Section 11.9. Representation of Functions as Power Series

Recall that the GST (Geometric Series Theorem) states that

\[
\sum_{n=0}^{\infty} q^n = \sum_{n=1}^{\infty} q^{n-1} = 1 + q + q^2 + \cdots + q^n + \cdots = \frac{1}{1-q}, \quad q \in (-1, 1).
\]

We proved this in Section 11.2. We have also observed in Section 11.2 (and also in E.2 of Section 11.8) that the geometric series diverges at every real number outside the interval \((-1, 1)\), so the interval \((-1, 1)\) is the interval of convergence for the geometric series. So the starting observation in this section is

(1) \[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad x \in (-1, 1).
\]

where \((-1, 1)\) is the interval of convergence of the power series (geometric series) at the right-hand side. Replacing \(x\) by \(-x\) in (1) we get

(2) \[
\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \quad x \in (-1, 1).
\]

We can also deduce easily that if \(a > 0\), then

(3) \[
\frac{1}{a-x} = \frac{1}{a(1-(x/a))} = \sum_{n=0}^{\infty} \left(\frac{x}{a}\right)^n = \sum_{n=0}^{\infty} \frac{1}{a^{n+1}} x^n, \quad x \in (-a, a),
\]

as the power series in (3) converges when \(|x/a| < 1\), that is, when \(x \in (-a, a)\). Note also that if \(x\) is outside the interval \((-a, a)\), then the power series diverges.

Replacing \(x\) by \(-x\) in (3) we get

(4) \[
\frac{1}{a+x} = \frac{1}{a-(-x)} = \sum_{n=0}^{\infty} \frac{1}{a^{n+1}} (-x)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1}} x^n \quad x \in (-a, a).
\]

Note also that if \(x\) is outside the interval \((-a, a)\), then the power series diverges.

The main tool to find power series representations of certain functions is the following theorem.

**Theorem (Differentiation and Integration of Power Series).** If

\[
f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad x \in (a-R, a+R),
\]

is a power series with radius of convergence \(R\), then \(f\) is differentiable (hence continuous) on the interval \((a-R, a+R)\), and we have
(a) \[ f'(x) = \sum_{n=1}^{\infty} c_n (x-a)^{n-1}, \quad x \in (a-R, a+R), \]

and

(b) \[ \int f(x) \, dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}, \quad x \in (a-R, a+R), \]

and the radii of convergence of the power series in both (a) and (b) remain \( R \).

Roughly speaking the “Differentiation and Integration of Power Series Theorem” tells us that it is legal to differentiate and integrate a power series term by term on the open interval of convergence.

**Differentiating the Geometric Series**

E.1. Differentiating the geometric series (1) we get

(5) \[ \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}, \quad x \in (-1, 1). \]

and \( R := 1 \) remains the radius of convergence in (5). Note that the summation starts from \( n = 1 \) as the derivative of the constant term in (1) is 0. Note also that the indices may be shifted to get

(5*) \[ \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^{n}, \quad x \in (-1, 1). \]

E.2. Differentiating the series (5) we get

(6) \[ \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^{n-2}, \quad x \in (-1, 1). \]

and \( R := 1 \) remains the radius of convergence in (5). Note that the summation starts from \( n = 2 \) as the derivative of the constant term in (5) is 0. Note also that the indices may be shifted to get

(6*) \[ \frac{2}{(1-x)^3} = \sum_{n=0}^{\infty} (n+2)(n+1) x^{n}, \quad x \in (-1, 1). \]

E.3. Combining the results in (6) and (5) we get

(7) \[ \sum_{n=1}^{\infty} n^2 x^n = \sum_{n=1}^{\infty} (n(n-1) + n) x^n = \sum_{n=1}^{\infty} n(n-1) x^n + \sum_{n=1}^{\infty} n x^n = \sum_{n=2}^{\infty} n(n-1) x^n + \sum_{n=1}^{\infty} n x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}, \quad x \in (-1, 1). \]
E.4. The special case $x = 1/2$ in (5) gives
\[ \sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} n x^n \bigg|_{x=1/2} = \frac{x}{(1-x)^2} \bigg|_{x=1/2} = \frac{1/2}{(1/2)^2} = 2. \]

E.5. The special case $x = -1/2$ in (5) gives
\[ \sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n} = \sum_{n=1}^{\infty} n x^n \bigg|_{x=-1/2} = \frac{x}{(1-x)^2} \bigg|_{x=-1/2} = \frac{-1/2}{(3/2)^2} = -\frac{2}{9}. \]

E.6. The special case $x = 1/3$ in (7) gives
\[ \sum_{n=1}^{\infty} \frac{n^2}{3^n} = \sum_{n=1}^{\infty} n^2 x^n \bigg|_{x=1/3} = \left( \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \right) \bigg|_{x=1/3} = \frac{2/9}{8/27} + \frac{1/3}{4/9} - \frac{6}{8} + \frac{3}{4} - \frac{3}{2} = \frac{8}{27} - \frac{3}{4} + \frac{3}{32}. \]

E.7. The special case $x = -1/3$ in (7) gives
\[ \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{3^n} = \sum_{n=1}^{\infty} n^2 x^n \bigg|_{x=-1/3} = \left( \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} \right) \bigg|_{x=-1/3} = \frac{2(-1/3)^2}{(4/3)^3} + \frac{-1/3}{(4/3)^2} = \frac{6}{64} - \frac{3}{16} = \frac{-3}{32}. \]

**Integrating the Geometric Series**

E.8. Integrating the geometric series
\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad x \in (-1, 1), \]
we get
\[ -\ln |1-x| = -\ln(1-x) = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \quad x \in (-1, 1), \]
and $R := 1$ remains the radius of convergence in (8). Putting $x = 0$ in (8) we get $0 = C + 0$ and hence we have $C = 0$ in (8), that is,
\[ \ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\frac{x}{1} - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots, \quad x \in (-1, 1). \]
Note also that the indices may be shifted to get

\[(10) \quad \ln(1 - x) = - \sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots, \quad x \in (-1, 1).\]

If we replace \(x\) by \(-x\) in (10) we get

\[(11) \quad \ln(1 + x) = \ln(1 - (-x)) = - \sum_{n=1}^{\infty} \frac{(-x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n},
= \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad x \in (-1, 1),\]

In fact, in advanced calculus it is proved that the power series always continuous on its interval of convergence, hence if follows from the AST and the continuity of the \(\ln(1 + x)\) function at 1 that plugging \(x = 1\) in the power series of \(\ln(1 + x)\) we get the value \(\ln(1 + 1) = \ln 2\). That is,

\[\ln 2 = \ln(1 + 1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\]

and

\[(12) \quad \ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad x \in (-1, 1].\]

E.9. The special case \(x = 1/2\) in (10) gives

\[- \ln 2 = \ln \frac{1}{2} = \ln \left(1 - \frac{1}{2}\right) = - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{2}\right)^n = - \sum_{n=1}^{\infty} \frac{1}{n2^n},\]

and hence

\[\sum_{n=1}^{\infty} \frac{1}{n2^n} = \ln 2.\]

E.10. The special case \(x = -1/2\) in (10) gives

\[\ln 3 - \ln 2 = \ln \frac{3}{2} = \ln \left(1 - \frac{1}{2}\right) = - \sum_{n=1}^{\infty} \frac{1}{n} \left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n},\]

and hence

\[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n} = \ln 3 - \ln 2.\]
E.11. Observe that replacing $x$ by $x^2$ in the geometric series

\[(2) \quad \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \quad x \in (-1, 1), \]

we get

\[(13) \quad \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad x \in (-1, 1). \]

Integrating (13) we get

\[
\arctan x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad x \in (-1, 1),
\]

and the radius of convergence remains $R = 1$. Plugging $x = 0$ in the formula above we get $0 = \arctan 0 = C + 0$, so $C = 0$. Hence

\[(14) \quad \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad x \in (-1, 1). \]

Shifting the indices we may rewrite this as

\[(14^*) \quad \arctan x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad x \in (-1, 1). \]

In fact, in advanced calculus it is proved that the power series always continuous on its interval of convergence, hence if follows from the AST and the continuity of the $\arctan x$ function at 1 that plugging $x = 1$ in the power series of $\arctan x$ we get the value $\arctan 1 = \pi/4$. That is,

\[
\frac{\pi}{4} = \arctan 1 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n-1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots.
\]

Similarly,

\[
-\frac{\pi}{4} = \arctan(-1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{-1}{2n-1} = -\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots,
\]

and hence

\[(15) \quad \arctan x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \quad x \in [-1, 1]. \]
E.12. Observe that the power series representation of the arctan \( x \) function is an alternating series satisfying the assumptions of the AST for every \( x \in [0, 1] \) with the notation

\[
b_n := \frac{x^{2n-1}}{2n-1} \geq 0, \quad n = 1, 2, \ldots
\]

Let

\[
s_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^{2k-1}}{2k-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1}
\]

be the \( n \)th partial sum of the power series in (15). Using the EE in the AST (Error Estimate in the Alternating Series Test), we get

\[
|\arctan x - s_n| = |\arctan x - \left( x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1}\right)|
\]

\[
\leq b_{n+1} = \frac{x^{2n+1}}{2n+1}, \quad x \in [0, 1].
\]

As a special case

\[
\left|\arctan \left( \frac{1}{10} \right) - \left( \frac{1}{10} - \frac{1}{3} \frac{1}{10^3} + \frac{1}{5} \frac{1}{10^5} - \frac{1}{7} \frac{1}{10^7}\right)\right| \leq \frac{1}{9} \frac{1}{10^9}.
\]

Hence to approximate \( \arctan(1/10) \) we have achieved a high accuracy by adding only the first four (non-zero) terms of the power series representation.

E.13. Find \( \sum_{k=0}^{\infty} \frac{(-1)^k}{3^k(2k+1)} \).

We have

\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{3^k(2k+1)} = \sqrt{3} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} \bigg|_{x=1/\sqrt{3}} = \sqrt{3} \arctan \left( \frac{1}{\sqrt{3}} \right) = \sqrt{3} \frac{\pi}{6} = \frac{\pi}{2\sqrt{3}}.
\]

E.14. Use the term by term Integration of Power Series Theorem to represent

\[
\int x^2 \ln(32 + x^5) \, dx
\]

as a power series. What is its radius of convergence? Write \( \int_0^1 x^2 \ln(32 + x^5) \, dx \) as an infinite series.

We have

\[
\ln(32 + x^5) = \ln \left( 32 \left( 1 + \frac{x^5}{32} \right) \right) = \ln 32 + \ln \left( 1 + \frac{x^5}{32} \right).
\]
By (11) we have
\[
\ln(1 + y) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{y^n}{n}, \quad |y| < 1,
\]
where \( R = 1 \) is the radius of convergence. Substituting \( y = x^5/32 \), we obtain
\[
\ln \left(1 + \frac{x^5}{32}\right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{x^5}{32}\right)^n}{n}, \quad \left|\frac{x^5}{32}\right| < 1,
\]
where \( R = 2 \) is the radius of convergence. Hence
\[
x^2 \ln(32 + x^5) = (\ln 32)x^2 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{5n+2}}{32^nn}, \quad |x| < 2,
\]
where \( R = 2 \) is the radius of convergence. Integrating term by term gives
\[
\int x^2 \ln(32 + x^5) \, dx = C + (\ln 32)x^3 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{5n+3}}{32^nn(5n + 3)}, \quad |x| < 2,
\]
where \( R = 2 \) is the radius of convergence.

E.15. Use the term by term Integration of Power Series Theorem to represent
\[
\int \frac{x^7}{27 + 8x^3} \, dx
\]
as a power series. What is its radius of convergence? Write \( \int_{0}^{1/2} \frac{x^7}{8x^3 + 27} \, dx \) as an infinite series.

We have
\[
\frac{1}{27 + 8x^3} = \frac{1}{27(1 - (-8x^3/27))}
\]
By (1) we have
\[
\frac{1}{1 - y} = \sum_{n=0}^{\infty} y^n, \quad |y| < 1,
\]
where \( R = 1 \) is the radius of convergence. Substituting \( y = -8x^3/27 \) and multiplying by \( 1/27 \), we obtain

\[
\frac{1}{27 + 8x^3} = \frac{1}{27(1 - (-8x^3/27))} = \sum_{n=0}^{\infty} \frac{1}{27} \left( \frac{-8x^3}{27} \right)^n, \quad \left| \frac{-8x^3}{27} \right| < 1,
\]

where \( R = 3/2 \) is the radius of convergence. Hence

\[
\frac{x^7}{27 + 8x^3} = \sum_{n=0}^{\infty} \frac{1}{27} \left( \frac{-8}{27} \right)^n x^{3n+7}, \quad |x| < 3/2,
\]

where \( R = 3/2 \) is the radius of convergence. Integrating term by term gives

\[
\int \frac{x^7}{27 + 8x^3} \, dx = C + \sum_{n=0}^{\infty} \frac{1}{27} \left( \frac{-8}{27} \right)^n \frac{x^{3n+8}}{3n+8}, \quad |x| < 3/2,
\]

where \( R = 3/2 \) is the radius of convergence.

We have

\[
\int_0^{1/2} \frac{x^7}{27 + 8x^3} \, dx = \left[ \sum_{n=0}^{\infty} \frac{1}{27} \left( \frac{-8}{27} \right)^n \frac{x^{3n+8}}{3n+8} \right]_0^{1/2} = \sum_{n=0}^{\infty} \frac{1}{27} \left( \frac{-1}{27} \right)^n \frac{1}{2^n(3n+8)}.
\]

E.16. Use the term by term Integration of Power Series Theorem to represent

\[
\int \frac{\arctan(x^3/27)}{x^2} \, dx
\]

as a power series. What is its radius of convergence? Write \( \int_1^2 \frac{\arctan(x^3/27)}{x^2} \, dx \) as an infinite series.

By (14) we have

\[
\arctan y = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{2n+1}, \quad |y| < 1.
\]

Substituting \( y = x^3/27 \), we obtain

\[
\arctan(x^3/27) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3/27)^{2n+1}}{2n+1}, \quad |x^3/27| < 1,
\]

where \( R = 3 \) is the radius of convergence. Hence

\[
\frac{\arctan(x^3/27)}{x^2} = \sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{27} \right)^{2n+1} \frac{1}{2n+1} x^{6n+1}, \quad |x| < 3,
\]
where \( R = 3 \) is the radius of convergence. Integrating term by term gives

\[
\int \frac{\arctan(x^3/27)}{x^2} \, dx = C + \sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{27} \right)^{2n+1} \frac{x^{6n+2}}{(2n + 1)(6n + 2)}, \quad |x| < 3,
\]

where \( R = 3 \) is the radius of convergence.

We have

\[
\int_{1}^{2} \frac{\arctan(x^3/27)}{x^2} \, dx = \left[ \sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{27} \right)^{2n+1} \frac{x^{6n+2}}{(2n + 1)(6n + 2)} \right]_{1}^{2} \\
= \sum_{n=0}^{\infty} (-1)^n \left( \frac{1}{27} \right)^{2n+1} \frac{2^{6n+2} - 1}{(2n + 1)(6n + 2)}.
\]