Review Problems before Exam 3

Section 11.4 Comparison Tests

E.1. Does $\displaystyle\sum_{n=9}^{\infty} \frac{\ln \ln n}{n}$ converge?

E.2. Does $\displaystyle\sum_{n=3}^{\infty} \frac{n^2+3n}{\sqrt[3]{n^{10}-4n^8}}$ converge?

E.3. Does $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{3}{n}\right)$ converge?

E.4. Does $\displaystyle\sum_{n=1}^{\infty} \frac{5^n+3^n}{6^n-4^n}$ converge?

E.5. Does $\displaystyle\sum_{n=1}^{\infty} (\pi/2 - \arctan n)$ converge?

E.6. Does $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+3n^2}}{\sqrt[3]{n^8+4n^6}}$ converge?

E.7. Does $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+3n^2}}{\sqrt[3]{n^7+6n^5}}$ converge?

E.8. Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^{\sqrt{n}}}$ converge?

E.9. Does $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^{5/4}}$ converge?

E.10. Does $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(3n)}{\sqrt[5]{n^4+n^2}}$ converge?

E.11. Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ converge?

E.12. For what values of p does $\displaystyle\sum_{n=1}^{\infty} \frac{n^4+n}{n^p+n^4}$ converge?

Section 11.5. The Alternating Series Test

E.1. Observe that $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ converges by the AST.

E.2*. We showed that in fact $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \ln 2$, but this is beyond possible exam 3 questions.

E.3. Observe that $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$ converges by the AST.

E.4. Observe that $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2^n + n^2}$ converges to a number s by the AST. How many terms do we need to add to guarantee an error less than 10^{-3} by the EE in the AST?

E.5. Can you apply the AST to the series $\sum_{n=1}^{\infty} (-1)^{n-1} \ln n$?

E.6. Can you apply the AST to the series $\sum_{n=1}^{\infty} (-1)^{n-1} (\ln(3n+2) - \ln(2n-1))$?

E.7. Can you apply the AST to the series $\sum_{n=1}^{\infty} (-1)^{n-1} (\ln(n+1) - \ln n)$?

Section 11.6. Absolute Convergence Test and Ratio Test

Absolute Convergence Test

E.1. Observe that the harmonic series with alternating signs, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is convergent by the AST, but it is not absolutely convergent.

E.2. Show that $\sum_{n=1}^{\infty} \frac{\cos(n\pi/200)}{n\sqrt{n}}$ converges.

E.3. Show that $\sum_{n=3}^{\infty} (-1)^{n-1} \frac{1}{n \ln n}$ is convergent by the AST, but it is not absolutely convergent.

E.4. Show that $\sum_{n=3}^{\infty} (-1)^{n-1} \frac{1}{\ln n}$ is convergent by the AST, but it is not absolutely convergent.

E.5. Show that $\sum_{n=3}^{\infty} (-1)^{n-1} \frac{\cos(n^3)}{n(\ln n)^2}$ is absolutely convergent, so it is convergent.

Ratio Test

E.1. Observe that the Ratio Test is inconclusive in the study of $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

E.2. Does $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$ converge?

E.3. Does $\sum_{n=1}^{\infty} \frac{(-2)^n n!}{n^n}$ converge?

E.4. Does $\sum_{n=1}^{\infty} \frac{(-3)^n n!}{n^n}$ converge?

E.5*. In E.3 and E.4 we have used

$$\lim_{n\to\infty} (1+1/n)^n = e = 2.71\ldots. \tag{1}$$

Please, do not forget it!

E.6. Does $\sum_{n=1}^{\infty} (-3)^n \frac{n!n!}{(2n)!}$ converge?

E.7. Does $\sum_{n=1}^{\infty} (-5)^n \frac{n!n!}{(2n)!}$ converge?

Review of the Tests to decide about convergence or divergence of a series

1. D (Definition)
2. GST (Geometric Series Theorem)
3. TF (Telescoping Feature)
4. TD (Test for Divergence)
5. IT (Integral Test) with EE in IT (Error Estimate in the Integral Test)
6. p-t (p-test)
7. CT (Comparison Test)
8. LCT (Limit Comparison Test)
9. AST (Alternating Series Test) with EE in AST (Error Estimate in the Alternating Series Test)
10. ACT (Absolute Convergence Test)
11. RT (Ratio Test)

11.8 Power Series

E.1. Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n n!}{n^n}\, x^n$.

E.2. Given $\displaystyle\sum_{n=0}^{\infty} x^n$, find the set (interval) of convergence.

E.3. Given $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(-2)^n}\, x^{3n}$, find the set (interval) of convergence.

E.4. Given $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^{n^2}}{(-3)^n n^3}$, find the set (interval) of convergence.

E.5. Given $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{3n} x^{5n+2}}{(n!)^2}$, find the set (interval) of convergence.

E.6. Given $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n (x-5)^n}{n+1}$, find the set (interval) of convergence.

E.7. Suppose that $\displaystyle\sum_{n=0}^{\infty} c_n (x+5)^n$ converges at $x = 6$ and diverges at $x = 8$. What can we deduce from these pieces of information in the light of the Interval of Convergence Theorem?

E.8. Given $\displaystyle\sum_{n=1}^{\infty} \frac{(x+5)^n}{3^n \ln(n+1)}$, find the set (interval) of convergence.

Section 11.9. Representation of Functions as Power Series

Recall that the GST (Geometric Series Theorem) states that

$$\sum_{n=0}^{\infty} q^n = \sum_{n=1}^{\infty} q^{n-1} = 1 + q + q^2 + \cdots + q^n + \cdots = \frac{1}{1-q}\,, \qquad q \in (-1,1)\,.$$

We proved it in Section 11.2, E.1 in Section 11.8 states that the interval $(-1,1)$ is the interval of convergence for the geometric series as it diverges at every real number outside the interval $(-1,1)$. So the starting observation in this section is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\,, \qquad x \in (-1,1)\,. \tag{1}$$

Replacing x by $-x$ in (1) we get

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n\,, \qquad x \in (-1,1)\,. \tag{2}$$

We can also deduce easily that if $a > 0$ then

$$\frac{1}{a-x} = \frac{1}{a(1-(x/a))} = \frac{1}{a}\sum_{n=0}^{\infty}(x/a)^n = \sum_{n=0}^{\infty}\frac{1}{a^{n+1}}\,x^n\,, \qquad x \in (-a,a)\,, \tag{3}$$

as the power series in (3) converges when $|x/a| < 1$, that is, $x \in (-a,a)$. Note also that it x is outside the interval $(-a,a)$, then the power series diverges.
Replacing x by $-x$ in (3) we get

$$\frac{1}{a+x} = \frac{1}{a-(-x)} = \sum_{n=0}^{\infty}\frac{1}{a^{n+1}}\,(-x)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{a^{n+1}}\,x^n \qquad x \in (-a,a)\,. \tag{4}$$

Note also that it x is outside the interval $(-a,a)$, then the power series diverges.

E.1. Differentiating the geometric series (2) we get

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}\,, \qquad x \in (-1,1)\,. \tag{5}$$

and 1 remains the radius of convergence in (5). Note that the summation starts from $n = 1$ as the derivative of the constant term in (2) is 0. Note also that the indices may be shifted to get

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n\,, \qquad x \in (-1,1)\,. \tag{5}$$

E.2. Differentiating the geometric series (5) we get

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}\,, \qquad x \in (-1,1)\,. \tag{6}$$

and 1 remains the radius of convergence in (5). Note that the summation starts from $n = 2$ as the derivative of the constant term in (5) is 0. Note also that the indices may be shifted to get

$$\frac{2}{(1-x)^3} = \sum_{n=0}^{\infty}(n+2)(n+1)x^n\,, \qquad x \in (-1,1)\,. \tag{6}$$

E.3. Combining the results in (5) and (6) we get

$$\sum_{n=1}^{\infty} n^2x^n = \sum_{n=1}^{\infty}(n(n-1)+n)x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}\,, \qquad x \in (-1,1)\,. \tag{7}$$

E.4. The special case $x = 1/2$ in (5) gives

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} nx^n \bigg|_{x=1/2} = \frac{x}{(1-x)^2}\bigg|_{x=1/2} = \frac{1/2}{(1/2)^2} = 2\,.$$

E.5. The special case $x = -1/2$ in (5) gives

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n} = \sum_{n=1}^{\infty} nx^n \bigg|_{x=-1/2} = \frac{x}{(1-x)^2}\bigg|_{x=-1/2} = \frac{-1/2}{(3/2)^2} = \frac{-2}{9}\,.$$

E.6. The special case $x = 1/3$ in (7) gives

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n^2}{3^n} &= \sum_{n=1}^{\infty} \frac{n^2}{3^n}\bigg|_{x=1/3} = \left(\frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}\right)\bigg|_{x=1/3} \\ &= \frac{1/3}{4/9} + \frac{2/9}{8/27} = \frac{3}{4} + \frac{6}{8} = \frac{3}{2}\,.\end{aligned}$$

E.7. The special case $x = -1/3$ in (7) gives

$$\begin{aligned}\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{3^n} &= \sum_{n=1}^{\infty} \frac{n^2}{3^n}\bigg|_{x=-1/3} = \left(\frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2}\right)\bigg|_{x=-1/3} \\ &= \frac{-1/3}{(4/3)^2} + \frac{2/9}{(4/3)^3} = \frac{-3}{16} + \frac{6}{64} = \frac{-3}{32}\,.\end{aligned}$$

E.8. Integrating the geometric series (1) we get

$$-\ln|1-x| = -\ln(1-x) = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\,, \qquad x \in (-1,1)\,, \tag{8}$$

and $R := 1$ remains the radius of convergence in (8). Putting $x = 0$ in (8) we get $0 = C + 0$ and hence we have $C = 0$ in (8), that is,

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}\,, \qquad x \in (-1,1)\,. \tag{9}$$

Note also that the indices may be shifted to get

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}\,, \qquad x \in (-1,1)\,. \tag{10}$$

If we replace x by $-x$ in (10) we get

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad x \in (-1,1), \tag{11}$$

In fact, in advanced calculus it is proved that the power series always continuous on its interval of convergence, hence if follows from the AST and the continuity of the $\ln(1+x)$ function at 1 that plugging $x = 1$ in the power series of $\ln(1+x)$ we get the value $\ln(1+1) = \ln 2$. That is,

$$\ln 2 = \ln(1+1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots.$$

and

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \qquad x \in (-1,1]. \tag{12}$$

E.9. The special case $x = 1/2$ in (10) gives

$$-\ln 2 = \ln \frac{1}{2} = \ln\left(1 - \frac{1}{2}\right) = -\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{2}\right)^n = -\sum_{n=1}^{\infty} \frac{1}{n2^n},$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = \ln 2.$$

E.10. The special case $x = 1/2$ in (11) gives

$$\ln 3 - \ln 2 = \ln \frac{3}{2} = \ln\left(1 + \frac{1}{2}\right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \left(\frac{1}{2}\right)^n = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n},$$

and hence

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n} = \ln 3 - \ln 2.$$

E.11. Observe that replacing x by x^2 in the geometric series (2) we get

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \qquad x \in (-1,1). \tag{13}$$

Integrating (13) we get

$$\arctan x = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\,, \qquad x \in (-1,1)\,.$$

Plugging $x = 0$ in the formula above we get $0 = \arctan 0 = C + 0$, so $C = 0$. Hence

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\,, \qquad x \in (-1,1)\,. \tag{14}$$

Shifting the indices we may rewrite this as

$$\arctan x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\,, \qquad x \in (-1,1)\,. \tag{14}$$

In fact, in advanced calculus it is proved that the power series always continuous on its interval of convergence, hence if follows from the AST and the continuity of the $\arctan x$ function at 1 that plugging $x = 1$ in the power series of $\arctan x$ we get the value $\arctan 1 = \pi/4$. That is,

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n-1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\,.$$

Similarly,

$$-\frac{\pi}{4} = \arctan(-1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{-1}{2n-1} = \frac{-1}{1} - \frac{-1}{3} + \frac{-1}{5} - \frac{-1}{7} + \frac{-1}{9} - \cdots\,,$$

and hence

$$\arctan x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\,, \qquad x \in [-1,1]\,. \tag{15}$$

E.12. Observe that the power series representation of the $\arctan x$ function is an alternating series satisfying the assumptions of the AST for every $x \in [0,1]$ with the notation

$$b_n := \frac{x^{2n-1}}{2n-1} \geq 0\,, \qquad n = 1, 2, \ldots\,.$$

Let

$$s_n = \sum_{k=1}^{n} (-1)^{k-1} \frac{x^{2k-1}}{2k-1} = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1}\,,$$

be the nth partial sum of the power series in (14). Using the EE inn the AST (Error Estimate in the Alternating Series Test, we get

$$
\begin{aligned}
|\arctan x - s_n| &= \left|\arctan x - \left(\frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1}\right)\right| \\
&\le b_{n+1} = \frac{x^{2n+1}}{2n+1}\,, \qquad x \in [0,1]\,.
\end{aligned}
$$

As a special case

$$
\left|\arctan\left(\frac{1}{10}\right) - \left(\frac{1}{10} - \frac{1}{3}\frac{1}{10^3} + \frac{1}{5}\frac{1}{10^5} - \frac{1}{7}\frac{1}{10^7}\right)\right| \le \frac{1}{9}\frac{1}{10^9}\,.
$$

Hence to approximate arctan(1/10) we have achieved a high accuracy by adding only the first four (non-zero) terms of the power series representation.

E.13. Find $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{3^k(2k+1)}$.

E.14. Use the term by term Integration of Power Series Theorem to represent

$$
\int x^2 \ln(32 + x^5)\, dx
$$

as a power series. What is its radius of convergence? Write $\displaystyle\int_0^1 x^2 \ln(32 + x^5)\, dx$ as an infinite series.

E.15. Use the term by term Integration of Power Series Theorem to represent

$$
\int \frac{x^7}{8x^3 + 27}\, dx
$$

as a power series. What is its radius of convergence? Write $\displaystyle\int_0^{1/2} \frac{x^5}{8x^3 + 27}\, dx$ as an infinite series.

E.16. Use the term by term Integration of Power Series Theorem to represent

$$
\int \frac{\arctan(x^3/27)}{x^2}\, dx
$$

as a power series. What is its radius of convergence? Write $\displaystyle\int_1^2 \frac{\arctan(x^3/27)}{x^2}\, dx$ as an infinite series.

Theorem (Coefficient Formula). *If f has a power series representation on an open interval centered at, that is,*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \,, \qquad |x-a| < R \,, \tag{1}$$

then its coefficients can be found by the formula

$$c_n = \frac{f^{(n)}(a)}{n!} \,, \qquad n = 0, 1, 2, \ldots \,. \tag{2}$$

Definition (Taylor and Maclauren Series). If f is a function such that the higher derivatives $f^{(n)}(x)\,,\ n = 0, 1, 2, \ldots$ exist whenever $|x-a| < R$ with some $R > 0$, then we define the so-called Taylor series (expansion) of f centered at a by

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \,, \tag{3}$$

that is,

$$f(x) \sim f^{(0)}(a) + \frac{f^{(1)}(a)}{1!}(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots \,.$$

A Taylor series expansion centered at 0 is often called Maclauren series.

E.1. If $f(x) = e^x$, then the Taylor series expansion of f centered at 0 is

$$e^x \sim \sum_{k=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \,. \tag{4}$$

E.2. If $f(x) = \sin x$, then the Taylor series expansion of f centered at 0 is

$$\sin x \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \,. \tag{5}$$

E.3. If $f(x) = \cos x$, then the Taylor series expansion of f centered at 0 is

$$\cos x \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \,. \tag{6}$$

Theorem. *We have*

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots, \qquad x \in (-\infty, \infty), \tag{7}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \qquad x \in (-\infty, \infty), \tag{8}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \qquad x \in (-\infty, \infty). \tag{9}$$

That is, for the functions $f(x) = e^x$, $f(x) = \sin x$, or $f(x) = \cos x$ the Taylor series expansion of $f(x)$ centered at 0 converges back to the function $f(x)$ for vevery $x \in (-\infty, \infty)$.

E.4. Let $f(x) = e^x$. Write down the 5th Taylor polynomial of $f(x)$ centered at 0.

$$T_5(x) = \sum_{k=0}^{5} \frac{x^n}{n!} = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}.$$

E.5. Let $f(x) = \sin x$. Write down the 5th and 6th Taylor polynomials of $f(x)$ centered at 0.

$$T_5(x) = T_6(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

E.6. Let $f(x) = \cos x$. Write down the 6th and 7th Taylor polynomials of $f(x)$ centered at 0.

$$T_6(x) = T_7(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

E.7. Find

$$\sum_{n=2}^{\infty} \frac{1}{n!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots.$$

E.8. Find

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!}.$$

E.9. Plug $x = \pi/6$, $x = \pi/4$, $x = \pi/3$, $x = \pi/2$, $x = \pi, \ldots,$ in the Taylor series expansions of $\sin x$ and $\cos x$ and use the Theorem to find the exact value of the infinite sum.

E.10. Show that (9) follows from (8) by using the term by term Differentiation of a Power Series Theorem (Section 10.9).

E.11. Show that

$$e^x = \sum_{n=0}^{\infty} \frac{e^3(x-3)^n}{n!} = 1 + \frac{e^3(x-3)^1}{1!} + \frac{e^3(x-3)^2}{2!} + \cdots + \frac{e^3(x-3)^n}{n!} + \cdots$$

for every $x \in (-\infty, \infty)$.

E.12. Let $f(x) = x^3 \sin(2x^2)$. Write down the 9th and 13th Taylor polynomials T_9 and T_{13} of f centered at 0 and compute $T_9(1)$ and $T_{13}(1)$.

E.13. Let $f(x) = x^2 \arctan x$. Find $f^{(9)}(0)$ and $f^{(10)}(0)$.

E.14. Let $f(x) = x^2 \sin(x^3)$. Find $f^{(71)}(0)$ and $f^{(75)}(0)$.

E.15. Let $f(x) = x^3 e^{-x^2}$. Find $f^{(73)}(0)$ and $f^{(74)}(0)$.

E.16. Let $f(x) = \ln x$. Find the Taylor series expansion of $f(x)$ centered at 3. Find its radius of convergence and show that it converges back to the function on the interval $(0, 6]$.

E.17. Find

$$\lim_{x \to 0} \frac{x - \frac{x^3}{6} - \sin x}{x^5}.$$

E.18. Find

$$\lim_{x \to 0} \frac{x^4 - \frac{x^{12}}{6} - \sin(x^4)}{x^{20}}.$$

E.19. In Section 11.9 we used the term by term Differentiation Theorem by differentiating the geometric series that

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n, \qquad x \in (-1, 1).$$

Now use the Product of Power Series Theorem to obtain this formula.

E.20. Find the Taylor series expansion of the function $f(x) = \dfrac{1}{x^2 - 3x + 2}$ centered at 0.

E.22. Find the Taylor series expansion of $f(x) := \dfrac{x^5}{(8 + x^3)^2}$ centered at 0.

E.23. Let $f(x) = \dfrac{x^5}{(8 + x^3)^2}$ centered at 0. Find $f^{(38)}(0)$.

E.24. Find the Taylor series expansion of $g(x) = \displaystyle\int \frac{x^5}{(8 + x^3)^2}\, dx$ centered at 0.

E.25. Find the Taylor series expansion of $f(x) = x^3 \ln(9 + x^2)$ centered at 0.

E.26. Let $f(x) = x^3 \ln(9 + x^2)$. Find $f^{(41)}(0)$.

E.27. Find the Taylor series expansion of $g(x) = \int x^3 \ln(9 + x^2)\, dx$ centered at 0.

E.28. Find the Taylor series expansion of the function $f(x) = (1+x)^k$ centered at 0, where k is a real number.

E.29. If k is not a nonnegative integer, then we can examine where the Taylor series expansion of the function $f(x) = (1 + x)^k$ centered at 0 converges back to the function. Show by the Ratio Test that if k is not a nonnegative integer, then the radius of convergence of the power series at the right-hand side of (10) is 1.

E.30. Find the Taylor series expansion of $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$ centered at 0.

E.31. Let $f(x) = \dfrac{1}{3x + 8}$.

(a) Find the Taylor series expansion of $f(x)$ centered at -2.

(b) What is the largest interval on which the Taylor series expansion of $f(x)$ centered at -2 converges back to the function $f(x)$. Justify your answer as clearly as you can.

Hint. Observe that

$$\frac{1}{3x + 8} = \frac{1}{3(x + 2) + 2} = \frac{1}{2} \frac{1}{1 - \dfrac{-3(x + 2)}{2}} .$$

E.32. Let $f(x) = (3x + 1)^{-2}$.

(a) Find the Taylor series expansion associated with the function f centered at 2.

(b) What is the largest interval on which the Taylor series expansion associated with f converges back to the function. Justify your answer as clearly as you can.

Hint. Observe that

$$f(x) = \frac{1}{49} \left(1 + \frac{3(x - 2)}{7}\right)^{-2} .$$

E.33. Let $f(x) = \ln(7x + 15)$.

(a) Find the Taylor series expansion associated with the function f centered at -2.

(b) What is the largest interval on which the Taylor series expansion associated with f converges back to the function f. Justify your answer as clearly as you can.

Hint. Observe that $f(x) = \ln(1 + 7(x + 2))$.

E.34. Find the exact value of $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{2^n n^2}{5^n}$.

Hint. First find

$$\sum_{n=0}^{\infty} n^2 x^n = \sum_{n=0}^{\infty} n(n - 1)x^n + \sum_{n=0}^{\infty} nx^n , \qquad x \in (-1, 1) .$$