11.3. Integral Test

Remark 1. If \( f \) is a non-negative continuous function defined on \([a, \infty)\), then

\[
\int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx
\]

always exists as an extended real number, and the above improper integral converges if \( \int_a^\infty f(x) \, dx < \infty \), and it diverges if \( \int_a^\infty f(x) \, dx = \infty \).

Remark 1 follows from the fact the non-negativity and continuity of the integrand \( f \) on \([a, \infty)\) imply that the function

\[
g(t) := \int_a^t f(x) \, dx
\]

is increasing on \([a, \infty)\).

Remark 2. If \( a_n \geq 0 \) for each \( n = 1, 2, \ldots \), then \( \sum_{n=1}^\infty a_n = \lim_{n \to \infty} s_n \) always exists as an extended real number, and the infinite sum converges if \( \sum_{n=1}^\infty a_n < \infty \), and it diverges if \( \sum_{n=1}^\infty a_n = \infty \).

Remark 2 follows from the fact that the non-negativity of the terms implies that the sequence \( \{s_n\} \) of the \( n \)th partial sums is increasing, so in the light of the MST (Monotone Sequence Theorem) \( \lim_{n \to \infty} s_n \) exists as an extended real number and it is a finite number \( L \) if \( \{s_n\} \) is bounded above, while it is \( \infty \) if \( \{s_n\} \) is not bounded above.

**IT (Integral Test).** Suppose \( f \) is continuous, positive, and decreasing on \([1, \infty)\). Let \( a_n := f(n) \). The infinite sum \( \sum_{n=1}^\infty a_n \) converges if and only if \( \int_1^\infty f(x) \, dx \) converges. This means that

(a) if \( \int_1^\infty f(x) \, dx < \infty \) then \( \sum_{n=1}^\infty a_n < \infty \),

and

(b) if \( \int_1^\infty f(x) \, dx = \infty \) then \( \sum_{n=1}^\infty a_n = \infty \).

**EE in the IT (Error Estimate in the Integral Test).** Suppose \( f \) is continuous, positive, and decreasing on \([1, \infty)\). Let \( a_n := f(n) \). Let

\[
s := \sum_{n=1}^\infty a_n < \infty \quad \text{and} \quad s_n := \sum_{k=1}^n a_k, \quad n = 1, 2, \ldots
\]
We have
\[ \int_{n+1}^\infty f(x) \, dx \leq s - s_n \leq \int_n^\infty f(x) \, dx, \quad n = 1, 2, \ldots. \]

It is not difficult to see why the IT and the EE in the IT is true. The proof is typically sketched in the lecture and you may read it as well in the book. The assumptions of the IT implies that
\[ \int_1^{n+1} f(x) \, dx \leq f(1) + f(2) + \cdots + f(n) = a_1 + a_2 + \cdots + a_n = s_n \]

and
\[ s_n - a_1 = f(2) + f(3) + \cdots + f(n) = a_2 + a_3 + \cdots + a_n \leq \int_1^n f(x) \, dx, \]

and hence
\[ \int_1^{n+1} f(x) \, dx \leq s_n \leq a_1 + \int_1^n f(x) \, dx. \]

Taking the limit when \( n \to \infty \) we get
\[ \int_1^\infty f(x) \, dx \leq s := \sum_{n=1}^{\infty} a_n \leq a_1 + \int_1^\infty f(x) \, dx, \]

and the IT follows. There is a similar geometrical reason behind the EE in the IT. We have
\[ s - s_n = (f(1) + f(2) + f(3) + \cdots) - (f(1) + f(2) + \cdots + f(n)) = f(n+1) + f(n+2) + f(n+3) + \cdots \]

and hence, as the picture shows
\[ \int_{n+1}^\infty f(x) \, dx \leq s - s_n = f(n+1) + f(n+2) + f(n+3) + \cdots \leq \int_n^\infty f(x) \, dx. \]

Sometimes for technical reasons we use the following version of the IT and EE in the IT. When the summation starts from \( n = m \) where \( m > 0 \) is an integer it will be convenient for us to define
\[ a_1 = a_2 = \cdots = a_{m-1} = 0 \]

and use the notation \( s_n \) to denote the \( n \)th partial sum accordingly.
IT (Integral Test). Suppose $f$ is continuous, positive, and decreasing on $[m, \infty)$. Let $a_n := f(n)$. The infinite sum $\sum_{n=m}^{\infty} a_n$ converges if and only if $\int_{m}^{\infty} f(x) \, dx$ converges. This means that

(a) If $\int_{m}^{\infty} f(x) \, dx < \infty$ then $\sum_{n=m}^{\infty} a_n < \infty$,

and

(b) If $\int_{m}^{\infty} f(x) \, dx = \infty$ then $\sum_{n=m}^{\infty} a_n = \infty$.

EE in the IT (Error Estimate in the Integral Test). Suppose $f$ is continuous, positive, and decreasing on $[m, \infty)$. Let $a_n := f(n)$. Let $s := \sum_{n=m}^{\infty} a_n < \infty$ and $s_n := \sum_{k=m}^{n} a_k$, $n = m, m + 1, \ldots$.

We have

$$\int_{n+1}^{\infty} f(x) \, dx \leq s - s_n \leq \int_{n}^{\infty} f(x) \, dx , \quad n = m, m + 1, \ldots .$$

E.1 ($p$-test.) We show that $\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty$ if $p > 1$, and $\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$ if $p \leq 1$. Observe that the function $f(x) = 1/x^p$ is continuous, positive, and decreasing on $[1, \infty)$ for $p \geq 0$. Recall that by E.1 in Section 7.8 we have

$$\int_{1}^{\infty} \frac{1}{x^p} \, dx < \infty , \quad p > 1 ,$$

while

$$\int_{1}^{\infty} \frac{1}{x^p} \, dx = \infty , \quad p \leq 1 .$$

So the IT gives the stated result when $p \geq 0$.

We have to examine the case $p < 0$ separately as in this case $f(x) = 1/x^p$ does not have the decreasing property. However, in the case $p < 0$ we have

$$\frac{1}{n^p} \geq 1 \quad \text{and} \quad s_n \geq n , \quad n = 1, 2, \ldots ,$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \lim_{n \to \infty} s_n \geq \lim_{n \to \infty} n = \infty .$$
We note that the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is called the harmonic series. This is the special case of \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) which is often referred to as the “p-series”. By the result of E.1 the harmonic series diverges to \( \infty \). The result in E.1 will be referred to as the \( p \)-test and you ought to be able to recall it by heart.

E.2. By E.1 we have \( \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty \) if \( p > 1 \). Compute what the EE in the IT gives in this case.

Using the definition of the improper integral (type I (a)), we have

\[
\int_{n}^{\infty} \frac{1}{x^p} \, dx = \lim_{t \to \infty} \int_{n}^{t} \frac{1}{x^p} \, dx = \lim_{t \to \infty} \int_{n}^{t} x^{-p} \, dx = \lim_{t \to \infty} \left[ \frac{x^{-p+1}}{-p+1} \right]_{n}^{t} = \lim_{t \to \infty} \left( \frac{t^{-p+1}}{-p+1} - \frac{n^{-p+1}}{-p+1} \right) = \left( 0 - \frac{n^{-p+1}}{-p+1} \right) = \frac{1}{(p-1)n^{p-1}}
\]

for all \( p > 1 \) as \( -p + 1 < 0 \) in the exponent of \( t \) and \( t \) tends to \( \infty \). We conclude

\[
\frac{1}{(p-1)(n+1)^{p-1}} \leq s - s_n \leq \frac{1}{(p-1)n^{p-1}}
\]

by the EE in the IT.

In the special case \( p = 3 \) the EE in the IT looks like

\[
\frac{1}{2(n+1)^2} \leq s - s_n \leq \frac{1}{2n^2}.
\]

E.3. How many terms do we need to approximate \( s = \sum_{n=1}^{\infty} \frac{1}{n^3} \) by its \( n \)th partial sum \( s_n = \sum_{k=1}^{n} \frac{1}{k^{3}} \) to achieve an error estimate not greater than \( 1/20000 \) guaranteed by the EE in the IT?

In the light of the EE in the IT computed in E.2 we have to solve for

\[
\frac{1}{2n^2} \leq \frac{1}{20000}.
\]

This means \( n \geq 100 \), that is, we need at least \( n = 100 \) terms to achieve

\[
s - s_n \leq \frac{1}{2n^2} \leq \frac{1}{20000}.
\]
E.4. We show that \( \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^p} < \infty \) if \( p > 1 \), and \( \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^p} = \infty \) if \( p \leq 1 \).

Observe that the function

\[ f(x) = \frac{1}{x(\ln x)^p} \]

is continuous, positive, and decreasing on \([3, \infty)\) for \( p \geq 0 \). Recall that by E.1 in Section 7.8 we have

\[ \int_{3}^{\infty} \frac{1}{x(\ln x)^p} \, dx = \int_{3}^{\infty} \frac{1}{u^p} \, du < \infty , \quad p > 1 , \]

while

\[ \int_{3}^{\infty} \frac{1}{x(\ln x)^p} \, dx = \int_{3}^{\infty} \frac{1}{u^p} \, du = \infty , \quad p \leq 1 , \]

So the IT gives the stated result when \( p \geq 0 \).

We have to examine the case \( p < 0 \) separately as in this case \( f(x) \) does not have the decreasing property. However, in the case \( p < 0 \) we have

\[ \frac{1}{n(\ln n)^p} \geq 1 \quad \text{and} \quad s_n \geq n - 2 , \quad n = 3, 4, \ldots , \]

and hence

\[ \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^p} = \lim_{n \to \infty} s_n \geq \lim_{n \to \infty} n - 2 = \infty . \]

E.5. By E.4 we have \( \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2} < \infty \). Compute what the EE in the IT gives in this case.

Using the definition of the improper integral (type I (a)), we have

\[ \int_{n}^{\infty} \frac{1}{x(\ln x)^2} \, dx = \lim_{t \to \infty} \int_{n}^{t} \frac{1}{x(\ln x)^2} \, dx = \lim_{t \to \infty} \int_{\ln n}^{\ln t} u^{-2} \, du = \lim_{t \to \infty} \left[ \frac{u^{-1}}{-1} \right]_{\ln n}^{\ln t} \]

\[ = \lim_{t \to \infty} \left( -\left( \frac{1}{\ln t} \right)^{-1} - \left( \frac{1}{\ln n} \right)^{-1} \right) = \lim_{t \to \infty} \left( -\frac{1}{\ln t} + \frac{1}{\ln n} \right) \]

\[ = 0 + \frac{1}{\ln n} = \frac{1}{\ln n} , \quad n = 3, 4, \ldots . \]

We conclude that

\[ \frac{1}{\ln (n + 1)} \leq s - s_n \leq \frac{1}{\ln n} , \quad n = 3, 4, \ldots , \]

by the EE in the IT.

E.6. How many terms do we need to approximate \( s = \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2} \) by its \( n \)th partial sum

\[ s_n = \sum_{k=3}^{n} \frac{1}{k(\ln k)^2} \]

to achieve an error estimate not greater than \( 1/20 \) guaranteed by the EE in the IT?
In the light of the EE in the IT computed in E.5 we have to solve for

\[ \frac{1}{\ln n} \leq \frac{1}{20}. \]

This means \( n \geq e^{20} \), that is, we need at least \( n = 485165196 \) terms to achieve

\[ s - s_n \leq \frac{1}{\ln n} \leq \frac{1}{20}. \]

The sum is very slowly convergent.