5.5 Integration by Substitution

**Substitution Rule for indefinite integrals (SR1).** If \( u = g(x) \) is differentiable on an interval \( J \), \( g' \) is continuous on \( J \), the range of \( g \) is an interval \( I \), and \( f \) is continuous on \( I \), then

\[
\int f(g(x))g'(x) \, dx = \int f(u) \, du.
\]

Roughly speaking it is permissible to operate with \( dx \) and \( du \) after the integral sign as if they were differentials:

\[
u = g(x), \quad \frac{du}{dx} = g'(x), \quad du = g'(x) \, dx.
\]

Following the short proof below must help your understanding.

**Proof.** Let

\[
\int f(u) \, du = F(u) + C, \quad u \in I,
\]

that is, \( F'(u) = f(u) \) for all \( u \in I \). Then

\[
\int f(g(x))g'(x) \, dx = \int F'(g(x))g'(x) \, dx = \int (F \circ g)'(x) \, dx
\]

\[
= (F \circ g)(x) + C = F(g(x)) + C
\]

\[
= F(u) + C, \quad u \in I.
\]

\( \square \)

Note that \( F \circ g \) is the composition function as you have learned in Math 151 defined by

\[
(F \circ g)(x) := F(g(x)).
\]

**Substitution Rule for definite integrals (SR2).** If \( u = g(x) \) is differentiable on an interval \([a, b] \), \( g' \) is continuous on \([a, b] \), the range of \( g \) is an interval \( I \), and \( f \) is continuous on the range of \( g \), then

\[
\int_a^b f(g(x))g'(x) \, dx = \int_{g(a)}^{g(b)} f(u) \, du.
\]

Note that for the case \( a > b \) we adopt the definition

\[
\int_a^b f(x) \, dx := (-1) \int_b^a f(x) \, dx.
\]

Following the short proof below must help your understanding.
Proof. Let
\[ \int f(u) \, du = F(u) + C , \]
that is, \( F'(u) = f(u) \) for all \( u \) in the range of \( g \). Then by the FTC we have
\[ \int_{g(a)}^{g(b)} f(u) \, du = [F(u)]_{g(a)}^{g(b)} = F(g(b)) - F(g(a)). \]

On the other hand, the FTC combined with the Chain Rule of differentiation gives
\[
\int_{a}^{b} f(g(x))g'(x) \, dx = \int_{a}^{b} F'(g(x))g'(x) \, dx = \int_{a}^{b} (F \circ g)'(x) \, dx = \left[ (F \circ g)(x) \right]_{a}^{b} = F(g(b)) - F(g(a)).
\]
□

E.1. Evaluate
\[ \int x^2 \sqrt{2x + 3} \, dx . \]

Let \( u = 2x + 3 \). Then \( du/dx = 2 \), \( du = 2 \, dx \), \( x = (u - 3)/2 \), \( dx = (1/2) \, du \). Using SR1 we get
\[
\int x^2 \sqrt{2x + 3} \, dx = \int \left( \frac{u - 3}{2} \right)^2 \sqrt{u} \cdot \frac{1}{2} \, du = \int \left( \frac{u^2}{4} - \frac{6u}{4} + \frac{9}{4} \right) \frac{u^{1/2}}{2} \, du = \int \left( \frac{u^{5/2}}{8} - \frac{3}{4} u^{3/2} + \frac{9}{8} u^{1/2} \right) \, du = \frac{u^{7/2}}{8 \cdot (7/2)} - \frac{3}{4} \frac{u^{5/2}}{5/2} + \frac{9}{8} \frac{u^{3/2}}{3/2}
\]
\[ = \frac{1}{28}(2x + 3)^{7/2} - \frac{3}{10}(2x + 3)^{5/2} + \frac{3}{4}(2x + 3)^{3/2} + C . \]

E.2. Evaluate
\[ \int \frac{x + 1}{x^2 + 2x} \, dx . \]

Let \( u = x^2 + 2x \). Then \( du/dx = 2x + 2 = 2(x + 1) \), so \( du = 2(x + 1) \, dx \). Using SR1 we get
\[
\int \frac{x + 1}{x^2 + 2x} \, dx = \int \frac{1}{2} \, du = \int \frac{1}{2} u^{-1} \, du = \frac{1}{2} \ln |u| = \frac{1}{2} \ln |x^2 + 2x| + C .
\]
E.3. Evaluate
\[ \int \tan^2 x \sec^2 x \, dx. \]

Let \( u = \tan x \). Then \( du/dx = \sec^2 x \), so \( du = \sec^2 x \, dx \). Using SR1 we get
\[ \int \tan^2 x \sec^2 x \, dx = \int u^2 \, du = \frac{u^3}{3} = \frac{1}{3} \tan^3 x + C. \]

E.4. Evaluate
\[ \int \tan x \, dx. \]
We substitute \( u = \cos x \). Then \( du/dx = -\sin x \), hence \((-1)du = \sin x \, dx \). We get
\[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int \frac{1}{u} (-1) du = (-1) \ln |u| = -\ln |\cos x| + C. \]

Note that \( \ln |\sec x| = \ln(1/|\cos x|) = \ln 1 - \ln |\cos x| = -\ln |\cos x| \).

E.5. Evaluate
\[ \int_0^1 (2x - 1)^{100} \, dx. \]
Let \( u = g(x) = 2x - 1 \). Then \( du/dx = 2 \), so \( dx = (1/2)du \). Using SR2 we get
\[ \int_0^1 (2x - 1)^{100} \, dx = \int_{g(0)}^{g(1)} u^{100} \frac{1}{2} \, du = \int_{-1}^1 u^{100} \frac{1}{2} \, du \]
\[ = \left[ \frac{u^{101}}{202} \right]_{-1}^1 = \frac{1}{202} - \frac{(-1)^{101}}{202} = \frac{2}{202} = \frac{1}{101}. \]

We can also proceed by SR1 rather than SR2. First we are less ambitious and calculate only the corresponding indefinite integral by using the substitution \( u = 2x - 1 \) as before:
\[ \int (2x - 1)^{100} \, dx = \int u^{100} \frac{1}{2} \, du = \frac{u^{101}}{202} = \frac{(2x - 1)^{101}}{202} + C, \]
and then we use the FTC to evaluate the original definite integral:
\[ \int_0^1 (2x - 1)^{100} \, dx = \left[ \frac{(2x - 1)^{101}}{202} \right]_0^1 = \frac{1}{202} - \frac{(-1)^{101}}{202} = \frac{2}{202} = \frac{1}{101}. \]
E.6. Evaluate
\[ \int_0^{\pi/4} \sin(4t) \, dt. \]

Let \( u = g(t) = 4t. \) Then \( du/dt = 4, \ dt = (1/4)du. \) Using SR2 we get
\[
\int_0^{\pi/4} \sin(4t) \, dt = \int_{g(0)}^{g(\pi/4)} (\sin u) \frac{1}{4} \, du = \int_0^{\pi} (\sin u) \frac{1}{4} \, du = \left[ -\frac{1}{4} \cos u \right]_0^{\pi} \\
= \left( -\frac{1}{4} \cos \pi \right) - \left( -\frac{1}{4} \cos 0 \right) = \left( -\frac{1}{4} (-1) \right) - \left( -\frac{1}{4} \cdot 1 \right) \\
= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\]

We can also proceed by SR1 rather than SR2. First we are less ambitious and calculate only the corresponding indefinite integral by using the substitution \( u = 4t \) as before:
\[
\int \sin(4t) \, dt = \int \sin u \frac{1}{4} \, du = \frac{1}{4} (-\cos u) = -\frac{1}{4} \cos u = -\frac{1}{4} \cos(4t) + C,
\]
and then we use the FTC to evaluate the original definite integral:
\[
\int_0^{\pi/4} \sin(4t) \, dt = \left[ -\frac{1}{4} \cos(4t) \right]_0^{\pi/4} = \left( -\frac{1}{4} \cos(4(\pi/4)) \right) - \left( -\frac{1}{4} \cos(4 \cdot 0) \right) \\
= \left( -\frac{1}{4} \cos \pi \right) - \left( -\frac{1}{4} \cos 0 \right) = \left( -\frac{1}{4} (-1) \right) - \left( -\frac{1}{4} \cdot 1 \right) \\
= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\]

E.7. Evaluate
\[ \int_1^4 \frac{1}{x^2} \sqrt{1 + \frac{1}{x}} \, dx. \]

Let \( u = 1 + 1/x = 1 + x^{-1}, \) Then \( du/dx = (-1)x^{-2} = -1/x^2, \) so \( du = (-1/x^2) \, dx. \) Using SR2 we get
\[
\int_1^4 \frac{1}{x^2} \sqrt{1 + \frac{1}{x}} \, dx = \int_2^5 \sqrt{u} (-1) \, du = \int_2^{5/4} -u^{1/2} \, du \\
= \left[ -u^{3/2} \right]_2^{5/4} = \frac{(- (5/4)^{3/2})}{3/2} - \frac{(-2^{3/2})}{3/2} \\
= 4\sqrt{2} \cdot 5\sqrt{5} = 12.
\]
E.8. Evaluate
\[ \int \sin^5 x \, dx . \]

One may naturally try the substitution \( u = \sin x \). Then
\[ \frac{du}{dx} = \cos x = \sqrt{1 - u^2} , \quad x \in (-\pi/2, \pi/2) , \]
and hence
\[ \int \sin^5 x \, dx = \int \frac{u^5}{\sqrt{1 - u^2}} \, du , \]
which looks a more complicated integral to evaluate.

If we write \( \sin^5 x = (\sin^2 x)^2 \sin x = (1 - \cos^2 x)^2 \sin x \), then the substitution \( u = \cos x \) looks more promising as \( du/dx = -\sin x \). Indeed, substituting \( u = \cos x \), we have \( \sin x \, dx = (-1)\, du \), and hence
\[
\int \sin^5 x \, dx = \int (1 - \cos^2 x)^2 \sin x \, dx = \int (1 - u^2)^2 (-1)\, du = \int (-u^4 + 2u^2 - 1) \, du
\]
\[ = -\frac{u^5}{5} + \frac{2u^3}{3} - u = -\frac{\cos^5 x}{5} + 2 \frac{\cos^3 x}{3} - \cos x + C . \]

We will learn more about this and similar tricks to evaluate the integral of certain trigonometric products later in Section 7.2.

E.9. Evaluate
\[ \int \sqrt{1 - x^2} \, dx . \]

Observe that the understood domain of the integrand is the interval \([-1, 1]\). We use the substitution \( u = \arcsin x \) (recall the graph of the function \( u = \arcsin x \)). Then we have \( x = \sin u \) and \( dx/du = \cos u \), so
\[
\int \sqrt{1 - x^2} \, dx = \int \sqrt{1 - \sin^2 u} \cos u \, du = \int \cos^2 u \, du = \int \frac{1 + \cos(2u)}{2} \, du
\]
\[ = \frac{u}{2} + \frac{\sin(2u)}{4} = \frac{u}{2} + \frac{2\sin u \cos u}{4} = \frac{1}{2} \arcsin x + \frac{1}{2} x \sqrt{1 - x^2} + C . \]

We will learn more about such trigonometric substitutions later in Section 7.3.

E.10. Observe that we can calculate the area of the disk of radius 1 centered at 0 as
\[
A = 2 \int_{-1}^{1} \sqrt{1 - x^2} \, dx = \left[ \arcsin x + x \sqrt{1 - x^2} \right]_{-1}^{1} = \arcsin 1 - \arcsin(-1)
\]
\[ = \frac{\pi}{2} - \left( -\frac{\pi}{2} \right) = \pi . \]
E.11. Evaluate
\[ \int \frac{5x - 3}{x^2 + 1} \, dx. \]

The trick is that in the numerator of the integrand we divide term by term and we write
\[ \int \frac{5x - 3}{x^2 + 1} \, dx = \int \frac{5x}{x^2 + 1} \, dx - \int \frac{3}{x^2 + 1} \, dx. \]

We can handle the first integral above by substituting \( u = x^2 + 1 \) to get
\[ \int \frac{5x}{x^2 + 1} \, dx = \int u^{-1} \frac{5}{2} \, du = \frac{5}{2} \ln |u| = \frac{5}{2} \ln |x^2 + 1|. \]

The second integral was already in the table of reversed differentiation formulas up to the constant multiple:
\[ \int \frac{3}{x^2 + 1} \, dx = 3 \arctan x. \]

We conclude that
\[ \int \frac{5x - 3}{x^2 + 1} \, dx = \int \frac{5x}{x^2 + 1} \, dx - \int \frac{3}{x^2 + 1} \, dx = \frac{5}{2} \ln |x^2 + 1| - 3 \arctan x + C. \]

The next example belongs to Section 7.4, but you may already have a look at it and ought to be able to follow.

E.12. Evaluate
\[ \int \frac{4x - 3}{x^2 - 6x + 13} \, dx. \]

The trick is that in the numerator of the integrand we separate a term that is constant times the derivative of the denominator: \( 4x - 3 = 2(2x - 6) + 9 \), and we write
\[ \int \frac{4x - 3}{x^2 - 6x + 13} \, dx = \int \frac{2(2x - 6) + 9}{x^2 - 6x + 13} \, dx = \int \frac{2(2x - 6)}{x^2 - 6x + 13} \, dx + \int \frac{9}{x^2 - 6x + 13} \, dx. \]

We can handle the first integral above by substituting \( u = x^2 - 6x + 13 \) to get
\[ \int \frac{2(2x - 6)}{x^2 - 6x + 13} \, dx = \int u^{-1} \frac{2}{2} \, du = 2 \ln |u| = 2 \ln |x^2 - 6x + 13|. \]

To handle the second integral above we use the substitution \( v = (x - 3)/2 \) to get
\[ \int \frac{9}{x^2 - 6x + 13} \, dx = 9 \int \frac{1}{4 (\frac{x-3}{2})^2 + 1} \, dv = \frac{9}{4} \int \frac{1}{v^2 + 1} \, 2dv = \frac{9}{2} \arctan v = \frac{9}{2} \arctan \frac{x - 3}{2}. \]

We conclude
\[ \int \frac{4x - 3}{x^2 - 6x + 13} \, dx = \int \frac{2(2x - 6) + 9}{x^2 - 6x + 13} \, dx = \int \frac{2(x - 6)}{x^2 - 6x + 13} \, dx + \int \frac{9}{x^2 - 6x + 13} \, dx = 2 \ln |x^2 - 6x + 13| + \frac{9}{2} \arctan \frac{x - 3}{2} (+C). \]
**Some Trigonometry**

We have

\[ \sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha \]

and

\[ \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta . \]

These formulas may be referred to as the addition formulas in trigonometry. Some simple consequences of them include

\[ 1 = \cos 0 = \cos(u - u) = \cos u \cos u - \sin u(-\sin u) = \cos^2 u + \sin^2 u , \]

that is,

\[ 1 = \cos^2 u + \sin^2 u , \]

\[ \cos(2u) = \cos^2 u - \sin^2 u , \]

\[ \cos^2 u = \frac{1 + \cos(2u)}{2} , \quad \sin^2 u = \frac{1 - \cos(2u)}{2} , \]

\[ \sin(2u) = 2 \sin u \cos u . \]