6.3. **Volume by the method of cylindrical shells**

Let $A$ be a region in the $xy$-plane bounded by the graphs of the functions $y = f(x)$ and $y = g(x)$, where $f$ and $g$ are continuous functions on the interval $[a, b]$ of the $x$-axis for which $g(x) \leq f(x)$ for all $a \leq x \leq b$. That is,

$$A := \{(x, y) : x \in [a, b], \ g(x) \leq y \leq f(x)\}.$$

Let $B$ be the three-dimensional body obtained by revolving the region about the line $x = L$ parallel to the $y$-axis. To avoid overlap we assume that either

1. $A$ is right to the line $y = L$, that is, $L \leq a < b$,

or

2. $A$ is left to the line $y = L$, that is, $a < b \leq L$.

We are interested in the volume $V(B)$ of the body $B$.

**Formula 1 (Volume by the shell(y) method).** In case (1) we have

$$V(B) = 2\pi \int_a^b (f(x) - g(x))(x - L) \, dx.$$

**Formula 2. (Volume by the shell(y) method).** In case (2) we have

$$V(B) = 2\pi \int_a^b (f(x) - g(x))(L - x) \, dx.$$

Unlike the volume by the washer method formulas these formulas do not follow from the volume by cross sections formula. You may have a look at the book to see how these formulas are arising, typically I outline the justification of these formulas in my lectures. Often times we revolve the region $A$ about the $y$-axis, that is, the line $x = 0$, so we apply the above formulas with $L = 0$. Formulas 1 and 2 are called volume by the shell(y) method formulas, where $y$ indicates that the region $A$ is revolved around a line parallel to the $y$-axis.
Why is Formula 1 correct? For simplicity let $L = 0 \leq a < b$. Let

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

and let

$$P_n := \{[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\}$$

be the $n$th partition. Recall that the norm of $P_n$ is defined by

$$\|P_n\| := \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}.$$ 

Let

$$x_i^* := \frac{x_i + x_{i-1}}{2}, \quad i = 1, 2, \ldots, n.$$ 

The body $B$ is approximated by the union of $n$ cylindrical shells, where the $i$th cylindrical shell ($i = 1, 2, \ldots, n$) is obtained by revolving the rectangle with base length $x_i - x_{i-1}$ and height $f(x_i^*) - g(x_i^*)$ around the $y$-axis, and the volume $V_i$ of the $i$th cylindrical shell is

$$V_i = (x_i^2 \pi - x_{i-1}^2 \pi)(f(x_i^*) - g(x_i^*)).$$

We have

$$\sum_{i=1}^{n} V_i = \sum_{i=1}^{n} (x_i^2 \pi - x_{i-1}^2 \pi)(f(x_i^*) - g(x_i^*))$$

$$= \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\pi(x_i^2 - x_{i-1}^2)$$

$$= \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))\pi(x_i + x_{i-1})(x_i - x_{i-1})$$

$$= \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))2\pi x_i^*(x_i - x_{i-1}),$$

and hence the volume $V$ of the body $B$ is

$$V = \lim_{\|P_n\| \to 0} \sum_{i=1}^{n} V_i$$

$$= \lim_{\|P_n\| \to 0} \sum_{i=1}^{n} (f(x_i^*) - g(x_i^*))2\pi x_i^*(x_i - x_{i-1})$$

$$= 2\pi \int_{a}^{b} (f(x) - g(x))x \, dx.$$
Now let \( A \) be a region in the \( xy \)-plane bounded by the graphs of the functions \( x = F(y) \) and \( x = G(x) \), where \( F \) and \( G \) are continuous functions on the interval \([c, d]\) of the \( y \)-axis for which \( G(y) \leq f(y) \) for all \( c \leq y \leq d \). That is,

\[
A := \{(x, y) : y \in [c, d], \ G(y) \leq x \leq F(y)\}.
\]

Let \( B \) be the three-dimensional body obtained by revolving the region about the line \( y = L \) parallel to the \( x \)-axis. To avoid overlap we assume that either

(3) \( A \) is above the line \( y = L \), that is, \( L \leq c < d \),

or

(4) \( A \) is below the line \( y = L \), that is, \( c < d \leq L \).

We are interested in the volume \( V(B) \) of the body \( B \).

Formula 3 (Volume by the shell(x) method). In case (3) we have

\[
V(B) = 2\pi \int_c^d (F(y) - G(y))(y - L) \, dy.
\]

Formula 4. (Volume by the washer(x) method). In case (4) we have

\[
V(B) = \pi \int_c^d (F(y) - G(y))(L - y) \, dy.
\]

Unlike the volume by the washer method formulas these formulas do not follow from the volume by cross sections formula. You may have a look at the book to see how these formulas are arising, typically I outline the justification of these formulas in my lectures. Often times we revolve the region \( A \) about the \( x \)-axis, that is, the line \( y = 0 \), so we apply the above formulas with \( L = 0 \). Formulas 1 and 2 are called volume by the shell(x) method formulas, where \( x \) indicates that the region \( A \) is revolved around a line parallel to the \( x \)-axis.
E.1. The volume of the ball $B$ of radius $r > 0$ centered at the origin is $V(B) := \left(4\pi/3\right)r^3$. We achieve this result again by setting up the formula by the shell(x) formula first, and then by the shell(y) formula.

The ball $B$ may be obtained by revolving the half disk $A$ bounded by the graph of the function

$$x = F(y) := \sqrt{r^2 - y^2}$$

from above and by the graph of the function

$$x = G(y) := -\sqrt{r^2 - y^2}$$

from below on the interval $[0, r]$ of the $y$-axis about the line $y = 0$ (the $x$-axis). Hence by the shell(x) formula we have

$$V(B) = 2\pi \int_{0}^{r} \left(\sqrt{r^2 - y^2} - (-\sqrt{r^2 - y^2})\right) y \, dy = 4\pi \int_{0}^{r} \sqrt{r^2 - y^2} y \, dy$$

$$= 4\pi \int_{r^2}^{0} u^{1/2}(-1/2) \, du = 4\pi \left[\frac{u^{3/2}}{3/2}(-1/2)\right]_{r^2}^{0} = 0 - \left(-\frac{4}{3}\pi r^3\right)$$

$$= \frac{4}{3}\pi r^3,$$

where we used the substitution $u = r^2 - y^2$ with $y \, dy = (-1/2) \, du$.

The ball $B$ may be obtained by revolving the half disk $A$ bounded by the graph of the function

$$y = f(x) := \sqrt{r^2 - x^2}$$

from above and by the graph of the function

$$y = g(x) := -\sqrt{r^2 - x^2}$$

from below on the interval $[0, r]$ of the $x$-axis about the line $x = 0$ (the $y$-axis). Hence by the shell(y) formula we have

$$V(B) = 2\pi \int_{0}^{r} \left(\sqrt{r^2 - x^2} - (-\sqrt{r^2 - x^2})\right) x \, dx = 4\pi \int_{0}^{r} \sqrt{r^2 - x^2} x \, dx$$

$$= 4\pi \int_{r^2}^{0} u^{1/2}(-1/2) \, du = 4\pi \left[\frac{u^{3/2}}{3/2}(-1/2)\right]_{r^2}^{0} = 0 - \left(-\frac{4}{3}\pi r^3\right)$$

$$= \frac{4}{3}\pi r^3,$$

where we used the substitution $u = r^2 - x^2$ with $x \, dx = (-1/2) \, du$. 

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E.2. Let $A$ be the disk in the $xy$-plane of radius $0 < r \leq R$ centered at the point $(0, R)$. The body $B$ called torus is obtained by revolving $A$ about the $x$-axis. We calculate the volume of $B$ by the shell(x) method.

Observe that the disk $A$ is bounded by the graph of the function

$$x = F(y) := \sqrt{r^2 - (y - R)^2}$$

from above, and the graph of the function

$$x = G(y) := -\sqrt{r^2 - (y - R)^2}$$

from below on the interval $[R-r, R+r]$ on the $y$-axis. Hence by the volume by the shell(x) method the volume $V(B)$ of the body $B$ may be calculated as

$$V(B) = 2\pi \int_{R-r}^{R+r} \left( \sqrt{r^2 - (y - R)^2} - \left( -\sqrt{r^2 - (y - R)^2} \right) \right) y \, dy$$

$$= 4\pi \int_{R-r}^{R+r} \sqrt{r^2 - (y - R)^2} y \, dy = 4\pi \int_{-r}^{r} \sqrt{r^2 - u^2} (u + R) \, du$$

$$= 4\pi \int_{-r}^{r} \left( \sqrt{r^2 - u^2} R + \sqrt{r^2 - u^2} u \right) \, du$$

$$= 4\pi R \int_{-r}^{r} \sqrt{r^2 - u^2} \, du + 4\pi \int_{-r}^{r} \sqrt{r^2 - u^2} u \, du$$

$$= 4\pi R \left( \frac{1}{2} r^2 \pi \right) + 0 = (2\pi R)(r^2 \pi) .$$

Here we used the substitution $u = y - R$ and the fact that the geometrical meaning of

$$\int_{-r}^{r} \sqrt{r^2 - x^2} \, dx$$

is the area of the half disk of radius $r$ centered at the origin above the $x$-axis, so

$$\int_{-r}^{r} \sqrt{r^2 - x^2} \, dx = \frac{1}{2} r^2 \pi .$$

We also used the substitution $v = r^2 - u^2$ with $u \, du = (-1/2) \, dv$ in observing that

$$\int_{-r}^{r} \sqrt{r^2 - u^2} u \, du = \int_{0}^{0} \sqrt{v} (-1/2) \, dv .$$

You may easily memorize the volume formula for the torus as the product of the area of the disk of radius $r$, that is $r^2 \pi$, multiplied by the the arc length travelled by its center, that is $2\pi R$ by revolving the disk about the $x$-axis.
E.3. Let $A$ be the region bounded by the line $x - 3y + 1 = 0$ and the curve given by the equation $y = \sqrt{x - 1}$ in the $xy$-plane.

Recall that the area of the region $A$ has been studied in Section 6.1. We have found the coordinates of the two intersection points: $(2, 1)$ and $(5, 2)$.

a] The body $B_1$ is obtained by revolving the region $A$ around the line $y = -1$.

a1] Set up the integral representing the volume of the body $B_1$ by using the washer method.

In part a1] we set $[a, b] := [2, 5]$, $y = f(x) := \sqrt{x - 1}$, $y = g(x) := (x + 1)/3$, and $L := -1$. Hence the volume by the washer(x) method gives

$$V(B_1) = \pi \int_2^5 \left( (\sqrt{x - 1} - (-1))^2 - \left( \frac{x + 1}{3} - (-1) \right)^2 \right) dx.$$

In part a2] we set $[c, d] := [1, 2]$, $x = F(y) := 3y - 1$, $x = G(y) := y^2 + 1$, and $L := -1$. Hence the volume by the shell(x) method gives

$$V(B_1) = 2\pi \int_1^2 ((3y - 1) - (y^2 + 1))(y - (-1)) dy.$$

b] The body $B_2$ is obtained by revolving the region $A$ around the line $y = 3$.

b1] Set up the integral representing the volume of the body $B_2$ by using the washer method.

In part b1] we set $[a, b] := [2, 5]$, $y = f(x) := \sqrt{x - 1}$, $y = g(x) := (x + 1)/3$ and $L := 3$. Hence the volume by the washer(x) method gives

$$V(B_2) = \pi \int_2^5 \left( 3 - \frac{x + 1}{3} \right)^2 - \left( 3 - \sqrt{x - 1} \right)^2 dx.$$

In part b2] we set $[c, d] := [1, 2]$, $x = F(y) := 3y - 1$, $x = G(y) := y^2 + 1$ and $L := 3$. Hence the volume by the shell(x) method gives

$$V(B_2) = 2\pi \int_1^2 ((3y - 1) - (y^2 + 1))(3 - y) dy.$$
c] The body $B_3$ is obtained by revolving the region $A$ around the line $x = -3$.

c1] Set up the integral representing the volume of the body $B_3$ by using the washer method.

c2] Set up the integral representing the volume of the body $B_3$ by using the shell method.

In part c1] we set $[c, d] := [1, 2]$, $x = F(y) := 3y - 1$, $x = G(y) := y^2 + 1$ and $L := -3$.
Hence the volume by the washer(y) method gives

$$V(B_3) = \pi \int_1^2 \left( ((3y - 1) - (-3))^2 - ((y^2 + 1) - (-3))^2 \right) dy .$$

In part c2] we set $[a, b] := [2, 5]$, $y = f(x) := \sqrt{x - 1}$, $y = g(x) := (x + 1)/3$ and $L := -3$.
Hence the volume by the shell(y) method gives

$$V(B_3) = 2\pi \int_2^5 \left( \sqrt{x - 1} - \frac{x + 1}{3} \right) (x - (-3)) \, dx .$$

d] The body $B_4$ is obtained by revolving the region $A$ around the line $x = 7$.

d1] Set up the integral representing the volume of the body $B_4$ by using the washer method.

d2] Set up the integral representing the volume of the body $B_4$ by using the shell method.

In part d1] we set $[c, d] := [1, 2]$, $x = F(y) := 3y - 1$, $x = G(y) := y^2 + 1$ and $L := 7$.
Hence the volume by the washer(y) method gives

$$V(B_4) = \pi \int_1^2 \left( (7 - (y^2 + 1))^2 - (7 - (3y - 1))^2 \right) dy .$$

In part d2] we set $[a, b] := [2, 5]$, $y = f(x) := \sqrt{x - 1}$, $y = g(x) := (x + 1)/3$ and $L := 7$.
Hence the volume by the shell(y) method gives

$$V(B_4) = 2\pi \int_2^5 \left( \sqrt{x - 1} - \frac{x + 1}{3} \right) (7 - x) \, dx .$$
E.4. Let $A$ be the region bounded by the curves given by the equations $y = x - 4$ and $x = y^2 - 2y$.

a] Let $B_1$ be the body obtained by revolving $A$ around the line $x = -2$. Set up the integral representing the volume of $B_1$. Specify the method you use.

b] Let $B_2$ be the body obtained by revolving $A$ around the line $x = 9$. Set up the integral representing the volume of $B_2$. Specify the method you use.

c] Let $B_3$ be the body obtained by revolving $A$ around the line $y = -3$. Set up the integral representing the volume of $B_3$. Specify the method you use.

d] Let $B_4$ be the body obtained by revolving $A$ around the line $y = 5$. Set up the integral representing the volume of $B_4$. Specify the method you use.

Solutions.

a] By the washer(y) method we have

$$V(B_1) = \pi \int_{-1}^{4} \left( (y + 4) - (y^2 - 2y) \right)^2 - (y^2 - 2y - (-2))^2 \, dy.$$ 

b] By the washer(y) method we have

$$V(B_2) = \pi \int_{-1}^{4} \left( 9 - (y^2 - 2y) \right)^2 - (9 - (y + 4))^2 \, dy.$$ 

c] By the shell(x) method we have

$$V(B_3) = 2\pi \int_{-1}^{4} \left( (y + 4) - (y^2 - 2y) \right)(y - (-3)) \, dy.$$ 

d] By the shell(x) method we have

$$V(B_4) = 2\pi \int_{-1}^{4} \left( (y + 4) - (y^2 - 2y) \right)(5 - y) \, dy.$$
E.5. Let $B_1$ be a ball of radius $r > 0$ and let $B_2$ be a ball of radius $r > 0$ centered at a point on the surface of $B_1$. What is the volume of the intersection $B_1 \cap B_2$?

Without loss of generality we may assume that the center of the ball $B_1$ is at the origin $(0, 0, 0)$ and the center of the ball $B_2$ is at the point $(r, 0, 0)$. Half of the intersection $B_1 \cap B_2$ may be viewed as the solid obtained by revolving the region $A$ bounded by the upper semi-circle $y = \sqrt{r^2 - x^2}$ the $x$-axis and the line $x = r/2$ about the $x$-axis.

Setting up the integral representing the volume of the intersection $B_1 \cap B_2$ by the washer-x method we obtain we set $f(x) := \sqrt{r^2 - x^2}$ and $g(x) = 0$ on the interval $[r/2, r]$, so

$$V = 2\pi \int_{r/2}^{r} \left( \left( \sqrt{r^2 - x^2} \right)^2 - 0^2 \right) \, dx = 2\pi \int_{r/2}^{r} (r^2 - x^2) \, dx$$

$$= 2\pi \left[ x^2 - \frac{x^3}{3} \right]_{r/2}^{r} = 2\pi \left( r^3 - \frac{r^3}{3} \right) - \left( \frac{r^3}{2} - \frac{r^3}{24} \right)$$

$$= \frac{5\pi r^3}{12}.$$

Setting up the integral representing the volume of the intersection of $B_1 \cap B_2$ by the shell-x method we obtain we set $F(y) := \sqrt{r^2 - y^2}$ and $G(y) := r/2$ and on the interval $[0, (\sqrt{3}/2)r]$, so

$$V = 4\pi \int_{0}^{(\sqrt{3}/2)r} \left( \sqrt{r^2 - y^2} - \frac{r}{2} \right) y \, dy$$

$$= 4\pi \int_{0}^{(\sqrt{3}/2)r} \sqrt{r^2 - y^2} y \, dy - 4\pi \int_{0}^{(\sqrt{3}/2)r} \frac{r}{2} y \, dy$$

$$= \frac{7\pi r^3}{6} - \frac{3\pi r^3}{4} = \frac{5\pi r^3}{12},$$

as

$$\int_{0}^{(\sqrt{3}/2)r} \sqrt{r^2 - y^2} y \, dy = \int_{r^2/4}^{r^2/4} u^{1/2}(-1/2) \, du = \left[ \frac{u^{3/2}}{3/2}(-1/2) \right]_{r^2/4}^{r^2/4}$$

$$= \frac{1}{3} \left( - \frac{r^3}{8} + r^3 \right) = \frac{7r^3}{24},$$

and

$$\int_{0}^{(\sqrt{3}/2)r} \frac{r}{2} y \, dy = \left[ \frac{r}{2} \frac{y^2}{2} \right]_{0}^{(\sqrt{3}/2)r} = \frac{3r^3}{16}.$$
E.6. Let $B_1$ and $B_2$ be the cylinders given by the

\[ B_1 := \{(x, y, z) : x^2 + y^2 \leq r^2\} \quad \text{and} \quad B_2 := \{(x, y, z) : x^2 + z^2 \leq r^2\}, \]

where $r > 0$. What is the volume of the intersection $B_1 \cap B_2$?

Let $x_0 \in [-r, r]$. Observe that the intersection of $B_1 \cap B_2$ and the plane $x = x_0$ is a square

\[ A(x_0) = \left\{ (x_0, y, z) : |y| \leq \sqrt{r^2 - x_0^2}, \ |z| \leq \sqrt{r^2 - x_0^2} \right\}, \]

so if $A(x_0)$ denotes the area of the region $A(x_0)$, then

\[ A(x_0) = \left(2\sqrt{r^2 - x_0^2}\right)^2 = 4(r^2 - x_0^2). \]

Hence the volume $V$ of the solid $B_1 \cap B_2$ by the cross section method is

\[ V = \int_{-r}^r A(x) \, dx = \int_{-r}^r 4(r^2 - x^2) \, dx = \left[ 4 \left( x^2 - \frac{x^3}{3} \right) \right]_{-r}^r = \frac{8r^3}{3} - \left( -\frac{8r^3}{3} \right) = \frac{16r^3}{3}. \]