6.4. Work

When a particle (mass point) moves on the $x$-axis we imagine that a force $f(x) := ma(t)$ is acting on the particle when it is at the position $x = s(t)$, where $m$ is the mass at the mass point and $a(t) = s''(t)$ is the acceleration of the particle.

Formula 1. The work $W$ done by a force function $f(x)$ by moving a particle from a point $a$ to a point $b$ on the $x$-axis is

$$W := \int_a^b f(x) \, dx,$$

where $f(x)$ is the force acting on the particle when it is at the point $x \in [a, b]$. Typically we assume that the force function $f(x)$ is continuous on $[a, b]$. It is explained in the book and in my lectures why this is a good formula. The work done by a constant force $F$ by moving a particle from a point $a$ to a point $b$ on the $x$-axis if $W := F(b - a)$. Formula 1 follows from the definition given in this simple case. Indeed, let us study the case when the particle moves from a point $a$ to the point $b$ on the $x$-axis in such a way that at the point $x$ the force $f(x)$ acting on the particle varies with $x$ and $f$ is continuous on $[a, b]$. Let

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b,$$

and let

$$P_n := \{[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\}$$

be the $n$th partition. Let $x_i^* \in [x_{i-1}, x_i]$. Recall that the norm of the $n$th partition $P_n$ is defined by

$$\|P_n\| := \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}.$$

Using the definition given in the case when the force $f$ acting on the particle is constant throughout the motion, the work $W_i$ done by a force function $f(x)$ by moving a particle from a point $x_{i-1}$ to the point $x_i$ is

$$W_i \approx f(x_i^*)(x_i - x_{i-1}).$$

So the the total work $W$ done by a force function $f(x)$ by moving a particle from a point $a$ to the point $b$ is

$$W = \sum_{i=1}^n W_i \approx \sum_{i=1}^n f(x_i^*)(x_i - x_{i-1}).$$

Now we define the work $W$ done by a force function $f(x)$ by moving a particle from a point $a$ to a point $b$ on the $x$-axis as

$$W := \lim_{\|P_n\| \to 0} \sum_{i=1}^n f(x_i^*)(x_i - x_{i-1}) = \int_a^b f(x) \, dx.$$

When we use SI units we measure mass in terms of kilograms (kg), distance in terms of meters (m), time in terms of seconds (s), acceleration in terms of m/s$^2$, force in terms of Newton = N = kg m/s$^2$, work in Joule = J = Nm = kg m$^2$/s$^2$.

When we use British units we measure distance in terms of feet (ft), time in terms of seconds (s), force in terms of pounds (lb), and work in terms of foot-pound (ft-lb).

We discuss two special cases of Formula 1.
Hooke’s Law

Hooke’s Law tells us that the force required to maintain a spring (ideal spring) stretched \( x \) units beyond its original length is a linear function of \( x \), that is, \( f(x) = kx \), where \( k \) depends only on the material of the spring. So when we stretch a spring from the position when it is stretched \( a \) units beyond its natural length to the position when it is stretched \( b \) units beyond its natural length, the work done is given by the formula

\[
W = \int_a^b kx \, dx.
\]

E.1. A force 40 N is required to hold a spring that has been stretched from its natural length of 10 cm to 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

Solution. By Hooke’s Law the force required to hold the spring \( x \) m beyond its natural length is given by the formula \( f(x) = kx \). We have \( f(0.05) = k(0.05) = 40 \), so \( k = 40/0.05 = 800 \). Hence \( f(x) = 800x \). So the work done in stretching the spring from 15 cm to 18 cm is

\[
W = \int_{0.05}^{0.08} 800x \, dx = \left[ 800 \frac{x^2}{2} \right]_{0.05}^{0.08} = 1.56 \text{ J}.
\]

E.2. 5 J work is done by stretching a spring from its natural length of 10 cm to 15 cm. How much work is done in stretching the spring from 15 cm to 18 cm?

Solution. By Hooke’s Law the force required to hold the spring \( x \) m beyond its natural length is given by the formula \( f(x) = kx \). We have

\[
5 = \int_0^{0.05} kx \, dx = \left[ \frac{kx^2}{2} \right]_0^{0.05} = \frac{k(25 - 0)}{20000} = \frac{25k}{20000},
\]

hence \( k = 4000 \) and \( f(x) = 4000x \). So the work done in stretching the spring from 15 cm to 18 cm is

\[
W = \int_{0.05}^{0.08} 4000x \, dx = \left[ 4000 \frac{x^2}{2} \right]_{0.05}^{0.08} = 7.8 \text{ J}.
\]
Cable is used to lift a weight

A cable with weight density $\delta$ lb/ft is used to lift $L$ lb of weight from level $z = a$ to level $z = b$ (that is, from $-a$ ft below the ground level to $-b$ ft below the ground level). The work done is given by the formula

$$W = \int_a^b (L + \delta(0 - z)) \, dz = \left[ Lz - \delta \frac{z^2}{2} \right]_a^b.$$  

This formula is a special case of Formula 1 as we imagine the weight $L$ as a mass point at the end of the cable, so the force $f(z)$ acting on the mass point at the $z$ level required to overcome the force due to gravity is $(L + \delta(0 - z))$ lb.

Analogously, if we use SI units and a cable with mass density $\rho$ kg/m is used to lift $L$ kg of weight from level $z = a$ to level $z = b$ (that is, from $-a$ ft below the ground level to $-b$ ft below the ground level). The work done is given by the formula

$$W = \int_a^b (Lg + \rho g(0 - z)) \, dz = g \int_a^b (L + \rho(0 - z)) \, dz = g \left[ Lz - \rho \frac{z^2}{2} \right]_a^b,$$

where $g = 9.81$ m/s$^2$ is the gravitational acceleration.

E.1. A cable with weight density 2 lb/ft is used to lift 800 lb coal up a mine-shaft 500 ft deep.

a] Find the work done by lifting the coal to the top.
b] Find the work done by lifting the first 200 ft of the cable to the top.

Solution to a]. The work $W$ done is

$$W = \int_{-500}^0 (800 + 2(0 - z)) \, dz = \left[ 800z - 2 \frac{z^2}{2} \right]_0^{-500} = 650000 \text{ ft-lb}.$$

Solution to b]. The work $W$ done is

$$W = \int_{-500}^{-300} (800 + 2(0 - z)) \, dz = \left[ 800z - 2 \frac{z^2}{2} \right]_{-500}^{-300} = 320000 \text{ ft-lb}.$$

3
Cable (or rope) is used to lift a weight (a similar consideration)

A cable (or rope) with weight density \( \delta \) lb/ft hangs from the top of a building of height \( h \) and it is used to lift \( L \) lb of weight from level \( z = a \) to level \( z = b \) (that is, from \( a \) ft above the ground level to \( b \) ft above the ground level). The work done is given by the formula

\[
W = \int_a^b (L + \delta(h - z)) \, dz = \left[ Lz - \delta \frac{(h - z)^2}{2} \right]^b_a.
\]

This formula is a special case of Formula 1 as we imagine the weight \( L \) as a mass point at the end of the cable, so force \( f(z) \) acting on the mass point at the \( z \) level required to overcome the force due to gravity is \( L + \delta(0 - z) \) lb.

Analogously, if we use SI units and a cable with mass density \( \rho \) kg/m is used to lift \( L \) kg of weight from level \( z = a \) to level \( z = b \) (that is, from \( a \) ft above the ground level to \( b \) ft above the ground level). The work done is given by the formula

\[
W = g \int_a^b (L + \rho(h - z)) \, dz = g \left[ Lz - \rho \frac{(h - z)^2}{2} \right]^b_a.
\]

E.1. A heavy rope, 50 ft long, weights 20 lb and hangs over the edge of a building 120 ft high.

a] How much work is done in pulling the rope to the top of the building?
b] How much work is done in pulling half the rope to the top of the building?

Solution to a]. Observe that \( L = 0 \) the weight density of the rope is \( \delta = 20/50 = 2/5 \) lb/ft. The work \( W \) done is

\[
\int_{70}^{120} \delta(120 - z) \, dz = \int_{70}^{120} \frac{2}{5} (120 - z) \, dz = \int_{50}^{0} \frac{2}{5} u (-1) \, du = \int_{0}^{50} \frac{2}{5} u \, du = \left[ \frac{u^2}{5} \right]_0^{50} = 500 \text{ ft-lb}.
\]

Solution to b]. Observe that \( L = 0 \) the weight density of the rope is \( \delta = 20/50 = 2/5 \) lb/ft. The work \( W' \) done is

\[
\int_{70}^{95} \delta(120 - z) \, dz = \int_{70}^{95} \frac{2}{5} (120 - z) \, dz = \int_{50}^{25} \frac{2}{5} u (-1) \, du = \int_{25}^{50} \frac{2}{5} u \, du = \left[ \frac{u^2}{5} \right]_{25}^{50} = \frac{1}{5} (500 - 125) = 375 \text{ ft-lb}.
\]
WORK DONE BY PUMPING THE LIQUID OUT OF A TANK

Let $B$ be a tank (partially) filled with some liquid with mass density $\rho(\text{kg/m}^3)$ or weight density $\delta(\text{lb/ft}^3)$. Let $a$ denote the smallest $z$ coordinate of a point in $B$, and let $b$ denote the largest $z$ coordinate of a point in $B$. Let $A(z_0)$ denote the intersection of the tank $B$ and the plane $z = z_0$ for any $z_0 \in [a, b]$. Suppose that there is an outlet at the $z = c$ level, where $c \geq b$. Suppose that the tank $B$ is partially filled up to the $z = d$ level, where $a \leq d \leq b$. Then the work $W$ in terms of ft lb done by pumping the liquid out of the outlet is defined by

$$W := \delta \int_{a}^{d} (c - z) A(z) \, dz,$$

if we use British units. When the liquid is water we have $\delta = 62.37(\text{lb/ft}^3)$.

Analogously, the work $W$ in terms of Joule = $J = \text{Nm} = \text{kg m}^2/\text{s}^2$ done by pumping the liquid out of the outlet is defined by

$$W := g \rho \int_{a}^{d} (c - z) A(z) \, dz,$$

if we use SI units. Here $g = 9.81\text{m/s}^2$ is the gravitational acceleration. When the liquid is water $\rho = 1000\text{kg/m}^3$. 
Why are these formulas correct? We justify the formula by using SI units. Let
\[ a = z_0 < z_1 < z_2 < \ldots < z_{n-1} < z_n = d \]
and let
\[ P_n = \{ [z_0, z_1], [z_1, z_2], \ldots , [z_{n-1}, z_n] \} \]
be the \( n \)th partition of the interval \([a, b]\) on the \( z \)-axis. Recall that the norm of the \( n \)th partition \( P_n \) is defined by
\[ \| P_n \| := \max \{ z_1 - z_0, z_2 - z_1, \ldots , z_n - z_{n-1} \} . \]
Let \( z_i^* = z_{i-1} \) for \( i = 1, 2, \ldots , n \).

We call the portion of the tank between the planes \( z = z_{i-1} \) and \( z = z_i \) the \( i \)th layer.

The volume \( V_i \) of the liquid in the \( i \)th layer is
\[ V_i \approx A(z_i^*)(z_i - z_{i-1}) . \]

The mass \( m_i \) in the \( i \)th layer is
\[ m_i \approx \rho A(z_i^*)(z_i - z_{i-1}) . \]

The force \( F_i \) required to raise the \( i \)th layer must overcome the force due to gravity, so
\[ F_i = gm_i \approx g\rho A(z_i^*)(z_i - z_{i-1}) . \]

Each particle in the \( i \)th layer must travel a distance \( \approx c - z_i^* \).

The work \( W_i \) done by raising the \( i \)th layer to the outlet level is
\[ W_i \approx g\rho A(z_i^*)(z_i - z_{i-1})(c - z_i^*) = g\rho A(z_i^*)(c - z_i^*)(z_i - z_{i-1}) . \]

The total work to pump the water out is approximately
\[ W \approx \sum_{i=1}^{n} g\rho A(z_i^*)(c - z_i^*)(z_i - z_{i-1}) . \]

Hence the work \( W \) in terms of \( J(= \text{kg m}^2/\text{s}^2) \) done by pumping the liquid out of the outlet is defined by
\[ W := \lim_{\| P_n \| \to 0} \sum_{i=1}^{n} g\rho A(z_i^*)(c - z_i^*)(z_i - z_{i-1}) = \rho g \int_a^d (c - z)A(z) \, dz , \]
assuming that the area of cross-section function \( A(z) \) is continuous on the interval \([a, d]\).
E.1. A tank has the shape of an inverted circular cone with height 10 m and base radius 4 m. It is filled with water up to the height 8 m. Find the work required to empty the tank by pumping all the water out of the outlet on the top.

Solution. We set $a = -10$, $d = -2$, $c = 0$. We have

$$W = g\rho \int_{-10}^{-2} (0 - z)A(z)\,dz.$$ 

Observe that the intersection of the tank and the plane $z = z_0$ is a disk of radius $r(z_0)$, where the theorem of similar triangles gives that

$$\frac{r(z_0)}{4} = \frac{z_0 - (-10)}{10}, \text{ hence } r(z_0) = \frac{4}{10}(z_0 + 10).$$

It follows that

$$A(z_0) = r(z_0)^2\pi = \left(\frac{4}{10}(z_0 + 10)\right)^2 \pi.$$ 

Putting this in the formula for $W$ we get

$$W = \rho g\pi \int_{-10}^{-2} (-z)\left(\frac{4}{10}(z + 10)\right)^2 \,dz = -g\rho\pi \frac{16}{100} \int_{-10}^{-2} (z^3 + 20z^2 + 100z) \,dz$$

$$= -9810 \frac{16}{100} \pi \left[ \frac{z^4}{4} + 20 \frac{z^3}{3} + 100 \frac{z^2}{2} \right]_{-10}^{-2} \approx 3.4 \cdot 10^6 (\text{J}).$$

We note that another way to set up the integral representing the work could be

$$W = g\rho \pi \int_0^8 (10 - z)\left(\frac{4z}{10}\right)^2 \,dz.$$
E.2. A tank has the shape of the lower half of a ball of radius 5 m centered at the origin. It is filled with water up to the height 3 m from the bottom. Find the work required to empty the tank by pumping all the water out of the outlet which is 1 m above the top of the tank.

Solution. We set \( a = -5, \ d = -2, \ c = 1 \). We have

\[
W = g\rho \int_{-2}^{-5} (1-z)A(z) \, dz.
\]

Observe that the intersection of the tank and the plane \( z = z_0 \) is a disk of radius \( r(z_0) \), where a simple geometrical consideration gives that

\[
r(z_0)^2 + |z_0|^2 = 5^2 \quad \text{that is} \quad r(z_0)^2 = 25 - z_0^2.
\]

It follows that

\[
A(z_0) = r(z_0)^2\pi = (25 - z_0^2)\pi.
\]

Putting this in the formula for \( W \) we get

\[
W = \rho g\pi \int_{-2}^{-5} (1-z)(25 - z^2) \, dz = \rho g\pi \int_{-2}^{-5} (z^3 - z^2 - 25z + 25) \, dz
\]
\[
= 9810\pi \left[ \frac{z^4}{4} - \frac{z^3}{3} - 25\frac{z^2}{2} + 25z \right]_{-5}^{-2} = 5275082.25 \text{(J)}.
\]

We note that another way to set up the integral representing the work could be

\[
W = \rho g\pi \int_{0}^{3} (6-z)(25 - (5-z)^2) \, dz = \rho g\pi \int_{0}^{3} (6-z)(10z - z^2) \, dz.
\]
E.3. Let $A$ be the region bounded by lines $x = 0$ and $z = 18$, and the curve $2z = x^2$ in the $xz$-plane, the length is measured in terms of meters. The tank $B$ is obtained by revolving $A$ around the $z$-axis, and an outlet is 2 m above the top of the tank. The tank is partially filled with liquid with mass density $\rho = 600$ (kg/m$^3$) up to the height 5 m from the bottom. Set up an integral representing the work done by pumping the liquid out through the outlet.

Solution. We set $a = 0$, $d = 5$, $c = 18 + 2 = 20$. We have

$$W = g\rho \int_0^5 (20 - z)A(z) \, dz.$$  

Observe that the intersection of the tank and the plane $z = z_0$ is a disk of radius $r(z_0)$, where

$$r(z_0) = \sqrt{2z_0}.$$  

It follows that

$$A(z_0) = r(z_0)^2 \pi = 2z_0 \pi.$$  

Putting this in the formula for $W$ we get

$$W = \rho g \int_0^5 (20 - z)(2z)\pi \, dz = (600) \cdot (9.81)\pi \int_0^5 (-2z^2 + 40z) \, dz$$

$$= 5886\pi \int_0^5 (-2z^2 + 40z) \, dz = 5886\pi \left[ \frac{-2z^3}{3} + 20z^2 \right]_0^5 = 7704755.983 (J).$$