laws are straightforward consequences of the corresponding limit laws.

Constant Multiple Rule.

\[ \sum_{n=1}^{\infty} (ca_n) = c \sum_{n=1}^{\infty} a_n . \]

Sum/Difference Law.

\[ \sum_{n=1}^{\infty} (a_n \pm b_n) = \left( \sum_{n=1}^{\infty} a_n \right) \pm \left( \sum_{n=1}^{\infty} b_n \right) . \]

E.1. We have

\[ \sum_{n=1}^{\infty} 1 = \lim_{n \to \infty} s_n = \lim_{n \to \infty} n = \infty . \]

E.2. We have

\[ \sum_{n=1}^{\infty} (-1)^{n-1} = \lim_{n \to \infty} s_n . \]

Observe that

\[ s_{2k} = (1 - 1) + (1 - 1) + \cdots + (1 - 1) = k(1 - 1) = 0 , \quad k = 1, 2, \ldots , \]

and

\[ s_{2k+1} = (1 - 1) + (1 - 1) + \cdots + (1 - 1) + 1 = k(1 - 1) + 1 = 1 , \quad k = 1, 2, \ldots . \]

Hence \( \lim_{k \to \infty} s_{2k} = 0 \) and \( \lim_{k \to \infty} s_{2k+1} = 1 \). So we have found two different subsequences of the sequence \( \{s_n\}_{n=1}^{\infty} \) of the \( n \)th partial sums with different limits, so the Subsequence Theorem formulated in Section 11.1 tells us that \( \lim_{n \to \infty} s_n \) does not exist, hence \( \sum_{n=1}^{\infty} (-1)^n \) diverges, and certainly NOT to \( \infty \) or \( -\infty \).
The special infinite sum
\[ \sum_{n=1}^{\infty} q^{n-1} = 1 + q^1 + q^2 + \cdots \]
is called a geometric sum or geometric series. The number \( q \) is called the quotient of the geometric series. It can be shown easily that
\[ \sum_{n=1}^{\infty} q^{n-1} = 1 + q^1 + q^2 + \cdots = \frac{1}{1-q}, \quad q \in (-1, 1). \]
Indeed, we have
\[
\begin{align*}
  s_n &= 1 + q + q^2 + q^3 + \cdots + q^{n-2} + q^{n-1} \\
  qs_n &= \quad q + q^2 + q^3 + \cdots + q^{n-2} + q^{n-1} + q^n.
\end{align*}
\]
Subtracting the second line above from the first one, we obtain
\[ s_n - qs_n = 1 - q^n, \quad s_n = \frac{1 - q^n}{1 - q}. \]
Hence
\[ \sum_{n=1}^{\infty} q^{n-1} = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{1 - q^n}{1 - q} = \frac{1}{1-q} \lim_{n \to \infty} (1 - q^n) = \frac{1}{1-q} (1 - 0) = \frac{1}{1-q}. \]
Here we used the fact that \( \lim_{n \to \infty} q^n = 0 \) by E.14 in Section 11.1.
It is also easy to see that \( s_n \geq n \) if \( q \geq 1 \), and hence
\[ \sum_{n=1}^{\infty} q^{n-1} = \lim_{n \to \infty} s_n \geq \lim_{n \to \infty} n = \infty, \quad q \geq 1, \]
that is, the geometric series diverges “nicely” to \( \infty \) if \( q \geq 1 \).
E.3. Show that the geometric series diverges if \( q \leq -1 \).
Hint. Use the Test for Divergence stated later in this section.
E.4. Find the value of \( \sum_{n=1}^{\infty} 3^n 5^{3-n} \).
We have
\[
\begin{align*}
  \sum_{n=1}^{\infty} 3^n 5^{3-n} &= \sum_{n=1}^{\infty} 3 \cdot 5^2 \left( \frac{3}{5} \right)^{n-1} = 75 \left( 1 + \left( \frac{3}{5} \right)^1 + \left( \frac{3}{5} \right)^2 + \cdots \right) \\
  &= \frac{75}{1 - 3/5} = \frac{75}{2/5} = \frac{375}{2}.
\end{align*}
\]
E.5. Write 4.157015701570... as a ratio of positive integers.

We have

\[
4.157015701570... = 4 + \frac{1570}{10000} + \frac{1570}{10000^2} + \frac{1570}{10000^3} + \cdots
\]

\[
= 4 + 1570 \sum_{n=1}^{\infty} \left( \frac{1}{10000} \right)^{n-1}
\]

\[
= 4 + \frac{1570}{10000} \left( 1 + \left( \frac{1}{10000} \right)^1 + \left( \frac{1}{10000} \right)^2 + \cdots \right)
\]

\[
= 4 + \frac{1570}{10000} \frac{1}{1 - \frac{1}{10000}} = 4 + \frac{1570}{9999} = 4\frac{1566}{9999}.
\]

**Telescoping Sums**

The infinite sum \( \sum_{k=1}^{\infty} a_k \) is called telescoping if \( a_k = b_k - b_{k+1} \) for all \( k = 1, 2, \ldots \). It is easy to calculate the \( n \)th partial sums of a telescoping sum as the cancellations imply that

\[
s_n = (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots + (b_{n-1} - b_n) + (b_n - b_{n+1}) = b_1 - b_{n+1}.
\]

Hence

\[
\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} s_n = \lim_{n \to \infty} (b_1 - b_{n+1}) = b_1 - \lim_{n \to \infty} b_{n+1}.
\]

if \( \lim_{n \to \infty} b_{n+1} \) exists. Note that each infinite sum is telescoping as

\[
a_k = s_k - s_{k-1} = b_k - b_{k+1} \quad \text{with} \quad s_0 := 0 \quad \text{and} \quad b_k := -s_{k-1}, \quad k = 1, 2, \ldots.
\]

However, typically we do not have an explicit formula for the \( k \)th partial sums, so some difficulty is involved to find the telescoping feature in a usable format.

Moral: Partial fraction decomposition often helps to recognize telescoping feature.

E.6. Find

\[
\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.
\]

Observe that

\[
a_k := \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.
\]

hence

\[
s_n = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{n-1} - \frac{1}{n} \right) + \left( \frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{n+1}.
\]

\[
= \frac{n}{n+1}.
\]
Therefore
\[
\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( 1 - \frac{1}{n+1} \right) = 1 - \lim_{n \to \infty} \frac{1}{n+1} = 1 - 0 = 1.
\]

E.7. Find
\[
\sum_{k=1}^{\infty} \frac{6}{k^2 + 3k}.
\]

Observe that
\[
a_k := \frac{6}{k^2 + 3k} = \frac{6}{k(k+3)} = \frac{2}{k} - \frac{2}{k+3},
\]

hence
\[
s_n = \frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \frac{2}{6} + \cdots + \frac{2}{n-2} + \frac{2}{n-1} + \frac{2}{n} - \frac{2}{2} - \frac{2}{3} - \frac{2}{4} - \frac{2}{5} - \frac{2}{6} - \cdots - \frac{2}{n-2} - \frac{2}{n-1} - \frac{2}{n} - \frac{2}{n+1} - \frac{2}{n+2} - \frac{2}{n+3}
\]
\[
= \frac{2}{1} + \frac{2}{2} + \frac{2}{3} - \frac{2}{n+1} - \frac{2}{n+2} - \frac{2}{n+3}.
\]

Therefore
\[
\sum_{k=1}^{\infty} \frac{6}{k^2 + 3k} = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( \frac{2}{1} + \frac{2}{2} + \frac{2}{3} - \frac{2}{n+1} - \frac{2}{n+2} - \frac{2}{n+3} \right)
\]
\[
= \frac{11}{3} - 0 - 0 - 0 = \frac{11}{3}.
\]

E.8. Find
\[
\sum_{k=1}^{\infty} \frac{6}{4k^2 - 1}.
\]

Observe that
\[
a_k := \frac{6}{4k^2 - 1} = \frac{6}{(2k-1)(2k+1)} = \frac{3}{2k-1} - \frac{3}{2k+1},
\]

hence
\[
s_n = \left( \frac{3}{1} - \frac{3}{3} \right) + \left( \frac{3}{3} - \frac{3}{5} \right) + \left( \frac{3}{5} - \frac{3}{7} \right) + \cdots
\]
\[
+ \left( \frac{3}{2n-3} - \frac{3}{2n-1} \right) + \left( \frac{3}{2n-1} - \frac{3}{2n+1} \right)
\]
\[
= \frac{3}{1} - \frac{3}{2n+1}.
\]
Therefore
\[ \sum_{k=1}^{\infty} \frac{6}{4k^2 - 1} = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( 3 - \frac{3}{2n + 1} \right) = 3 - 0 = 3. \]

E.9. Find
\[ \sum_{k=1}^{\infty} \ln \left( \frac{k}{k+1} \right). \]

Observe that
\[ a_k := \ln \left( \frac{k}{k+1} \right) = \ln k - \ln(k+1), \]
hence
\[ s_n = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) + \cdots + (\ln(n-1) - \ln n) + (\ln n - \ln(n+1)) = \ln 1 - \ln(n+1) = -\ln(n+1). \]

Therefore
\[ \sum_{k=1}^{\infty} \ln \left( \frac{n}{n+1} \right) = \lim_{n \to \infty} s_n = \lim_{n \to \infty} -\ln(n+1) = -\infty. \]

So the infinite sum diverges (nicely) to \(-\infty\).

E.10. Find
\[ \sum_{k=2}^{\infty} \ln \left( \frac{k^2}{k^2 - 1} \right). \]

Observe that
\[ a_k := \ln \left( \frac{k^2}{k^2 - 1} \right) = \ln \left( \frac{k^2}{(k-1)(k+1)} \right) = \ln \left( \frac{k}{k-1} \right) - \ln \left( \frac{k+1}{k} \right), \]
hence
\[ s_n = \left( \ln \left( \frac{2}{1} \right) - \ln \left( \frac{3}{2} \right) \right) + \left( \ln \left( \frac{3}{2} \right) - \ln \left( \frac{4}{3} \right) \right) + \left( \ln \left( \frac{4}{3} \right) - \ln \left( \frac{5}{4} \right) \right) + \cdots + \left( \ln \left( \frac{n-1}{n-2} \right) - \ln \left( \frac{n}{n-1} \right) \right) + \left( \ln \left( \frac{n}{n-1} \right) - \ln \left( \frac{n+1}{n} \right) \right) = \ln \left( \frac{2}{1} \right) - \ln \left( \frac{n+1}{n} \right) = \ln 2 - \ln \left( \frac{n+1}{n} \right). \]
Therefore
\[
\sum_{k=2}^{\infty} \ln \left( \frac{k^2}{k^2 - 1} \right) = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( \ln 2 - \ln \left( \frac{n+1}{n} \right) \right)
\]
\[
= \lim_{n \to \infty} \left( \ln 2 - \ln \left( 1 + \frac{1}{n} \right) \right) = \ln 2 - \lim_{x \to 1} \ln x = \ln 2 - \ln 1 = \ln 2.
\]
In the last step we used the fact that the \( \ln x \) function is continuous at 1. So the infinite sum converges to \( \ln 2 \).

**Test for Divergence**

**Theorem.** If \( \sum_{n=1}^{\infty} a_n \) converges, then \( \lim_{n \to \infty} a_n = 0 \).

**Proof.** We have
\[
a_n = (a_1 + a_2 + \cdots + a_{n-1} + a_n) - (a_1 + a_2 + \cdots + a_{n-1}) = s_n - s_{n-1}, \quad n = 2, 3, \ldots.
\]
Let
\[
\sum_{n=1}^{\infty} a_n = L.
\]
The Difference Law gives
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} (s_n - s_{n-1}) = \left( \lim_{n \to \infty} s_n \right) - \left( \lim_{n \to \infty} s_{n-1} \right) = L - L = 0.
\]
\(\square\)

The above theorem can be reformulated as follows.

**TD (Test for Divergence).** If \( \lim_{n \to \infty} a_n \) does not exist or \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{n=1}^{\infty} a_n \) diverges.

E.11. Show that \( \sum_{n=1}^{\infty} \arctan(\sqrt{n}) \) diverges.

We have
\[
\lim_{n \to \infty} \arctan(\sqrt{n}) = \lim_{u \to \infty} \arctan u = \frac{\pi}{2} \neq 0,
\]
and hence
\[
\sum_{n=1}^{\infty} \arctan(\sqrt{n})
\]
diverges by the TD.

E.12. Show that \( \sum_{n=1}^{\infty} (\ln(2n + 5) - \ln(3n - 2)) \) diverges.
We have
\[
\lim_{n \to \infty} (\ln(2n + 5) - \ln(3n - 2)) = \lim_{n \to \infty} \ln \left( \frac{2n + 5}{3n - 2} \right) = \ln \left( \lim_{n \to \infty} \frac{2n + 5}{3n - 2} \right) = \ln \left( \frac{2 + 5/n}{3 - 2/n} \right) = \ln \frac{2 + 0}{3 - 0} = \ln \frac{2}{3} \neq 0,
\]
and hence
\[
\sum_{n=1}^{\infty} (\ln(2n + 5) - \ln(3n - 2))
\]
diverges by the TD.

E.13. Show that \( \sum_{n=1}^{\infty} \frac{n}{\sqrt{1 + 3n^2}} \) diverges.

We have
\[
\lim_{n \to \infty} \frac{n}{\sqrt{1 + 3n^2}} = \lim_{n \to \infty} \frac{1}{\sqrt{1/n^2 + 3}} = \frac{1}{\sqrt{0 + 3}} = \frac{1}{\sqrt{3}} \neq 0,
\]
and hence
\[
\sum_{n=1}^{\infty} \frac{n}{\sqrt{1 + 3n^2}}
\]
diverges by the TD.

In E.2 we have observed that
\[
\sum_{n=1}^{\infty} (-1)^{n-1} = \lim_{n \to \infty} s_n
\]
diverges as
\[
s_{2k} = (1 - 1) + (1 - 1) + \cdots + (1 - 1) = k(1 - 1) = 0, \quad k = 1, 2, \ldots,
\]
and
\[
s_{2k+1} = (1 - 1) + (1 - 1) + \cdots + (1 - 1) + 1 = k(1 - 1) + 1 = 1, \quad k = 1, 2, \ldots,
\]
and hence \( \lim_{k \to \infty} s_{2k} = 0 \) and \( \lim_{k \to \infty} s_{2k+1} = 1 \). So we have found two different subsequences of the sequence \( \{s_n\}_{n=1}^{\infty} \) of the \( n \)th partial sums with different limits, so the Subsequence Theorem formulated in Section 11.1 tells us that \( \lim_{n \to \infty} s_n \) does not exist, hence \( \sum_{n=1}^{\infty} (-1)^n \) diverges, and certainly NOT to \( \infty \) or \( -\infty \).

E.14. Show that \( \sum_{n=1}^{\infty} (-1)^{n-1} \) diverges by the TD.
Indeed, the Subsequence Theorem implies that \( \lim_{n \to \infty} (-1)^n \) does not exist as

\[
\lim_{n \to \infty} a_{2n} = \lim_{n \to \infty} 1 = 1
\]

and

\[
\lim_{n \to \infty} a_{2n+1} = \lim_{n \to \infty} -1 = -1.
\]

Hence by the TD \( \sum_{n=1}^{\infty} (-1)^{n-1} \) diverges.

E.15. Show that if \( |q| \geq 1 \), then \( \sum_{n=1}^{\infty} (q^n) \) diverges by the TD.

Indeed, \( \lim_{n \to \infty} |q|^n = \infty \) if \( |q| > 1 \), while \( \lim_{n \to \infty} |q|^n = 1 \) if \( |q| = 1 \). Hence \( \lim_{n \to \infty} |q|^n \neq 0 \) whenever \( |q| \geq 1 \), and \( \sum_{n=1}^{\infty} (-1)^{n-1} \) diverges by the TD.

E.16. Observe that the divergence of

\[
\sum_{n=1}^{\infty} \ln \left( \frac{n}{n+1} \right)
\]

does not follow from the TD as

\[
\lim_{n \to \infty} \ln \left( \frac{n}{n+1} \right) = \lim_{n \to \infty} \ln \left( 1 - \frac{1}{n+1} \right) = \lim_{x \to 1} \ln x = \ln 1 = 0.
\]

However, in E.9 we have found, by using the telescoping feature of the sum, that this infinite series diverges to \( -\infty \).

E.17. Observe that the TD cannot be applied to show the divergence of

\[
\sum_{n=2}^{\infty} \ln \left( \frac{n^2}{n^2 - 1} \right),
\]

as

\[
\lim_{n \to \infty} \ln \left( \frac{n^2}{n^2 - 1} \right) = \lim_{n \to \infty} \ln \left( 1 + \frac{1}{n^2 - 1} \right) + \lim_{x \to 1} \ln x = \ln 1 = 0.
\]

In fact, the above sum converges to \( \ln 2 \) by using the telescoping feature of the sum, as we have seen it in E.10.

E.18. Observe that the combination of the Constant Multiple Rule, E.5, the Sum Rule, and the formula for the geometric series with quotient \( q \in (-1, 1) \) gives

\[
\sum_{n=1}^{\infty} \left( \frac{3}{n(n+1)} + \frac{5}{2^n} \right) = 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + 5 \sum_{n=1}^{\infty} \frac{1}{2^n} = 3 \cdot 1 + 5 \cdot \frac{1}{2 - \frac{1}{2}} = 3 + \frac{5}{2} = \frac{11}{2} = 8.
\]
E.19. Evaluate
\[ \sum_{n=1}^{\infty} \frac{3 \cdot 2^{n+1} + 2 \cdot (-3)^{n+2}}{5^n} \].

We have
\[
\begin{align*}
\sum_{n=1}^{\infty} \frac{3 \cdot 2^{n+1} + 2 \cdot (-3)^{n+2}}{5^n} &= \sum_{n=1}^{\infty} \frac{3 \cdot 2^{n+1}}{5^n} + \sum_{n=1}^{\infty} \frac{2 \cdot (-3)^{n+2}}{5^n} \\
&= \sum_{n=1}^{\infty} \frac{3 \cdot 2^2}{5} \left( \frac{2}{5} \right)^{n-1} + \sum_{n=1}^{\infty} \frac{2 \cdot (-3)^3}{5} \left( \frac{-3}{5} \right)^{n-1} \\
&= \frac{12}{5} \left( 1 + \left( \frac{2}{5} \right) + \left( \frac{2}{5} \right)^2 + \cdots \right) - \frac{54}{5} \left( 1 + \left( \frac{-3}{5} \right) + \left( \frac{-3}{5} \right)^2 + \cdots \right) \\
&= \frac{12}{5} \frac{1}{1 - 2/5} - \frac{54}{5} \frac{1}{1 - (-3/5)} = \frac{12}{5} \frac{1}{3/5} - \frac{54}{5} \frac{1}{8/5} \\
&= \frac{12}{3} - \frac{54}{8} = 4 - \frac{27}{4} = \frac{16 - 27}{4} = -\frac{11}{4}.
\end{align*}
\]