Commutative Algebra Problems
From Atiyah & McDonald

Thomas Yahl

Chapter 1

1. Let A be a ring and $x \in A$ be nilpotent. Let $n > 0$ be such that $x^n = 0$ and notice

$$(1+x)\left(\sum_{i=0}^{n-1}(-1)^i x^i\right) = 1 + x^n = 1.$$

Therefore, $1+x$ is a unit.

For any unit $a \in A$, we may write $a + x = a(1 + a^{-1}x)$ where a is a unit and $1 + a^{-1}x$ is a unit since $a^{-1}x$ is nilpotent ($(a^{-1}x)^n = a^{-n}x^n = 0$). Since the product of units is a unit, $a + x$ is then a unit for any unit $a \in A$.

2a. Let A be a ring and

$$f(x) = \sum_{i=0}^{n} a_i x^i \in A[x].$$

Assume a_0 is a unit and a_i is nilpotent for $i \geq 1$. Since $a_i x^i$ is nilpotent and the sum of nilpotents is nilpotent, $\sum_{i=1}^{n} a_i x^i$ is nilpotent. Therefore,

$$f(x) = a_0 + \sum_{i=1}^{n} a_i x^i$$

is the sum of a unit and a nilpotent. Therefore, $f(x)$ is a unit.

Assume now that $f(x)$ is a unit. Let

$$g(x) = \sum_{j=0}^{m} b_j x^j \in A[x]$$

be such that $f(x)g(x) = 1$. We must have

$$1 = \left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{m} b_j x^j\right) = \sum_{k=0}^{n+m} c_k x^k,\ c_k = \sum_{i+j=k} a_i b_j.$$

Immediately from this, $a_0 b_0 = c_0 = 1$ so a_0 is a unit. Now it will be shown that when $f(x)$ is nonconstant, the leading term is nilpotent. By (strong) induction, it will be shown that $a_n^{r+1} b_{m-r} = 0$. The base case follows immediately considering the term $a_n b_m = c_{n+m} = 0$. Assuming the above for all values up to r, we have

$$a_n^r \sum_{i+j=n+m-r} a_i b_j = a_n^r c_{n+m-r} = 0.$$

Notice now that each term with $i \neq n$ can be written as

$$a_n^r a_i b_j = a_n^r a_i b_{m-(r-n+i)} = a_i a_n^{n-i-1}(a_n^{r-n+i+1} b_{m-(r-n+i)}) = 0$$

by our (strong) inductive assumption. Therefore, it follows that $a_n^{r+1} b_{m-r} = 0$ (the term where $i = n$) and our induction is complete. When $r = m$, $a_n^{m+1} b_0 = 0$. Multiplying both sides by b_0^{-1}, $a_n^{m+1} = 0$. Therefore, a_n is nilpotent and so $a_n x^n$ is nilpotent.

Notice now that

$$f(x) - a_n x^n$$

is a unit since $f(x)$ is a unit and $-a_n x^n$ is nilpotent. Following the above proceedure, a_{n-1} is nilpotent and $a_{n-1}x^{n-1}$ is nilpotent. Continuing this, we see that a_i is nilpotent for $i \geq 1$. (Formally, use induction)

2b. Let $f(x)$ now be nilpotent. Let $n > 0$ be such that $(f(x))^n = 0$. The constant term of $(f(x))^n$ is exactly a_0^n so that a_0 is nilpotent. Notice that $1 + f(x)$ is a unit so that a_i is necessarily nilpotent for $i \geq 1$ from the above.

Let $a_0, \ldots, a_n$ be nilpotent. Since $a_i x^i$ is nilpotent for each $0 \leq i \leq n$ and the sum of nilpotents are nilpotent, we have

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

is nilpotent.

2c. If there exists $a \in A$ not equal to zero such that $af(x) = 0$, then $f(x)$ is, by definition, a zero divisor.

Suppose that $f(x) \in A[x]$ is a zero divisor. Let

$$g(x) = \sum_{j=0}^{m} b_j x^j$$

be of minimal degree such that $f(x)g(x) = 0$. By (strong) induction is will be shown that $a_{n-r}g(x) = 0$. For the base case, consider $a_n g(x)$. Since $a_n b_m = c_{n+m} = 0$, we have that $\deg(a_n g(x)) = m-1$ and $f(x)(a_n g(x)) = a_n f(x)g(x) = 0$. Since g was of minimal degree, this implies that $a_n g(x) = 0$. Assume now that $a_{n-i}g(x)$ for $0 \leq i \leq r$. Notice

$$\sum_{i+j=m+n-r-1} a_i b_j = c_{m+n-r-1} = 0.$$

For $i > n-r-1$, $a_i b_j = 0$ by our inductive assumption. There are no terms with $i < n-r-1$ so that we must have $a_{n-r-1}b_m = 0$. Therefore, $a_{n-r-1}g(x)$ has degree $m-1$ and satisfies $f(x)(a_{n-r-1}g(x)) = a_{n-r-1}f(x)g(x) = 0$. By minimality of $g(x)$, we must again have $a_{n-r-1}g(x) = 0$. This completes our induction. Since $a_i g(x) = 0$ for all $0 \leq i \leq n$, we necessarily have $a_i b_0 = 0$ for all $0 \leq i \leq n$ (or any coefficient from $g(x)$).

2d. Let $f(x), g(x) \in A[x]$. If $f(x)$ is not primitive, then $(a_0, \ldots, a_n) \subset (1)$. The coefficients of $f(x)g(x)$ are given by the sums

$$c_k = \sum_{i+j=k} a_i b_j \in (a_0, \ldots, a_n).$$

Therefore, $(c_0, \ldots, c_{n+m}) \subseteq (a_0, \ldots, a_n) \subset (1)$. This implies that $f(x)g(x)$ is not primitive.

Assume now that $f(x)$ and $g(x)$ are both primitive, that is, $(a_0, \ldots, a_n) = (b_0, \ldots, b_m) = (1)$. If $f(x)g(x)$ is not primitive, $(c_0, \ldots, c_{n+m}) \neq (1)$. Since every proper ideal is contained in a maximal ideal, $(c_0, \ldots, c_{n+m}) \subseteq M$ where M is a maximal ideal of A. Let i_0 and j_0 be minimal such that $a_{i_0} \notin M$ and $b_{j_0} \notin M$ (they cannot all be in M since it is not equal to (1)). Consider

$$\sum_{i+j=i_0+j_0} a_i b_j = c_{i_0+j_0} \in (c_0, \ldots, c_{n+m}) \subseteq M.$$

For indices such that $i > i_0$, $j < j_0$ and so $a_i b_j \in (c_0, \ldots, c_{n+m}) \subseteq M$. Similarly, for indices such that $i < i_0$, $a_i b_j \in (c_0, \ldots, c_{n+m}) \subseteq M$. Therefore, we have

$$a_{i_0} b_{j_0} = c_{i_0+j_0} - \sum_{\substack{i+j=i_0+j_0 \\ i \neq i_0, j \neq j_0}} a_i b_j \in M.$$

Since M is maximal, it is prime. This is a contradiction because it implies that either $a_{i_0} \in M$ or $b_{j_0} \in M$. This implies that $f(x)g(x)$ is necessarily primitive as well.

3. By induction on n, it will be shown that the above results carry over to the polynomial ring $A[x_1, \ldots, x_n]$. The base case is exactly the last problem, so assume the above for a fixed n and consider $A[x_1, \ldots, x_{n+1}] = A[x_1, \ldots, x_n][x_{n+1}]$.

   From the previous problem, any unit $f(x_1, \ldots, x_{n+1}) \in A[x_1, \ldots, x_n][x_{n+1}]$ necessarily has coefficients $a_i \in A[x_1, \ldots, x_n]$ such that a_0 is a unit and a_i is nilpotent for $i \geq 1$. By our inductive hypothesis, a_0 then has unit constant term and nilpotent coefficients otherwise and a_i has all nilpotent coefficients. Therefore,

$$f(x_1, \ldots, x_n) = \sum_i a_i(x_1, \ldots, x_n) x_{n+1}^i$$

   has a unit constant coefficient and all nilpotent coefficients otherwise. Conversely, if the constant coefficient of $f(x_1, \ldots, x_{n+1})$ is a unit $a \in A$ and the rest are nilpotent, the first problem lets us build

$$f(x_1, \ldots, x_{n+1}) = a + \sum_{i_1, \ldots, i_{n+1}} a_{i_1, \ldots, i_{n+1}} x_1^{i_1} \ldots x_{n+1}^{i_{n+1}}$$

   as the sum of a unit and nilpotent, which is then a unit.

   Following the same process as above, a nilpotent $f(x_1, \ldots, x_{n+1}) \in A[x_1, \ldots, x_n][x_{n+1}]$ then has all nilpotent coefficients. Similarly, if all the coefficients are nilpotent, each monomial is nilpotent, and so $f(x_1, \ldots, x_{n+1})$ is nilpotent.

   If there is an $a \in A$ such that $af(x_1, \ldots, x_{n+1}) = 0$, then $f(x_1, \ldots, x_{n+1})$ is a zero divisor. Conversely, if

$$f(x) = \sum_i a_i(x_1, \ldots, x_n) x_{n+1}^i$$

   is a zero divisor, there is an element $g(x_1, \ldots, x_n) \in A[x_1, \ldots, x_n]$ such that

$$a_i(x_1, \ldots, x_n) g(x_1, \ldots, x_n) = 0.$$

   Therefore, each $a_i \in A[x_1, \ldots, x_n]$ is a zero divisor. By our inductive hypothesis, there exists $b_i \in A$ such that $b_i a_i(x_1, \ldots, x_n) = 0$. The product $b = \prod_i b_i$ then satisfies $bf(x_1, \ldots, x_{n+1}) = 0$.

   This completes the use of induction.

   For the last part of the problem, the proof follows almost verbatim, with obvious modifications. In the first part, each c_i is a linear combination of the coefficients of f and so the ideal they generate is a still a subset of the ideal formed by the coefficients of f. For the other direction, the idea of the proof is the same, it just amounts to check indices.

4. The nilradical, $\mathfrak{N}$, is always a subset of the Jacobson radical, $\mathfrak{J}$, by the alternative definitions ($a \in \mathfrak{N}$ iff a is nilpotent and $a \in \mathfrak{J}$ iff $1 - ba$ is a unit for all $b \in A$). For the opposite inclusion, let $f(x) \in A[x]$ be such that $f(x) \in \mathfrak{J}$. Since $1 + xf(x)$ is a unit, the coefficients of the nonconstant terms are necessarily nilpotent. Since the coefficients of the nonconstant terms of $1 + xf(x)$ are the coefficients of $f(x)$, the coefficients of $f(x)$ are nilpotent. This then implies that $f(x)$ is nilpotent, that is, $f(x) \in \mathfrak{N}$. This shows the opposite inclusion holds and so $\mathfrak{N} = \mathfrak{J}$.

5a. Let $A[[x]]$ be the ring of formal power series. Let $f(x) \in A[[x]]$ be a unit. Let $g(x) \in A[[x]]$ be such that $f(x)g(x) = 1$. Writing out $f(x)$ and $g(x)$ in terms of coefficeints, the constant term of the product is exactly the product of the constant terms and is equal to 1. Therefore, the constant term of $f(x)$ is a unit.

   Conversely, let

$$f(x) = \sum_i a_i x^i \in A[[x]]$$

   be such that a_0 is a unit. Define $b_0 = a_0^{-1}$ and recursively define

$$b_k = -a_0^{-1} \sum_{i=0}^{k-1} a_{k-i} b_i, \ k \geq 1.$$

Let

$$g(x) = \sum_j b_j x^j \in A[[x]]$$

and notice that by definition

$$a_0 b_0 = 1, \quad \sum_{i+j=k} a_i b_j = 0.$$

Therefore,

$$f(x)g(x) = \sum_k c_k x^k = 1$$

and so $f(x) \in A[[x]]$ is a unit.

5b. Let $f(x) \in A[[x]]$ be nilpotent (with coefficients a_i). The claim is that the coefficient of the lowest degree term is nilpotent. This follows immediately from considering powers of $f(x)$ and looking at the term of lowest degree. It is exactly the term of lowest degree in $f(x)$ raised to the same power. Therefore, if $f(x)$ is nilpotent, so is the term of lowest degree. Since x^k is not a zero divisor for any $k \geq 0$ (consider the evaluation map $x \mapsto 1$), it must be that the coefficient is zero and so is nilpotent.

With the above in mind, it is easy to prove by induction that every coefficient of $f(x)$ is nilpotent. The base case, the constant coefficient of $f(x)$, follows exactly from the above. Assume that the first k coefficients of $f(x)$ are nilpotent. Then $\sum_{i=0}^{k-1} a_i x^i \in A[[x]]$ is nilpotent and so

$$f(x) - \sum_{i=0}^{k-1} a_i x^i = \sum_{i \geq k} a_i x^i$$

is nilpotent. Using the above, a_k is then nilpotent as well. Therefore, a_i is nilpotent for $i \geq 0$.

(Note: This proof works in $A[x]$ as well, but it not the most efficient)

5c. Let $f(x) \in A[[x]]$ be such that $f(x) \in \mathfrak{J}$. Then for all $b \in A$, $1 - bf(x)$ is a unit (in $A[[x]]$) and so the constant term $1 - ba_0$ is a unit in A. This implies that $a_0 \in \mathfrak{J}$.

Assume now that $a_0 \in \mathfrak{J}$. For all $g(x) \in A[[x]]$, $1 - f(x)g(x)$ has constant term given by one minus the product of a_0 and the constant term of $g(x)$. Since $a_0 \in \mathfrak{J}$, this implies that the constant term is a unit and so $1 - f(x)g(x)$ is a unit as well. Therefore, $f(x) \in \mathfrak{J}$.

5d. Let $\mathfrak{m} \subset A[[x]]$ be a maximal ideal and $\mathfrak{m}^c \subseteq A$ be its contraction under the inclusion map $i : A \hookrightarrow A[[x]]$. Notice first that $(x) \subseteq \mathfrak{m}$ since otherwise, $\mathfrak{m} \subset \mathfrak{m} + (x) \subset (1)$. The latter inclusion can be seen to be proper by writing $1 = m(x) + xf(x)$ with $m(x) \in M$ and $f(x) \in (x)$. This implies the constant term of $m(x)$ is 1 and so any element of $A[[x]]$ can be written with coefficients in M by solving a triangular system of equations. In particular, if

$$m(x) = 1 + \sum_{i \geq 1} m_i x^i, \;\; f(x) = \sum_{i \geq 0} a_i x^i \in A[[x]],$$

we may write

$$f(x) = m(x) \sum_{j \geq 0} c_j x^j \in \mathfrak{m}, \;\; c_0 = a_0, \;\; c_{k+1} = a_{k+1} - \sum_{j=0}^{k} c_j m_{k-j+1}.$$

This follows from the fact that

$$\sum_{i+j=n} m_i c_j = \sum_{i=0}^{n} m_i c_{n-i} = c_n + \sum_{i=1}^{n} m_i c_{n-i} = a_n - \sum_{j=0}^{n-1} c_j m_{n-j} + \sum_{i=1}^{n} m_i c_{n-i} = a_n.$$

This implies $\mathfrak{m} = A[[x]]$ and contradicts the maximality of M. Therefore, $(x) \subseteq \mathfrak{m}$. Now define the map $\varphi : A[[x]] \mapsto A/\mathfrak{m}^c$ by $\varphi(\sum_i a_i x^i) = [a_0]$. The kernel of this map is given by

$$\ker\varphi = \left\{ \sum_{i\geq 0} a_i x^i \in A[[x]] : a_0 \in \mathfrak{m}^c = i^{-1}(\mathfrak{m}) \right\}.$$

If $i(a_0) \in \mathfrak{m}$ then $(x) \subseteq \mathfrak{m}$ implies we have $\sum_i a_i x^i \in \mathfrak{m}$ so that $\ker\varphi \subseteq \mathfrak{m}$. Similarly, for any $\sum_i a_i x^i \in \mathfrak{m}$, we may write

$$a_0 + \sum_{i\geq 1} a_i x^i = \sum_{i\geq 0} a_i x^i \in \mathfrak{m} \implies a_0 \in \mathfrak{m} \implies a_0 \in \mathfrak{m}^c.$$

Therefore, the opposite inclusion holds as well and so we have that $\ker\varphi = \mathfrak{m}$. By the first isomorphism theorem, $A[[x]]/\mathfrak{m} \simeq A/\mathfrak{m}^c$ as rings. Since $A[[x]]/\mathfrak{m}$ is a field, this implies that $A/\mathfrak{m}^c$ is a field and so $\mathfrak{m}^c$ is maximal.

Clearly, $(\mathfrak{m}^c, x) \subseteq \mathfrak{m}$ (considering $\mathfrak{m}^c = i(\mathfrak{m}^c)$). Conversely, for any element

$$\sum_{i\geq 0} a_i x^i = a_0 + x \sum_{i\geq 1} a_i x^{i-1} \in \mathfrak{m},$$

since $\ker\varphi = \mathfrak{m}$ from above, we have exactly that $a_0 \in \mathfrak{m}^c$ and so the above is an element of $(\mathfrak{m}^c, x)$. Therefore, we have $\mathfrak{m} = (\mathfrak{m}^c, x)$.

5e. Consider the map $\pi : A[[x]] \mapsto A$ and let $P \subseteq A$ be a prime ideal. Let $\pi^{-1}(P)$ be a prime ideal of $A[[x]]$ and notice

$$(\pi^{-1}(P))^c = i^{-1}(\pi^{-1}(P)) = (\pi \circ i)^{-1}(P) = P.$$

Therefore, every prime ideal $P \subseteq A$ is a contraction of a prime ideal of $A[[x]]$.

6. Let A be a ring such that every ideal not contained in the nilradical $\mathfrak{N}$ contains a nonzero idempotent element. As in one of the previous problems, the nilradical is always a subset of the Jacobson radical (x nilpotent implies $1 - xy$ a unit for all y). Assume that equality does not hold. We then have that $\mathfrak{N} \subset \mathfrak{J}$ and so by assumption, there exists a nonzero idempotent $e \in \mathfrak{J}$. Note that since $e \in \mathfrak{J}$, $1 - e$ is a unit. We then have

$$e(1-e) = e - e^2 = e - e = 0 \implies e = 0$$

This contradiction implies that we must have $\mathfrak{N} = \mathfrak{J}$.

7. Let A be a ring such that for every $x \in A$, there exists $n \geq 1$ such that $x^n = x$ and let $P \subseteq A$ be a prime ideal. Consider the quotient A/P, which is an integral domain since P is prime, and let $[x] \in A/P$ be nonzero. For $n \geq 1$ such that $x^n = x$, we see

$$[x]([x]^{n-1} - 1) = [x^n - x] = 0.$$

Since $[x] \neq 0$, $[x]^{n-1} = 1$ and so $[x][x]^{n-2} = 1$. Therefore, every nonzero element of A/P is a unit. This implies that A/P is a field and so P is maximal. Since P was arbitrary, every prime ideal of A is maximal.

8. Let A be a nonzero ring. Ordering the prime ideals of A under reverse inclusion, $I \leq J$ iff $J \subseteq I$, it suffices by Zorn's lemma to show that every chain of prime ideals $I_1 \leq I_2 \leq \ldots$ (that is, $\ldots \subseteq I_2 \subseteq I_1$) has a maximal element. Let $I = \cap_n I_n$ be an ideal of A. Clearly, $I \subseteq I_n$ so that $I_n \leq I$ and I is a maximal element if I is prime.

Suppose that $a, b \in A$ are such that $ab \in I$. If $b \notin I$, there exists $n \geq 1$ such that $b \notin I_n$. Then since $ab \in I \subseteq I_n$, this implies $a \in I_n$ and $a \in I_k$ for $1 \leq k \leq n$ (by inclusion). Since $b \notin I_n$, $b \notin I_k$ for $k \geq n$. Then $ab \in I \subseteq I_k$ implies $a \in I_k$ for $k \geq n$. Therefore, $a \in I_k$ for $k \geq 1$ and so $a \in I$. Therefore, I is prime so that by Zorn's lemma, the set of prime ideals has a maximal element under reverse inclusion, that is, a minimal element under inclusion.

9. Assume $\mathfrak{a}$ is the intersection of prime ideals $\mathfrak{p}_i$. If $x^n \in \mathfrak{a}$, then $x^n \in \mathfrak{p}_i$ for all i so that $x \in \mathfrak{p}$ for all i and so $x \in \cap \mathfrak{p}_i = \mathfrak{a}$. That is, $r(\mathfrak{a}) = \mathfrak{a}$.

   Conversely, let $r(\mathfrak{a}) = \mathfrak{a}$. If $\pi : A \mapsto A/\mathfrak{a}$ is the projection map, we may write $r(\mathfrak{a}) = \pi^{-1}(\mathfrak{N})$ where $\mathfrak{N}$ is the nilradical. Since $\mathfrak{N}$ is the intersection of all prime ideals in $A/\mathfrak{a}$, which is in a one to one correspondence with the prime ideals of A that contain $\mathfrak{a}$ (via $\mathfrak{p} \leftrightarrow \pi^{-1}(\mathfrak{p})$), we have that

$$\mathfrak{a} = r(\mathfrak{a}) = \bigcap_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{a} \subseteq \mathfrak{p}}} \mathfrak{p}.$$

10. $(i) \implies (ii)$ Let $\mathfrak{p}$ be the unique prime ideal of A and $a \in A$. If a is not a unit, then (a) is a proper ideal and is so contained in a maximal ideal, which is then prime and hence, $\mathfrak{p}$. Therefore, $a \in \mathfrak{p}$. Since $\mathfrak{N}$ is the intersection of all prime ideals of A, $\mathfrak{N} = \mathfrak{p}$ and so every non-unit of A is nilpotent.

    $(ii) \implies (iii)$ Let $[a] \in A/\mathfrak{N}$ be nonzero. Since any representative $a \in A$ is a unit, we have

$$[a][a^{-1}] = [aa^{-1}] = [1].$$

    Therefore, $[a]$ is a unit. Since $[a]$ was arbitrary, this implies that $A/\mathfrak{N}$ is a field.

    $(iii) \implies (i)$ If there is more than one prime ideal, $\mathfrak{N}$ is properly contained in some prime ideal. Then $A/\mathfrak{N}$ is not a field since the prime ideals of A containing $\mathfrak{N}$ are in a one-to-one correspondence with the ideals of $A/\mathfrak{N}$ (and there is some prime ideal containing $\mathfrak{N}$ so there are nontrivial ideals of $A/\mathfrak{N}$).

11a. Let A be a Boolean ring and $x \in A$. Notice

$$x + 1 = (x + 1)^2 = x^2 + 2x + 1 = (x + 1) + 2x \implies 2x = 0.$$

11b. Let $\mathfrak{p}$ be a prime ideal of A and consider $A/\mathfrak{p}$. For any $x \in A/\mathfrak{p}$, $x \neq 0$, we see

$$0 = x^2 - x = x(x - 1) \implies x = 1.$$

Therefore, there are only two elements of $A/\mathfrak{p}$, 0 and 1. Since every nonzero element is a unit, $A/\mathfrak{p}$ is a field and so $\mathfrak{p}$ is maximal. (Note: One can use a previous problem to show that $\mathfrak{p}$ is maximal)

11c. Let $\mathfrak{a} = (a_0, \ldots, a_n)$ be a finitely generated ideal of A. First, we will find a more appropriate generating set for $\mathfrak{a}$. Define $v_1 = a_1$ and

$$v_j = a_j + a_j \sum_{i=1}^{j-1} v_i, \; 1 < j \leq n.$$

Clearly, $v_j \in \mathfrak{a}$ so that $(v_1, \ldots, v_n) \subseteq \mathfrak{a}$. Similarly, $a_1 \in (v_1, \ldots, v_n)$ and

$$a_j = v_j - a_j \sum_{i=1}^{j-1} v_i \in (v_1, \ldots, v_n)$$

for $1 < j \leq n$. Therefore, $\mathfrak{a} \subseteq (v_1, \ldots, v_n)$ so that $\mathfrak{a} = (v_1, \ldots, v_n)$. The utility of this new generating set will now be shown. It will be shown by induction on j that for $i < j$, $v_i v_j = 0$. The base case, $j = 1$, is trivial since there are no indices less than 1. Assume that for a fixed j, $v_i v_j = 0$ for $i < j$. For $i < j + 1$, we see

$$\begin{aligned} v_i v_{j+1} &= v_i \left(a_{j+1} + a_{j+1} \sum_{k=1}^{j} v_k \right) \\ &= v_i a_{j+1} + a_{j+1} \sum_{k=1}^{j} v_i v_k \\ &= v_i a_{j+1} + v_i^2 a_{j+1} + a_{j+1} \sum_{k<i} v_i v_k + a_{j+1} \sum_{i<k} v_i v_k \\ &= 2 v_i a_{j+1} = 0. \end{aligned}$$

Using this, we then have that $v_i v_j = \delta_{ij} v_j$. Let $v = \sum v_i$. Clearly, $v \in (v_1, \ldots, v_n) = \mathfrak{a}$ and

$$v_i = \sum_{j=1}^{n} v_i v_j = v_i v \in (v).$$

Therefore, $\mathfrak{a} = (v_1, \ldots, v_n) \subseteq (v)$. Combining these inclusions, we have that $\mathfrak{a} = (v)$.

12. Let A be a local ring and $e \in A$ be an idempotent such that $e \neq 1$. Since $0 = e(1-e)$, e cannot be a unit, since this would imply $e = 1$. Therefore, e is contained in the unique maximal ideal of A, which is necessarily equal to the Jacobson radical $\mathfrak{J}$. Therefore, $1 - e$ is necessarily a unit and the above equation implies that $e = 0$. Therefore, the only idempotents of a local ring are $e = 0, 1$.

13. Let K be a field and Σ be the set of irreducible, monic polynomials in one variable and coefficients in K. Let A be the ring of polynomials in indeterminants x_f indexed by $f \in \Sigma$ and $\mathfrak{a}$ be the ideal of A generated by polynomials of the form $g_f((x_h)_{h\in\Sigma}) = f(x_f)$ for $f \in \Sigma$. Assume there exists $g_{f_i} \in A$ such that

$$1 = \sum_{i=1}^{n} g_{f_i}.$$

Since each element of the right hand side is nonconstant, $\deg_{f_i} g_{f_i} > 1$. Since $\deg_{f_i} g_{f_j} = 0$ for $i \neq j$, the right hand side has degree $\deg_{f_i} g_{f_i}$ as a polynomial in f_i. Since the left hand side is constant, this is a contradiction. Therefore, $1 \notin \mathfrak{a}$. Therefore, $\mathfrak{a}$ is contained in some maximal ideal $\mathfrak{m}$. Let $K' = A/\mathfrak{m}$. Notice that K is a subfield of K' via $K \hookrightarrow A \to A/\mathfrak{m}$, which is injective since it is nontrivial (1 doesn't map to 0). For any $f \in \Sigma$, $f([x_f]) = [f(x_f)] = [0]$ so that every irreducible, monic polynomial in $K[x]$ factors over K'. Since every polynomial can be uniquely written as a product of irreducible, monic factors, every polynomial of $K[x]$ factors over K'.

Let $K^{(1)} = K'$ and $K^{(n+1)} = (K^{(n)})'$. Let $L = \cup K^{(n)}$. L is a field since for any $x, y \in L$, $x, y \in K^{(n)}$ for n sufficiently large so that $x \pm y$, xy, x/y are in L. For a polynomial $F \in L[x]$, the coefficients are in $K^{(n)}$ for some sufficiently large n. Then F factors over $K^{(n+1)}$ into factors of lesser degree. For k sufficiently large, F then factors into linear factors over $K^{(n+k)}$ and hence, over L. Therefore, every polynomial in $L[x]$ splits into linear factors over L. Let $\overline{K} \subseteq L$ be the set of elements of L that are algebraic over K. Since algebraic extensions of algebraic extensions are algebraic, $\overline{K}$ is an algebraic extension of K. For any polynomial with coefficients in $\overline{K}$, it splits into linear factors over L and since each root is then algebraic over $\overline{K}$, it is algebraic over K and hence, in $\overline{K}$. Therefore, $\overline{K}$ is an algebraic closure of K.

14. Let A be a ring and Σ be the set of ideals whose elements are all zero-divisors. By Zorn's lemma, it suffices to show that any chain

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \ldots$$

of elements in Σ has an upper bound. Let $\mathfrak{a} = \cup \mathfrak{a}_i$. It is easy to see that $\mathfrak{a}$ is an ideal and every element is an element of $\mathfrak{a}_i$ for i sufficiently large and hence a zero divisor in A. Since $\mathfrak{a}_i \subseteq \mathfrak{a}$, $\mathfrak{a}$ is an upper bound of this chain and by Zorn's lemma, maximal elements of Σ under inclusion exist. Let $\mathfrak{b}$ be a maximal element of Σ under inclusion and $ab \in \mathfrak{b}$. Then there exists $c \in A$ such that $abc = 0$. If $a \notin \mathfrak{b}$, then $a(bc) = 0$ implies that $bc = 0$ so that b is a zero divisor. By maximality, $b \in \mathfrak{b}$ since otherwise, $\mathfrak{b} \subset (b, \mathfrak{b})$. Therefore, $\mathfrak{b}$ is prime.

The union of all maximal elements of Σ is the set of all zero-divisors of A and a union of prime ideals. This completes the proof.

15a. Let A be a ring and X be the set of all prime ideals of X and for a subset $E \subseteq A$, define $V(E)$ to be the set of prime ideals containing E. Let $\mathfrak{a}$ be the ideal generated by a set E. Since $E \subseteq \mathfrak{a}$, any ideal containing $\mathfrak{a}$ contains E so that $V(\mathfrak{a}) \subseteq V(E)$. Conversely, any ideal containing E necessarily contains $\mathfrak{a}$ (as the smallest ideal containing E). Therefore, $V(E) \subseteq V(\mathfrak{a})$. Therefore, $V(E) = V(\mathfrak{a})$.

Similar to above, since $\mathfrak{a} \subseteq r(\mathfrak{a})$, $V(r(\mathfrak{a})) \subseteq V(\mathfrak{a})$. Conversely, since $r(\mathfrak{a})$ is the intersection of all prime ideals containing $\mathfrak{a}$, it is contained in every prime ideal that contains $\mathfrak{a}$. That is, $V(\mathfrak{a}) \subseteq V(r(\mathfrak{a}))$. Together, we have that $V(r(\mathfrak{a})) = V(\mathfrak{a})$.

15b. Clearly, every prime ideal contains 0 so $V(0) = X$. Since no prime ideal is all of A, $V(1) = \emptyset$.

15c. Let $\{E_i\}_{i\in I}$ be any family of subsets of A. Notice that for $P \in X$ such that $\cup E_i \subseteq P$, $E_i \subseteq P$ for each P so that $P \in \cap V(E_i)$. Conversely, for $P \in \cap V(E_i)$, $E_i \subseteq P$ for each $i \in I$ so that $\cup E_i \subseteq P$. Therefore, $P \in V(\cup E_i)$. Combining these, we have that

$$V\left(\bigcup_{i\in I} E_i\right) = \bigcap_{i\in I} V(E_i).$$

15d. Notice that $x^m \in \mathfrak{a}\cap\mathfrak{b}$ if and only if $x^m \in \mathfrak{a}$ and $x^m \in \mathfrak{b}$. That is, $r(\mathfrak{a}\cap\mathfrak{b}) = r(\mathfrak{a})\cap r(\mathfrak{b})$. Now $\mathfrak{ab} \subseteq \mathfrak{a}\cap\mathfrak{b}$ implies $r(\mathfrak{ab}) \subseteq r(\mathfrak{a}\cap\mathfrak{b})$. For $x^m \in \mathfrak{a}\cap\mathfrak{b}$, $x^{2m} \in \mathfrak{ab}$ so that $x \in r(\mathfrak{ab})$. Therefore, the opposite inclusion holds and

$$r(\mathfrak{ab}) = r(\mathfrak{a}\cap\mathfrak{b}) = r(\mathfrak{a})\cap r(\mathfrak{b}).$$

Using that $V(\mathfrak{a}) = V(r(\mathfrak{a}))$, we then have that

$$V(\mathfrak{a}\cap\mathfrak{b}) = V(\mathfrak{ab}).$$

Any ideal that contains $\mathfrak{a}$ or $\mathfrak{b}$ contains $\mathfrak{ab}$. Therefore,

$$V(\mathfrak{a})\cup V(\mathfrak{b}) \subseteq V(\mathfrak{ab}) = V(\mathfrak{a}\cap\mathfrak{b}).$$

Conversely, let $P \in X$ be such that $\mathfrak{a}\cap\mathfrak{b} \subseteq P$. If $\mathfrak{a} \not\subseteq P$ and $\mathfrak{b} \not\subseteq P$, we may take an element of each not in P and multiply them. The result is in $\mathfrak{a}\cap\mathfrak{b}$, but not in P (by primality). Therefore, either $\mathfrak{a} \subseteq P$ or $\mathfrak{b} \subseteq P$. This implies that the opposite inclusion holds as well and so we have

$$V(\mathfrak{a}\cap\mathfrak{b}) = V(\mathfrak{a})\cup V(\mathfrak{b}).$$

17a. For $f \in A$, let $X_f \subseteq X$ be the complement of $V((f))$. These are open sets of X (since from the above, the sets $V(E)$ satisfy the axioms of closed sets for a topology). Note that from the above, any open set U can be written as the complement of $V(E)$ for some subset $E \subseteq A$. For any $f \in E$, we then have that $V(E) \subseteq V((f))$ so that $X_f \subseteq U$. For any two open sets X_f and X_g, $X_f \cap X_g = X_{fg}$ from below. With these two properties, the sets of the form X_f constitute a base for the Zariski topology on X.

Let $f, g \in A$. $X_f \cap X_g = X_{fg}$ amounts to showing that $V((fg)) = V((f)(g)) = V((f)) \cup V((g))$, but this follows exactly from the fact that

$$V(\mathfrak{ab}) = V(\mathfrak{a}\cap\mathfrak{b}) = V(\mathfrak{a})\cup V(\mathfrak{b}).$$

Therefore, $X_f \cap X_g = X_{fg}$.

17b. If f is nilpotent, f is contained in every prime ideal and hence, $V((f)) = X$ and $X_f = \emptyset$. Conversely, if $X_f = \emptyset$ and $V((f)) = X$, then f is contained in every prime ideal and hence, in the nilradical. Therefore, f is nilpotent.

17c. If f is a unit, then $V((f)) = \emptyset$ so that $X_f = X$. Conversely, if f is not a unit, then (f) is a proper ideal and hence, contained in some maximal (and therefore, prime) ideal. This implies that $V((f)) \neq \emptyset$ so that $X_f \neq X$.

17d. If $r((f)) = r((g))$, we immediately have that $V((f)) = V((g))$ and $X_f = X_g$. If $X_f = X_g$, then $V((f)) = V((g))$ and a prime ideal $P \in X$ contains f if and only if it contains g. Since $r((f))$ is the intersection of all prime ideals that contain (f), we may then say that $r((f)) = r((g))$.

17e. Since $\{X_f\}_{f\in A}$ is a basis of the Zariski topology on X, it suffices to show that any open cover by sets of this form has a finite subcover (since every open set contains a set of this form). Let $\{X_{f_i}\}_{i\in I}$ be an open cover of X by elements of our basis. We then have

$$\emptyset = \bigcap_{i\in I} V((f_i)) = V\left(\bigcup_{i\in I}(f_i)\right) = V\left(\left(\bigcup_{i\in I}(f_i)\right)\right) = V((f_i)_{i\in I}).$$

Therefore, we must have that $(f_i)_{i\in I} = (1)$ (since otherwise, it is proper and contained in some maximal and hence, prime, ideal). We may then write

$$1 = \sum_{j=1}^{k} g_j f_{i_j},\ g_f \in A.$$

This implies that $(f_{i_j})_{j=1}^k = (1)$ and so

$$\emptyset = V((f_{i_j})_{j=1}^k) = V\left(\bigcup_{i\in I}(f_i)\right) = \bigcap_{i\in I} V((f_{i_j})).$$

Therefore, $X = \cup X_{f_{i_j}}$ and so there is a finite refinement of this open cover. Therefore, X is quasi-compact.

17f. Let $f \in A$ and X_f be an open set. Let $\{X_{f_i}\}_{i\in I}$ be an open cover as above. We see

$$X_f \subseteq \bigcup_{i\in I} X_{f_i} \iff V((f_i)_{i\in I}) \subseteq V((f)).$$

That is, every prime ideal containing $(f_i)_{i\in I}$ contains (f). This implies $r((f)) \subseteq r((f_i)_{i\in I})$. Then $f \in r((f)) \subseteq r((f_i)_{i\in I})$ so that

$$f^n = \sum_{j=1}^k g_j f_{i_j},\ g_j \in A.$$

Therefore, $f \in r((f_{i_j})_{j=1}^k)$ so that $V((f_{i_j})_{j=1}^k) \subseteq V((f))$ and so $X_f \subseteq \cup X_{f_{i_j}}$. Therefore, X_f is quasi-compact.

17g. If $U = \cup X_{f_i}$ is a finite union of open, quasi-compact sets, then U is open and quasi-compact (from basic topology). Conversely, if U is an open, quasi-compact set, U can be written as a union of X_{f_i}, $U = \cup X_{f_i}$ since these sets form a basis for the Zariski topology. Letting $\{X_{f_i}\}$ be our open cover, we get a finite refinement of this open cover, of which contains U and is a subset of U. Therefore, U is this finite union of $X_{f_{i_j}}$.

18a. If $\mathfrak{p} \in X$ is maximal, clearly, $V(\mathfrak{p}) = \{\mathfrak{p}\}$ is a closed subset of X. Conversely, if $\mathfrak{p}$ is not maximal, it is contained in some prime ideal. Since every closed set has the form $V(E)$ for some subset $E \subseteq A$, we then have that every closed set containing $\mathfrak{p}$ contains this prime ideal containing $\mathfrak{p}$. Therefore, $\{\mathfrak{p}\}$ is not closed.

18b. Clearly, any closed set containing $\mathfrak{p}$ necessarily contains all prime ideals that contain $\mathfrak{p}$. Therefore, $V(\mathfrak{p}) \subseteq \overline{\{\mathfrak{p}\}}$. Conversely, $V(\mathfrak{p})$ is a closed set containing $\mathfrak{p}$. Therefore, $\overline{\{\mathfrak{p}\}} \subseteq V(\mathfrak{p})$. Therefore, we have that $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$.

18c. $\mathfrak{q} \in \overline{\{\mathfrak{p}\}} \iff \mathfrak{p} \subseteq \mathfrak{q}$ Clearly, if $\mathfrak{p} \subseteq \mathfrak{q}$, then every closed set containing $\mathfrak{p}$ contains $\mathfrak{q}$ so that $\mathfrak{q} \in \overline{\{\mathfrak{p}\}}$. Conversely, if $\mathfrak{q} \in \overline{\{\mathfrak{p}\}}$, then $\mathfrak{q}$ is contained in every closed set that contains $\mathfrak{p}$. In particular, $\mathfrak{q} \in V(\mathfrak{p})$, which implies that $\mathfrak{p} \subseteq \mathfrak{q}$.

18d. Let $\mathfrak{p}, \mathfrak{q} \in X$ be distinct. If there are no neighborhoods of X that contain one of these ideals and not the other, then all open and closed sets that contain one contain the other. In particular, $q \in \overline{\{\mathfrak{p}\}}$ and $p \in \overline{\{\mathfrak{q}\}}$. This imply that $\mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{p}$ so that $\mathfrak{p} = \mathfrak{q}$. This contradiction shows that some separating neighborhood necessarily exists for either $\mathfrak{p}$ or $\mathfrak{q}$ so that the Zariski topology is a T0 space.

19. Let A be a ring, $X = \mathrm{Spec}(A)$ and $\mathfrak{N}$ be the nilradical of A. Assume $\mathfrak{N} \in X$ and let U and V be nonempty open subsets of X. If $U = V = X$, then they intersect. Otherwise, the complement of the open set $U \cap V$ is a nonempty closed set and so has the form $V(E)$ for some $E \subseteq A$. Since $V(E) \neq X$, there is some prime ideal that does not contain E. Therefore, $E \not\subseteq \mathfrak{N}$. This implies that $\mathfrak{N} \in U \cap V$ and so the intersection is nonempty and X is irreducible.

Conversely, assume that $\mathfrak{N}$ is not prime. Then there exists $a, b \in A$ such that $ab \in \mathfrak{N}$, but $a, b \notin \mathfrak{N}$. Since $a, b \notin \mathfrak{N}$, we have that $V(a) \neq X \neq V(b)$. Notice that $V(a) \neq V(b)$ since $V(a) = V(b)$ implies

$$V(a) = V((a)) \cap V((b)) = V((a)(b)) = V(ab) = X$$

since $ab \in \mathfrak{N}$. However, this contradicts $a \notin \mathfrak{N}$. From this, it follows that $X_a, X_b \neq \emptyset$, $X_a \neq X_b$, and

$$X_a \cap X_b = X_{ab} = \emptyset.$$

Therefore, X is not irreducible.

20a. Let X be a topological space and Y an irreducible subspace. Consider $\overline{Y}$. For any nonempty, open sets $U, V \subseteq \overline{Y}$, there exists open sets $\widetilde{U}, \widetilde{V} \subseteq X$ such that $U = \widetilde{U} \cap \overline{Y}$ and $V = \widetilde{V} \cap \overline{Y}$. Then, $U' = \widetilde{U} \cap Y$ and $V' = \widetilde{V} \cap Y$ are open subsets of Y. They are easily checked to be nonempty (for any point $x \in U$ every neighborhood intersects Y since $x \in \overline{Y}$. Then the points of this neighborhood in Y are in U' also). Then there exists an element of $U' \cap V'$, but this element is also then in $U \cap V$ as desired.

20b. By Zorn's lemma, it suffices to show that for an ascending chain of irreducible subspaces

$$Y_1 \subseteq Y_2 \subseteq \dots,$$

that $Y = \cup Y_i$ is an irreducible subspace as well. Let $U, V \subseteq Y$. For sufficiently large n, we have that $U, V \subseteq Y_n$ so that $U \cap V \neq \emptyset$ since Y_n is irreducible. The result then follows.

20c. From the first part, if Y is a maximal irreducible subspace, $\overline{Y}$ is an irreducible subspace so that by maximality $\overline{Y} \subseteq Y$ which implies $\overline{Y} = Y$ and Y is closed. By repeating the previous part with ascending chains of irreducible subspaces containing a point $x \in X$, we see that every point $x \in X$ is contained in a maximal irreducible subspace.

If X is Hausdorff, singletons are closed (and of course, irreducible). If a subset $Y \subseteq X$ has more than one point, there exists disjoint neighborhoods of any two points, which implies that Y is not irreducible. Therefore, the irreducible subspaces are exactly the singletons. Since there are no inclusions among singletons, these are exactly the maximal irreducible subspaces as well.

20d. Let A be a ring and $X = \operatorname{Spec}(A)$. Let $\mathfrak{p}$ be a minimal prime ideal and consider $V(\mathfrak{p})$. For any two nonempty open sets $U_1, U_2 \subseteq V(\mathfrak{p})$, we may write

$$U_i = V(\mathfrak{p}) \setminus V(E_i)$$

for some subset $E_i \subseteq A$. Notice that $\mathfrak{p} \notin V(E_1) \cup V(E_2)$ since $\mathfrak{p} \in V(E_i)$ implies that $V(\mathfrak{p}) \subseteq V(E_i)$ so that U_i is empty. Therefore,

$$\mathfrak{p} \in V(\mathfrak{p}) \setminus (V(E_1) \cup V(E_2)) = U \cap V.$$

Therefore, $V(\mathfrak{p})$ is an irreducible subspace of X.

Assume that $V(\mathfrak{p}) \subset Y$ for some subspace $Y \subseteq X$. Then there exists some ideal $\mathfrak{q} \in Y \setminus V(\mathfrak{p})$. That is, $\mathfrak{p} \not\subseteq \mathfrak{q}$. Notice

$$V\left(\bigcap_{\substack{\mathfrak{q} \in Z \\ \mathfrak{p} \not\subseteq \mathfrak{q}}} \mathfrak{q}\right) \cup V(\mathfrak{p}) = V\left(\bigcap_{\mathfrak{q} \in Z} \mathfrak{q}\right) = Z.$$

Here, we have written Z as the union of two distinct (not hard to see since neither is the whole space) closed sets. Taking the complement, we get two open sets with empty intersection. Therefore, Y is not irreducible. This implies that $V(\mathfrak{p})$ is maximal and that the irreducible components of X are $V(\mathfrak{p})$ for minimal prime ideals $\mathfrak{p}$.

Conversely, if Y is a maximal irreducible closed subspace, $Y = V(\mathfrak{a}) = \operatorname{Spec}(A/\mathfrak{a})$ for some ideal $\mathfrak{a}$ which is irreducible if and only if $\mathfrak{N}_{A/\mathfrak{a}}$ is prime. This ideal is prime if and only if its contraction is prime, but its contraction is exactly $r(\mathfrak{a})$, which is prime if and only if $\mathfrak{a}$ is prime. Then $Y = V(\mathfrak{a}) = V(r(\mathfrak{a}) = V(\mathfrak{p})$ for some prime p. It is clear that this is maximal if and only if $\mathfrak{p}$ is minimal.

21a. Let $\phi : A \mapsto B$ be a ring homomorphism, $X = \operatorname{Spec}(A)$, and $Y = \operatorname{Spec}(B)$. Let $\phi^* : Y \mapsto X$ be the map $\phi^*(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$. Fix $f \in A$. Notice that for $\mathfrak{p} \in Y_{\phi(f)} = Y \setminus V(\phi(f))$, $\phi(f) \notin \mathfrak{p}$ implies $f \notin \phi^{-1}(\mathfrak{p}) = \phi^*(\mathfrak{p})$. Therefore, $\phi^*(\mathfrak{p}) \in X_f$ so that $\mathfrak{p} \in (\phi^*)^{-1}(X_f)$. Conversely, for $\mathfrak{p} \in (\phi^*)^{-1}(X_f)$, $\phi^*(\mathfrak{p}) \in X_f$ implies $f \notin \phi^*(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$. Therefore, $\phi(f) \notin \mathfrak{p}$ and $\mathfrak{p} \in Y_{\phi(f)}$. Combining inclusions, we get that

$$(\phi^*)^{-1}(X_f) = Y_{\phi(f)}.$$

Since the open sets X_f form a basis for the topology on Y and their preimages are open in X, ϕ^* is continuous.

21b. Let $\mathfrak{a}$ be an ideal of A. For $\mathfrak{b} \in (\phi^*)^{-1}(V(\mathfrak{a}))$, $\phi^{-1}(\mathfrak{b}) = \phi^*(\mathfrak{b}) \supseteq \mathfrak{a}$ implies that $\mathfrak{b} \supseteq \mathfrak{a}^e$ so that $\mathfrak{b} \in V(\mathfrak{a}^e)$. That is, $(\phi^*)^{-1}(V(\mathfrak{a})) \subseteq V(\mathfrak{a}^e)$. Conversely, let $\mathfrak{b} \in V(\mathfrak{a}^e)$, that is, $\mathfrak{b} \supseteq \mathfrak{a}^e$. This implies

$$\phi^*(\mathfrak{b}) = b^c \supseteq \mathfrak{a}^{ec} \supseteq \mathfrak{a} \implies \mathfrak{b} \in (\phi^*)^{-1}(V(\mathfrak{a})).$$

Therefore, $V(\mathfrak{a}^e) \subseteq (\phi^*)^{-1}(V(\mathfrak{a}))$. Combining inclusions, we get the result.

21c. Let $\mathfrak{b}$ be an ideal of B. Clearly, $\phi^*(V(\mathfrak{b})) \subseteq V(\mathfrak{b}^c)$ so that $\overline{\phi^*(V(\mathfrak{b}))} \subseteq V(\mathfrak{b}^c)$. For the opposite inclusion, let $\phi^*(V(\mathfrak{b})) \subseteq V(\mathfrak{a})$. This implies that $\mathfrak{a} \subseteq \phi^{-1}(\mathfrak{q})$ for every $\mathfrak{q} \in Y$ such that $\mathfrak{b} \subseteq \mathfrak{q}$. That is,

$$\mathfrak{a} \subseteq \bigcap_{\substack{\mathfrak{q}\in Y \\ \mathfrak{b}\subseteq\mathfrak{q}}} \phi^{-1}(\mathfrak{q}).$$

Therefore,

$$V\left(\bigcap_{\substack{\mathfrak{q}\in Y \\ \mathfrak{b}\subseteq\mathfrak{q}}} \phi^{-1}(\mathfrak{q})\right) \subseteq V(\mathfrak{a}).$$

However,

$$V(\mathfrak{b}^c) = V(r(\phi^{-1}(\mathfrak{b}))) = V(\phi^{-1}(r(\mathfrak{b}))) = V\left(\phi^{-1}\left(\bigcap_{\substack{\mathfrak{q}\in Y \\ \mathfrak{b}\subseteq\mathfrak{q}}} \mathfrak{q}\right)\right) = V\left(\bigcap_{\substack{\mathfrak{q}\in Y \\ \mathfrak{b}\subseteq\mathfrak{q}}} \phi^{-1}(\mathfrak{q})\right).$$

From this, $V(\mathfrak{b}^c)$ is contained in the closure $\overline{\phi^*(V(\mathfrak{b}))}$. Therefore, the result holds.

21d. Assume now that $\phi : A \mapsto B$ is surjective. Consider the induced map $\phi^* : Y \mapsto X$. For every $\mathfrak{b} \in Y$, $\phi^*(\mathfrak{b}) = \phi^{-1}(\mathfrak{b}) \supseteq \ker\phi$ so that $\phi^*(\mathfrak{b}) \in V(\ker\phi)$ so that we may consider ϕ^* as a map $\phi^* : Y \mapsto V(\ker\phi)$. As noted in a previous problem, this map is continuous.

Define a map $\varphi : V(\ker\phi) \mapsto Y$ by $\varphi(\mathfrak{a}) = \mathfrak{a}^e = \phi(\mathfrak{a})$ (because ϕ is surjective, this is in fact an ideal). To show that this actually maps a prime ideal $\mathfrak{a} \in V(\ker\phi)$ to an element of Y, let $b_1 = \phi(a_1)$ and $b_2 = \phi(a_2)$ be such that $\phi(a_1a_2) = b_1b_2 = \phi(a_3) \in \mathfrak{a}^e = \phi(\mathfrak{a})$. This implies $a_1a_2 = a_3 + k$ for some $k \in \ker\phi$, but $\ker\phi \subseteq \mathfrak{a}$ so this implies $a_1a_2 \in \mathfrak{a}$. Since $\mathfrak{a}$ is prime, this implies that either $a_1 \in \mathfrak{a}$ or $a_2 \in \mathfrak{a}$ so that either $b_1 = \phi(a_1) \in \mathfrak{a}^e$ or $b_2 = \phi(a_2) \in \mathfrak{a}^e$. Therefore, $\mathfrak{a}^e$ is prime. It is easy to see that since ϕ is surjective, $\mathfrak{a}^{ec} = \mathfrak{a}$ and $\mathfrak{b}^{ce} = \mathfrak{b}$, so that $\varphi^{-1} = \phi^*$.

All that remains is to show that φ is continuous, it will be shown that the preimage of a closed set is closed. Let $V(\mathfrak{b}) \subseteq Y$ be closed. Notice

$$\varphi^{-1}(V(\mathfrak{b})) = \phi^*(V(\mathfrak{b})) = \{\mathfrak{p}^c : \mathfrak{b} \subseteq \mathfrak{p}\} = V(\mathfrak{b}^c).$$

Therefore, φ is continuous and so $\phi^* : Y \mapsto V(\ker\phi)$ is a homeomorphism. In particular, if $\pi : A \mapsto A/\mathfrak{N}$ is the quotient map, $\pi^* : \mathrm{Spec}(A/\mathfrak{N}) \mapsto \mathrm{Spec}(A)$ is a homeomorphism.

21e. Notice the following.

$$\overline{\phi^*(Y)} = \overline{\phi^*(V(\{0\}))} = V(\{0\}^c) = V(\ker\phi).$$

In the case that $\ker\phi \subseteq \mathfrak{N}$ (in particular, if ϕ is injective), then $\overline{\phi^*(Y)} = V(\ker\phi) = V(\mathfrak{N}) = X$. Therefore, $\phi^*(Y)$ is dense in X. Conversely, if $V(\ker\phi) = \overline{\phi^*(Y)} = X$, then every prime ideal contains $\ker\phi$. Taking the intersection over all of them, we have $\ker\phi \subseteq \mathfrak{N}$.

21f. This follows immediately from the definition since for two maps $\phi : A \mapsto B$, $\psi : B \mapsto C$, and element $\mathfrak{p} \in C$,

$$(\psi \circ \phi)^*(\mathfrak{p}) = (\psi \circ \phi)^{-1}(\mathfrak{p}) = \phi^{-1}(\psi^{-1}(\mathfrak{p})) = (\phi^* \circ \psi^*)(\mathfrak{p}).$$

21g. Let A be an integral domain with one non-zero prime ideal $\mathfrak{p}$ (which is then maximal as well), K be the field of fractions of A, $B = (A/\mathfrak{p}) \times K$, and $\phi : A \mapsto B$ be defined by

$$\phi(x) = (\overline{x}, x).$$

From our description of A, we have that $\mathrm{Spec}(A) = \{\{0\}, \mathfrak{p}\}$. It is clear (since $A/\mathfrak{p}$ and K are fields) that the only ideals of B are $\langle(1,0)\rangle$ and $\langle(0,1)\rangle$. Since their respective quotients are fields, they are both maximal and hence, prime. Therefore, $\mathrm{Spec}(B) = \{\langle(1,0)\rangle, \langle(0,1)\rangle\}$. We see

$$\phi^*(\langle(1,0)\rangle) = \{0\},\ \phi^*(\langle(0,1)\rangle) = \mathfrak{p}.$$

From this and the preceding problems, ϕ^* is clearly a bijective continuous map. To show that ϕ^* is not a homeomorphism, it suffices to show that ϕ^* is not an open map. Since $\langle(1,0)\rangle$ is maximal, $V(\langle(1,0)\rangle) = \{\langle(1,0)\}$ and its complement $\{\langle(0,1)\rangle\}$ is therefore open. We see

$$\phi^*(\{\langle(0,1)\rangle\}) = \{\mathfrak{p}\}.$$

To show that this set is not open, notice that $\{0\} \cap \{\mathfrak{p}\} = \emptyset$. However, $\mathrm{Spec}(A)$ is irreducible since A is an integral domain ($\mathfrak{N} = \{0\}$ is a prime ideal). If $\{\mathfrak{p}\}$ were open, we should then reach a contradiction. Therefore, ϕ^* is not a homeomorphism.

22. Let $A = \prod_{i=1}^n A_i$, $\pi_i : A \mapsto A_i$ be the projection map onto the i-th coordinate, and $e_i = (0, \ldots, 1, \ldots, 0)$. Notice that for every prime ideal $\mathfrak{p}$ of A, there exists i such that $e_i \notin \mathfrak{p}$ (since otherwise, $\mathfrak{p} = A$ is not prime). Therefore, $e_j \in \mathfrak{p}$ for each $j \neq i$ since $e_i e_j = 0 \in \mathfrak{p}$. Therefore, $\ker \pi_i = A_1 \times \ldots \{0\} \times \ldots A_n \subseteq \mathfrak{p}$. That is, $\mathfrak{p} \subseteq V(\ker \pi_i)$ for some i.

    Let $X = \mathrm{Spec}(A)$, $X_i = V(\ker \pi_i)$, and $Y_i = \mathrm{Spec}(A_i)$. From a previous problem, we know that the map $\pi_i^* : Y_i \mapsto X_i$ is a homeomorphism, so the X_i are canonically isomorphic to $Y_i = \mathrm{Spec}(A_i)$. From the remarks above, we know that $X = \cup X_i$. It is easy to see that $V(\ker \pi_i) \cap V(\ker \pi_j) = \emptyset$ for $i \neq j$ since any prime ideal containing both $\ker \pi_i$ and $\ker \pi_j$ is necessarily all of A and so not prime. Each X_i is closed, but also,

    $$X_i = X \setminus \left(\bigcup_{j \neq i} X_j \right)$$

    is open (since the union is finite and so closed). Therefore, each X_i is a connected component of X.

    $(ii) \implies (i)$ Obvious from the above.

    $(ii) \implies (iii)$ Take $e = (1, 0)$.

    $(iii) \implies (ii)$ Let $e \neq 0, 1$ be an idempotent of A so that $1 - e$ is another idempotent. Define a map $\phi : A \mapsto (A/(e)) \times (A/(1-e))$ by $\phi(a) = (a, a)$. Since $(e) + (1 - e) = 1$, by the Chinese remainder theorem, this map is surjective with kernel $\ker \phi = (e) \cap (1 - e) = (e)(1 - e) = \{0\}$. Therefore, this map is injective as well. Therefore, $A \simeq (A/(e)) \times (A/(1 - e))$.

    $(i) \implies (iii)$ If $X = V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{ab})$, then no prime ideal contains $\mathfrak{a}$ and $\mathfrak{b}$. This implies that $\mathfrak{a} + \mathfrak{b} = 1$ (otherwise, find a maximal (prime) ideal that contains it). We also have that all prime ideals contain $\mathfrak{ab}$ so that $\mathfrak{ab} \subseteq \mathfrak{N}$. Since $\mathfrak{a} + \mathfrak{b} = 1$, there exists $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ such that $a + b = 1$. Since $ab \in \mathfrak{ab}$, there exists n such that $(ab)^n = 0$. Since $r((a)^n) = (a)$ and $r((b)^n) = (b)$ are coprime, $(a)^n + (b)^n = 1$. Then let $e \in (a)^n$ and $1 - e \in (b)^n$. We have $e(1 - e) \in (ab^n) = 0$. If $e = 1$, then $1 \in (a)^n$ implies $1 \in (a)$ is a contradiction. Similarly, if $e = 0$, $1 \in (b)^n$ implies $1 \in (b)$ is a contradiction. Therefore, nontrivial idempotents exist.

23a. Let A be a Boolean ring and $X = \mathrm{Spec}(A)$. If $X_f = \emptyset$, it is open and closed. If $X_f \neq 0$, by definition, the set $X_f = X \setminus V(f)$ is open. To show that it is also closed, notice that $X_f = X/V(f) = V(1 - f)$. This is because $V(f) \cup V(1 - f) = V(f(1 - f)) = V(0) = X$ and $V(f) \cap V(1 - f) = V((f, 1 - f)) = V(1) = \emptyset$.

23b. Let $f_1, \ldots, f_n \in A$. Notice

$$X_{f_1} \cup \ldots \cup X_{f_n} = X/(V(f_1) \cap \ldots \cap V(f_n)) = X/V((f_1, \ldots, f_n)).$$

From problem 12, every finitely generated ideal of a Boolean ring is principal. Therefore, there exists $f \in X$ such that $(f_1, \ldots, f_n) = (f)$ and $V((f_1, \ldots, f_n)) = V(f)$. Then the above becomes

$$X_{f_1} \cup \ldots \cup X_{f_n} = X_f.$$

23c. Let $Y \subseteq X$ be open and closed. Since Y is open, it can be written as $Y = \cup_i X_{f_i}$ for some $f_i \in A$. Since X is quasi-compact and Y is closed, Y is quasi-compact (from the standard argument in topology). Therefore, in the union above, we may take finitely many sets X_{f_i}. That is, $Y = \cup_{i=1}^n X_{f_i}$. From the previous problem, we have that $Y = X_f$ for some $f \in A$. Therefore, the sets X_f are the only open and closed subsets of X.

23d. Let $\mathfrak{p}, \mathfrak{q} \in X$ be such that $\mathfrak{p} \neq \mathfrak{q}$. Without loss of generality, there then exists $f \in \mathfrak{p}$ and $f \notin \mathfrak{q}$. That is, $\mathfrak{p} \in V(f) = X_{1-f}$ and $\mathfrak{q} \in X \setminus V(f) = X_f$. Notice $X_f \cap X_{1-f} = X_{f(1-f)} = \emptyset$. Therefore, we have disjoint neighborhoods of $\mathfrak{p}$ and $\mathfrak{q}$ in X, so X is Hausdorff. Since X is quasi-compact, this implies that X is a compact Hausdorff space.

24. I don't know much/any lattice theory, but the first problem is just a scramble of notation, and the result is well-known.

25. For a Boolean lattice L, consider $X = \mathrm{Spec}(A(L))$. From problem 23, the open-and-closed subsets are exactly the sets X_f, for $f \in A$. These of course are in correspondence with the elements of A, which are in correspondence with the elements of the lattice, as desired. All that remains is to show that the map induced by this correspondence preserves the lattice structure of the subsets of X. For X_f, X_g, we see

$$X_f \cap X_g = X_{fg} \leftrightarrow fg = f \wedge g$$
$$X_f \cup X_g = X_{f+g+fg} \leftrightarrow f + g + fg = f \vee g,$$

where $X_f \cup X_g = X_{f+g+fg}$ is shown in problem 11c (the ideal (f, g) is finitely generated and so is principal. Its generator can be easily computed to be $f + g + fg$). The result follows.

26a. Let $\mathfrak{m} \in \mathrm{Max}(C(X))$ and let $V(\mathfrak{m}) = \{x \in X : \forall f \in \mathfrak{m}, f(x) = 0\}$. If $V(\mathfrak{m}) = \emptyset$, then for every $x \in X$, there exists $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$ and hence, $f_x(t) \neq 0$ in some neighborhood U_x of x. By compactness, there exists a finite cover of X by the sets U_x, say $X = \cup_{i=1}^n U_{x_i}$. Consider

$$f(t) = \sum_{i=1}^n f_{x_i}^2(t).$$

It is clear that f does not vanish on all of X so that f is a unit in $\mathfrak{m}$, contradicting maximality. Therefore, $V(\mathfrak{m})$ is nonempty.

For any $x \in V(\mathfrak{m})$, $\mathfrak{m} \subseteq \mathfrak{m}_x$. By maximality, we then necessarily have that $\mathfrak{m} = \mathfrak{m}_x$ so that the map $\phi : X \mapsto \mathrm{Max}(C(X))$ that sends $x \mapsto \mathfrak{m}_x$ is surjective.

26b. It is well known that any compact Hausdorff space is normal. From this, Urysohn's lemma applies to X. Therefore, any two points may be separated by a continuous function. That is, for $x \neq y$, we may find a function $f \in C(X)$ such that $f(x) = 0$ and $f(y) = 1$. Therefore, $\mathfrak{m}_x \neq \mathfrak{m}_y$. Therefore, $\phi : X \mapsto \mathrm{Max}(C(X))$ is injective.

26c. For $f \in C(X)$, define

$$U_f = \{x \in X : f(x) \neq 0\},\ \overline{U_f} = \{\mathfrak{m} \in \mathrm{Max}(C(X)) : f \notin \mathfrak{m}\}.$$

Notice

$$\phi(U_f) = \{\mathfrak{m}_x \in \mathrm{Max}(C(X)) : f(x) \neq 0\}.$$

Since every $\mathfrak{m} \in \mathrm{Max}(C(X))$ has the form $\mathfrak{m}_x$ for some $x \in X$, we may then write

$$\phi(U_f) = \{\mathfrak{m} \in \mathrm{Max}(C(X)) : f \notin \mathfrak{m}\} = \overline{U_f}.$$

These are open sets in X and $\mathrm{Max}(C(X))$ respectively (the latter because it is the complement of $V(f)$ in $\mathrm{Max}(C(X))$). By Urysohn's lemma, the sets U_f constitute a basis for the topology on X and so ϕ is an open map. Therefore, the set-theoretic inverse of ϕ is continuous and so ϕ is a homeomorphism.

27. Let k be an algebraically closed field and

$$f_\alpha(t_1, \ldots, t_n) = 0$$

be a system of polynomial equations with solution set X. Let $I(X)$ be the set of polynomials who vanish identically on X and $P[x] = k[t_1, \ldots, t_n]/I(X)$. Since constant functions are in $P[x]$, the map $p_x : P[x] \mapsto k$ defined by $p_x(f) = f(x)$ is surjective and so the kernel $\mathfrak{m}_x$ is maximal so we have a mapping $\phi : X \mapsto \mathrm{Max}(P[x])$ as before sending $x \mapsto \mathfrak{m}_x$.

To see that ϕ is injective, let $x \neq y$. Then $x_i \neq y_i$ for some coordinate t_i and so the polynomial $f(t_1, \ldots, t_n) = t_i - x_i$ vanishes at x_i but not at y_i. Therefore, $\mathfrak{m}_x \neq \mathfrak{m}_y$.

28. Let $f_1, \dots, f_m \in k[t_1, \dots, t_n]$ and let $f : k^n \mapsto k^m$ be defined by $f(x) = (f_1(x), \dots, f_m(x))$. For affine algebraic varieties $X \subseteq k^n$ and $Y \subseteq k^m$, if f restricted to X maps into Y, we have a regular map $f : X \mapsto Y$. The induced map $\phi^* : P[Y] \mapsto P[X]$ is defined by $\phi^*(f) = f \circ \phi$.

    For a regular map $\phi : X \mapsto Y$, the induced map is easily seen to be a k-algebra homomorphism. For a k-algebra homomorphism $\eta : P[Y] \mapsto P[X]$ and $s_1, \dots, s_m$ the coordinate functions of k^m, $\eta(s_i) \in P[x]$ necessarily agrees with some polynomial $\eta_i \in k[t_1, \dots, t_n]$ (and the choice does not matter). Then consider the map $\phi : X \mapsto Y$ defined by $\phi(x) = (\eta_1(x), \dots, \eta_m(x))$. To see that this maps into Y, let $f_\alpha \in k[s_1, \dots, s_m]$ be any set of defining functions for Y. Notice

$$f_\alpha(\phi(x)) = f_\alpha(\eta_1(x), \dots, \eta_m(x)) = (\eta \circ f_\alpha)(s_1(x), \dots, s_m(x)) = \eta(0) = 0.$$

    The third to last equality is because η is a k-algebra homomorphism and so its value at f_α is determined on the coordinate functions s_i. $f_\alpha(s_1(x), \dots, s_m(x)) = 0$ because $(s_1(x), \dots, s_m(x)) \in Y$. Notice that in the same vein of proof, for $f \in P[Y]$,

$$\phi^*(f)(x) = f(\phi(x)) = \eta(f)(x).$$

Chapter 2

1. Let $n, m \in \mathbb{Z}_{>0}$ and define a map $\varphi : \mathbb{Z} \mapsto \mathbb{Z}_n \otimes \mathbb{Z}_m$ by

$$\varphi(1) = [1]_n \otimes [1]_m.$$

    By Bezout's lemma, there exists $\alpha, \beta \in \mathbb{Z}$ such that

$$\gcd(n, m) = \alpha n + \beta m.$$

    We then have

$$\varphi(\gcd(n, m)) = \alpha n([1]_n \otimes [1]_m) + \beta m([1]_n \otimes [1]_m) = \alpha([n]_n \otimes [1]_m) + \beta([1]_n \otimes [m]_m) = 0.$$

    Therefore, $(\gcd(n, m)) \subseteq \ker \varphi$ and so there is an induced map $\widetilde{\varphi} : \mathbb{Z}_{\gcd(n,m)} \mapsto \mathbb{Z}_n \otimes \mathbb{Z}_m$ defined by

$$\widetilde{\varphi}([1]_{\gcd(n,m)}) = ([1]_n \otimes [1]_m).$$

    Define another map $\phi : \mathbb{Z}_n \otimes \mathbb{Z}_m \mapsto \mathbb{Z}_{\gcd(n,m)}$ by

$$\phi([a]_n \otimes [b]_m) = [ab]_{\gcd(n,m)}.$$

    It is easy to check that this map is well-defined (that is, it is determined by a middle-linear map). It is simple to check that this map is an inverse to $\widetilde{\varphi}$ so that $\widetilde{\varphi}$ is an isomorphism.

    In the case that $\gcd(n, m) = 1$, we have that $\mathbb{Z}_n \otimes \mathbb{Z}_m \simeq \{0\}$.

2. Let $\mathfrak{a} \subseteq A$ be an ideal and M an A-module. Consider the exact sequence

$$0 \to \mathfrak{a} \xrightarrow{i} A \xrightarrow{\pi} A/\mathfrak{a} \to 0.$$

    Since the tensor functor is right exact, we have that the following sequence is exact.

$$\mathfrak{a} \otimes M \xrightarrow{i \otimes \mathrm{Id}} A \otimes M \xrightarrow{\pi \otimes \mathrm{Id}} (A/\mathfrak{a}) \otimes M \to 0$$

    In particular, there is a canonical isomorphism $A \otimes M \mapsto M$ defined by $a \otimes m \mapsto am$ so that we may consider this as an exact sequence

$$\mathfrak{a} \otimes M \to M \to (A/\mathfrak{a}) \otimes M.$$

    Notice that the image of the first map (after the identification of $A \otimes M$ with M) is exactly $\mathfrak{a}M$. Since the latter map is surjective, we have the following from the isomorphism theorem.

$$M/\mathfrak{a}M \simeq (A/\mathfrak{a}) \otimes M$$

3. Let A be a local ring with (unique) maximal ideal $\mathfrak{m}$ and quotient $k = A/\mathfrak{m}$ and M, N be a A-modules such that $M \otimes_A N = \{0\}$. We then have

$$\begin{aligned} M_k \otimes_k N_k &= (k \otimes_A M) \otimes_k (k \otimes_A N) \\ &= (M \otimes_A k) \otimes_k (k \otimes_A N) \\ &= M \otimes_A (k \otimes_k k) \otimes_A N \\ &= k \otimes_A (M \otimes_A N) = k \otimes_A \{0\} = \{0\}. \end{aligned}$$

Since M_k and N_k are vector spaces, this implies that either $M_k = 0$ or $N_k = 0$ (since the tensor product of free modules is free of rank equal to the product of their ranks). Without loss of generality, assume that $M_k = \{0\}$. We see

$$M/\mathfrak{a}M \simeq (A/\mathfrak{a}) \otimes_A M = k \otimes_A M = M_k = 0 \implies \mathfrak{a}M = M.$$

Since $\mathfrak{a} = \mathfrak{J}$ (it is the only maximal ideal) Nakayama's lemma implies that $M = 0$.

4. It is simple to check that

$$0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$$

is exact if and only if

$$0 \to \oplus A_i \xrightarrow{f} \oplus B_i \xrightarrow{g} \oplus C_i \to 0$$

is exact, where $f = (f_i)_{i \in I}$ and $g = (g_i)_{i \in I}$. From this and the fact that

$$A \otimes (\oplus_{i \in I} M_i) = \oplus_{i \in I}(A \otimes M_i),$$

the result follows immediately.

5. As an A-module, $A[x] \simeq \oplus_{i=1}^{\infty} A$ and so from the previous problem, it is flat as an A-module. For any short exact sequence of A-algebras

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0,$$

we may regard them as A-modules and get the following exact sequence of A-modules.

$$0 \to M_1 \otimes A[x] \xrightarrow{f \otimes \mathrm{Id}} M_2 \otimes A[x] \xrightarrow{g \otimes \mathrm{Id}} M_3 \otimes A[x] \to 0$$

It is clear (since f and g are A-algebra homomorphisms) that $f \otimes \mathrm{Id}$ and $g \otimes \mathrm{Id}$ are in fact A-algebra homomorphisms as well.

6. For an A-module M, it is clear that $M[x]$ inherits an $A[x]$-module structure where multiplication by an element of $A[x]$ is given exactly by their respective polynomials in $A[x]$ (since the coefficients of the product will stay in M, this works).

Define a map $\varphi : A[x] \otimes_A M \mapsto M[x]$ of $A[x]$-algebras by

$$\varphi\left(\left(\sum_{i=0}^{n} a_i x^i\right) \otimes_A m\right) = \sum_{i=0}^{n} a_i m x^i.$$

This map is well-defined since the corresponding map $A[x] \times M \mapsto M[x]$ is $A[x]$-linear and also a ring homomorphism. Define another map $\phi : M[x] \mapsto A[x] \otimes_A M$ by

$$\phi\left(\sum_{i=0}^{n} m_i x^i\right) = \sum_{i=0}^{n} x^i \otimes_A m_i.$$

This map is also easily verified to be $A[x]$-linear and also a ring homomorphism. It is then checked that these maps are inverses of one another so that $A[x] \otimes_A M \simeq M[x]$ as $A[x]$-algebras.

7. Simply notice that $A[x]/\mathfrak{p}[x] \simeq (A/\mathfrak{p})[x]$ is an integral domain since $A/\mathfrak{p}$ is an integral domain. Therefore, $\mathfrak{p}[x]$ is a prime ideal of $A[x]$.

   This does not hold for maximal ideals. For instance, let (2) be a maximal ideal of $\mathbb{Z}$. The ideal $(2)[x] \subseteq (2, x)$ is not maximal.

8a. From the definition of flatness of a module, for any exact sequence, we may tensor first by M and then by N, preserving exactness. The result is the same as tensoring by $M \otimes_A N$.

8b. Since N is exact as a B-module, tensoring with respect to B preserves exact sequences. Considering these maps as maps from/to the tensor product with respect to A gives the desired exact sequence (since the modules themselves don't change, the exactness conditions remain. They are easily A-linear).

9. Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

   be an exact sequence of R-modules where A and C are finitely generated by $(a_i)_{i=1}^n$ and $(c_j)_{j=1}^m$ respectively. The claim is that B is generated by $(f(a_i), b_j)_{i,j}$ where $b_j \in g^{-1}(c_j)$ is any element. Let $b \in B$. Since

$$g(b) = \sum_{j=1}^m r_j c_j,$$

   we may write

$$b = \sum_{j=1}^m r_j b_j + k, \;\; k \in \ker g = \operatorname{Im} f.$$

   Since f is injective, we may write $g = f(a)$ for some $a \in A$ where

$$a = \sum_{i=1}^n s_i a_i.$$

   Then we have that

$$b = \sum_{j=1}^m r_j b_J + \sum_{i=1}^n s_i f(a_i).$$

   Since this set of generators is finite, B is finitely generated.

10. Let $\mathfrak{a} \subseteq \mathfrak{J}$ be an ideal of A and M, N be A-modules where N is finitely generated. If $u : M \mapsto N$ is such that the induced map $\overline{u} : M/\mathfrak{a}M \mapsto N/\mathfrak{a}N$ is surjective, then the composition $M \to M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective. By definition of the induced map, the composition above is equal to the composition $M \to N \to N/\mathfrak{a}N$, and so this composition is surjective. This implies that $N = \mathfrak{a}N + u(M)$. Since $\mathfrak{a} \subseteq \mathfrak{J}$, Nakayama's lemma implies that $u(M) = N$ and so u is surjective.

11. Let A be a non-zero ring. If $\phi : A^n \mapsto A^m$ is an isomorphism, let $\mathfrak{m}$ be a maximal ideal of A and consider the induced map

$$\mathrm{Id} \otimes \phi : (A/\mathfrak{m}) \otimes_A A^n \mapsto (A/\mathfrak{m}) \otimes_A A^m.$$

   It is clear that this map is a bijective A-module homomorphism with inverse $\mathrm{Id} \otimes \phi^{-1}$ (by the composition of such maps). This map preserves the $A/\mathfrak{m}$-module structure of each as well and so gives a linear transformation of $(A/\mathfrak{m}) \otimes_A A^n$ to $(A/\mathfrak{m}) \otimes_A A^m$ as $A/\mathfrak{m}$-vector spaces.

   All that remains now is to compute the dimension of $(A/\mathfrak{m}) \otimes_A A^n$. Consider $\{1 \otimes e_i\}_{i=1}^n$. These are clearly linearly independent and span and so constitute a basis. Therefore, $\dim((A/\mathfrak{m}) \otimes_A A^n) = n$ and the result follows from the result for vector spaces.

   If $\phi : A^n \mapsto A^m$ is surjective, since the tensor functor is right exact, $\mathrm{Id} \otimes \phi : (A/\mathfrak{m}) \otimes_A A^n \mapsto (A/\mathfrak{m}) \otimes_A A^m$ is also surjective. From our knowledge of vector spaces, this implies that $n \geq m$.

   It is proven with a bit of work in Hungerford's text that every commutative ring with identity has the invariant dimension property. For an injective map $\phi : A^n \mapsto A^m$, ϕ induces an isomorphism of A^n with a submodule of A^m. By isomorphism invariance of dimension and monotonicity of dimension, we necessarily have that $n \leq m$.

12. Let M be a finitely generated A-module and $\phi : M \mapsto A^n$ be surjective. We may write $\{e_i\}_{i=1}^n$ as a generating set for A^n and $\{u_i\}_{i=1}^n$ such that $\phi(u_i) = e_i$. Let $(u_i)_{i=1}^n$ be the submodule of M generated by the u_i and $N = \ker\phi \oplus (u_i)_{i=1}^n$. Consider the following diagram with rows exact and φ defined by $\varphi(k,u) = k+u$.

$$\begin{array}{c} 0 \to \ker\phi \xrightarrow{i'} N \xrightarrow{\phi\circ\pi} A^n \to 0 \\ \downarrow \mathrm{Id} \qquad \downarrow \varphi \qquad \downarrow \mathrm{Id} \\ 0 \to \ker\phi \xrightarrow{i} M \xrightarrow{\phi} A^n \to 0 \end{array}$$

By the five lemma, the map φ is then an isomorphism so that $M \simeq \ker\phi \oplus (u_i)_{i=1}^n$. If we consider the surjective projection map onto $\ker\phi$, the images of the u_i then generate $\ker\phi$. Therefore, $\ker\phi$ is finitely generated.

13. Let $f : A \mapsto B$ be a ring homomorphism and N a B-module considered as an A-module (via f). Consider $N_B = B \otimes_A N$ and $g : N \mapsto N_B$ defined by $g(y) = 1 \otimes y$. To see that this is injective, define a new map $p : N_B \mapsto N$ defined by $p(b\otimes y) = by$ (well-defined is easy to see). Notice that $p \circ g = \mathrm{Id}_N$ so that g has a left-inverse. This immediately implies that g is injective. Now consider the short exact sequence

$$0 \to \ker p \xrightarrow{i} N_B \xrightarrow{g\circ p} \operatorname{Im} g \to 0.$$

The latter map is surjective since p is surjective. Notice that this short exact sequence is actually split exact with the map from $\operatorname{Im}(g) \mapsto N_B$ being just the inclusion map (since $g\circ p \circ i = \mathrm{Id}_{\operatorname{Im} g}$). Therefore, $N_B \simeq \ker p \oplus \operatorname{Im} g$.

14. There isn't really anything to do for this problem.

15. Let $\pi : \oplus M_i \mapsto \lim M_i = M$ be the projection map. For $x \in M$, we may write

$$x = \pi\left(\sum_{i\in I} x_i\right),$$

where $x_i \in M_i$ (considering M_i as part of the direct sum of the M_i) only finitely many $x_i \neq 0$. Let $j \in I$ be such that $i \leq j$ for every $i \in I$ such that $x_i \neq 0$. We then have

$$x = \pi\left(\sum_{i\leq j} x_i\right) = \pi\left(\sum_{i\leq j} \mu_{ij}(x_i)\right) = \mu_j\left(\sum_{i\leq j} \mu_{ij}(x_i)\right),$$

since the difference

$$\sum_{i\leq j} x_i - \sum_{i\leq j}\mu_{ij}(x_i) = \sum_{i\leq j}(x_i - \mu_{ij}(x_i)) \in D = \ker\pi$$

and $\sum_{i\leq j}\mu_{ij}(x_i) \in M_j$.

If $\mu_i(x) = 0$, then $x \in \ker\mu_i = D \cap M_i$. Written as an element of $D \subseteq \oplus M_i$, we have

$$x = \sum_{k=1}^n (x_{i_k} - \mu_{i_k j_k}(x_{i_k})).$$

Here, we consider various reductions. The easiest being, we may assume that $x_{i_k} \neq 0$ for each index i_k and $j_k \neq i_k$ for any k. We may assume that each $i_k \neq i_l$ for $k \neq l$ since we may add them together otherwise. We may assume that $i \leq i_k$ for each k. To see this, consider the set of minimal elements i_l in $\{i_k\}$ (since this set is finite, they exist). Since $x \in M_i$, the M_{i_l} component must vanish for $i_l \neq i$, but the only way this is possible is if $x_{i_l} = 0$ since each i_l appears in the sum only once and i_l is minimal, there are no elements of the form $x_{i_k} - \mu_{i_k i_l}(x_{i_k})$. Therefore, i is the only minimal element, which implies that $i \leq i_k$ for each index i_k (otherwise, consider a maximal descending chain. It must end since there are finitely many i_k. i must be the minimal elelment and so $i \leq i_k$). At this point, we can reduce to the case that $i_k = i$ for all k. For this, consider $\{i_k\}_{i_k\neq i}$ and the minimal elements of this set as before. If i_l is minimal in this set, since the i_l coordinate vanishes, either $x_{i_l} = 0$ or $x_{i_l} = \mu_{ii_l}(x_i)$ (since $i \leq i_l$ is the only one less than i_l), in which case, we may write the following.

$$x_{i_l} - \mu_{i_l j_l}(x_{i_l}) = \mu_{ii_l}(x_i) - \mu_{ij_l}(x_i) = (x_i - \mu_{ij_l}(x_i)) - (x_i - \mu_{ii_l}(x_i))$$

Since the indexing set is finite and there is always minimal elements of the set above, this process is finite and so terminates. Therefore, we can change the sum so that $i_k = i$ for all k. We now have

$$x = \sum_{k=1}^{n} (x_i - \mu_{ij_k}(x_i)).$$

Since each j_k appears only once and the j_k component must vanish, we then have that $\mu_{ij_k}(x_i) = 0$ for each k in the sum. In particular, $x = nx_i$ so that for any $j = j_k$, we have

$$\mu_{ij}(x) = n\mu_{ij}(x_i) = 0.$$

16. Let (M_i, μ_{ij}) be a directed system as in the previous problems, $M = \lim M_i$ be the direct limit, and N be an module such that for all indices i, there exists $\alpha_i : M_i \mapsto N$ such that $\alpha_i = \alpha_j \circ \mu_{ij}$. By the universal property of the direct sum, there exists a unique homomorphism $\phi : \oplus M_i \mapsto N$ defined by

$$\phi((m_i)_{i \in I}) = \sum_{i \in I} \alpha_i(m_i).$$

Notice for $x_i \in M_i$ and $i \leq j$,

$$\phi(x_i - \mu_{ij}(x_i)) = \alpha_i(x_i) - (\alpha_j \circ \mu_{ij})(x_i) = \alpha_i(x_i) - \alpha_i(x_i) = 0.$$

Therefore, $D \subseteq \ker \phi$ (where $D = \ker \pi$ and $\pi : \oplus M_i \mapsto M$) and so there is an induced map $\alpha : M = (\oplus M_i)/D \mapsto N$. This map makes the following diagram commute for each index i.

$$\begin{tikzcd}
M_i \arrow[d, "\iota_i"'] \arrow[dr, "\alpha_i"] & \\
\oplus M_i \arrow[r, "\phi"] \arrow[d, "\pi"'] & N \\
M \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

Since $\mu_i = \pi \circ \iota_i$, we have that $\alpha_i = \alpha \circ \mu_i$ as desired. For uniqueness, assume there exists another map $\beta : M \mapsto N$ such that $\alpha_i = \beta \circ \mu_i$ for every index $i \in I$. Since every element of M can be written in the form $\mu_i(x_i)$ for some index $i \in I$, it is easy to see that for every $m \in M$, $m = \mu_i(x_i)$,

$$\alpha(m) = (\alpha \circ \mu_i)(x_i) = \alpha_i(x_i) = (\beta \circ \mu_i)(x_i) = \beta(m).$$

Therefore, uniqueness follows. From this, our definition of the direct limit satisfies this condition.

Let N be another module satisfying the above condition (which implicitly includes maps $\eta_i : M_i \mapsto N$). Since there are maps $\mu_i : M_i \mapsto M$ and $\eta_i : M_i \mapsto N$, there are induced maps $\phi : N \mapsto M$ and $\varphi : M \mapsto N$ such that for all $i \in I$,

$$\mu_i = \phi \circ \eta_i, \;\; \eta_i = \varphi \circ \mu_i.$$

Consider the maps $\mu_i : M_i \mapsto M$. By the universal property, there is a map $\mathrm{Id}_M : M \mapsto M$, that satisfies

$$\mu_i = \mathrm{Id}_M \circ \mu_i.$$

However, using the relations above, $\mu_i = \phi \circ \varphi \circ \mu_i$. Therefore, $\phi \circ \varphi = \mathrm{Id}_M$. Similarly, $\varphi \circ \phi = \mathrm{Id}_N$. Therefore, $M \simeq N$ as desired.

17. Let $\{M_i\}_{i \in I}$ be a collection of submodules of some A-module with I ordered by $i \leq j$ if $M_i \subseteq M_j$, in which case, let $\mu_{ij} : M_i \mapsto M_j$ is the inclusion map. Assume for every $i, j \in I$, there exists k such that $i \leq k$ and $j \leq k$ so that I is directed upwards. Then define $M = \lim M_i$. Let $\alpha_i : M_i \mapsto \sum M_i$ be the inclusion map.

Clearly, $\alpha_i = \alpha_j \circ \mu_{ij}$ and so by the universal property of direct limits, there exists a map $\alpha : M \mapsto \sum M_i$ such that $\alpha_i = \alpha \circ \mu_i$. From the description above, α is defined exactly by

$$\alpha\left([(m_i)_{i\in I}]\right) = \sum_{i\in I} \alpha_i(m_i) = \sum_{i\in I} m_i.$$

Define another map $\phi : \sum M_i \mapsto M$ by

$$\phi\left(\sum_{i\in S} m_i\right) = \sum_{i\in S} \mu_i(m_i),$$

where $S \subseteq I$ is finite and $m_i \in M_i$. To see that this is well-defined, let $\sum_{i\in S} m_i = \sum_{j\in T} n_j$ where both indexing sets are finite and $m_i \in M_i$ and $n_j \in M_j$. Take an index $k \in I$ such that $S, T \le k$ and notice

$$\sum_{i\in S} \mu_i(m_i) = \sum_{i\in S} \mu_k(\mu_{ik}(m_i)) = \mu_k\left(\sum_{i\in S} \mu_{ik}(m_i)\right) = \mu_k\left(\sum_{i\in S} m_i\right) = \mu_k\left(\sum_{j\in T} n_j\right)$$
$$= \mu_k\left(\sum_{j\in T} \mu_{jk}(n_j)\right) = \sum_{j\in T} \mu_k(\mu_{jk}(n_j)) = \sum_{j\in T} \mu_j(n_j).$$

It's easy to see that ϕ is a homomorphism and that these maps are inverses and so α is an isomorphism. As for the latter equality, notice first that $\cup M_i \subseteq \sum M_i$. Conversely, for a finite subset $S \subseteq I$ and

$$m = \sum_{i\in S} m_i \in \sum M_i,\ m_i \in M_i,$$

there exists some upper bound $S \le j$ so that $\sum_{i\in S} M_i \subseteq M_j$ (by induction on $|S|$). Then $m \in M_j \subseteq \cup M_i$ so that $\sum M_i \subseteq \cup M_i$. Combining inclusions, we have that $\sum M_i = \cup M_i$.

18. Let (M_i, μ_{ij}) and (N_i, ν_{ij}) be two directed systems over the directed set I with limits $M = \lim M_i$ and $N = \lim N_i$. Let $\Phi : \mathbf{M} \mapsto \mathbf{N}$ be a family of homomorphisms $\phi_i : M_i \mapsto N_i$ such that $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$. Consider the maps $\alpha_i = \nu_i \circ \phi_i : M_i \mapsto N$. The satisfy

$$\alpha_j \circ \mu_{ij} = \nu_j \circ \phi_j \circ \mu_{ij} = \nu_j \circ \nu_{ij} \circ \phi_i = \nu_i \circ \phi_i = \alpha_i.$$

Therefore, there is an induced map $\phi : M \mapsto N$ that satisfies $\alpha_i = \phi \circ \mu_i$. That is,

$$\nu_i \circ \phi_i = \phi \circ \mu_i.$$

Uniqueness again follows similar to uniqueness of the universal property of direct limits. If there are two such functions, they take the same values since the range of the family of maps μ_i covers M.

19. Let (M_i, μ_{ij}), (N_i, ν_{ij}), and (P_i, ω_{ij}) be directed systems over the same directed set I, $f_i : M_i \mapsto N_i$ and $g_i : N_i \mapsto P_i$ be module homomorphisms such that $f_j \circ \mu_{ij} = \nu_{ij} \circ f_i$ and $g_j \circ \nu_{ij} = \omega_{ij} \circ g_i$, and

$$0 \to M_i \xrightarrow{f_i} N_i \xrightarrow{g_i} P_i \to 0$$

be exact for every $i \in I$. From the previous problem, there are induced module homomorphisms $f : M \mapsto N$ and $g : N \mapsto P$, where $M = \lim M_i$, $N = \lim N_i$, and $P = \lim P_i$. Notice that for every $i \in I$, these maps satisfy

$$g \circ f \circ \mu_i = g \circ \nu_i \circ f_i = \omega_i \circ g_i \circ f_i = 0.$$

Since the range of the family of maps μ_i covers M, we necessarily have that $g \circ f = 0$, so that $\operatorname{Im} f \subseteq \ker g$. To show that this is in fact an equality, let $(n_i)_{i\in I} \in \ker g$. We may write $(n_i)_{i\in I} = \nu_i(n_i)$ for some $n_i \in M_i$. Then

$$0 = g((n_i)_{i\in I}) = (g \circ \nu_i)(n_i) = (\omega_i \circ g_i)(n_i) = \omega_i(g_i(n_i)).$$

This implies there exists $i \le j$ such that

$$0 = \omega_{ij}(g_i(n_i)) = (\omega_{ij} \circ g_i)(n_i) = (g_j \circ \nu_{ij})(n_i) = g_j(\nu_{ij}(n_i)).$$

Therefore, $\nu_{ij}(n_i) \in \ker g_j = \operatorname{Im} f_j$. From this, we may write $\nu_{ij}(n_i) = f_j(m_j)$ for some $m_j \in M_j$. We see

$$f(\mu_j(m_j)) = (\nu_j \circ f_j)(m_j) = \nu_j(\nu_{ij}(n_i)) = \nu_i(n_i) = (n_i)_{i\in I}.$$

Therefore, $(n_i)_{i\in I} \in \operatorname{Im} f$ and so $\operatorname{Im} f = \ker g$ as desired. Therefore, the induced sequence

$$M \xrightarrow{f} N \xrightarrow{g} P$$

is exact as well.

20. Let (M_i, μ_{ij}) be a directed system over I with limit $M = \lim M_i$. Consider the new system $(M_i \times N, \mu_{ij} \times \mathrm{Id})$. If there are maps $\alpha_i : M_i \times N \mapsto Q$, then define $\alpha : M \times N \mapsto Q$ by

$$\alpha((m_i)_{i\in S}, n) = \sum_{i\in S} \alpha_i(m_i, n).$$

This map is easily verified to be a module homomorphism. Notice that $\alpha_i = \alpha \circ (\mu_i \times \mathrm{Id})$ so that $M \times N \simeq \lim(M_i \times N)$ by the universal property.

Now consider the directed system $(M_i \otimes N, \mu_{ij} \otimes \mathrm{Id})$ with limit $P = \lim(M_i \otimes N)$. By the family of maps $\mu_i \otimes \mathrm{Id} : M_i \otimes N \mapsto M \otimes N$, there is an induced homomorphism $\phi : P \mapsto M \otimes N$ defined by

$$\phi([(m_i \otimes n_i)_{i\in S}]) = \sum_{i\in S} \mu_i(m_i) \otimes n_i.$$

Consider the maps $g_i : M_i \times N \mapsto M_i \otimes N$ simply defined by $g_i(m_i, n) = m_i \otimes n$. Since

$$(\mu_{ij} \otimes \mathrm{Id}) \circ g_i = g_j \circ (\mu_{ij} \times \mathrm{Id}),$$

there is an induced map of limits $g : M \times N \mapsto P$ defined on by

$$g([(m_i)_{i\in I}], n) = [(m_i \otimes n)_{i\in I}].$$

This map is A-bilinear and so induces a A-module homomorphism $\varphi : M \otimes N \mapsto P$ defined on elementary tensors by

$$\varphi([(m_i)_{i\in I}] \otimes n) = [(m_i \otimes n)_{i\in I}].$$

We see for an elementary tensor

$$(\phi \circ \varphi)([(m_i)_{i\in S}] \otimes n) = \phi\left([(m_i \otimes n)_{i\in S}]\right) = \sum_{i\in S} \mu_i(m_i) \otimes n = [(m_i)_{i\in S}] \otimes n.$$

Therefore, $\phi \circ \varphi = \mathrm{Id}$. Similarly,

$$(\varphi \circ \phi)([m_i \otimes n_i]_{i\in S}) = \varphi\left(\sum_{i\in S} \mu_i(m_i) \otimes n_i\right) = \sum_{i\in S} [(\mu_i(m_i) \otimes n_i)_{i\in I}] = [m_i \otimes n_i]_{i\in S}.$$

Therefore, $\varphi \circ \phi = \mathrm{Id}$ and so $P \simeq M \otimes N$. That is,

$$\varinjlim(M_i \otimes N) = \left(\varinjlim M_i\right) \otimes N.$$

21. Let (A_i, α_{ij}) be a family of rings indexed by the directed set I. As $\mathbb{Z}$-modules, we may form the limit $\mathbb{Z}$-module $A = \lim A_i$. Define multiplication on A as follows. For $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$, there exists $i, j \in I$ and $a_i \in A_i$ and $b_j \in A_j$ such that $(a_i)_{i\in I} = \alpha_i(a_i)$ and $(b_i)_{i\in I} = \alpha_j(b_j)$. Define

$$(a_i)_{i\in I} \cdot (b_i)_{i\in I} = \alpha_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j))$$

for any $i, j \leq k$. To see that this is well-defined, assume $\alpha_i(a_i) = \alpha_k(a_k)$ and $\alpha_j(b_j) = \alpha_l(b_l)$. We would like to show

$$\alpha_m(\alpha_{im}(a_i)\alpha_{jm}(b_j)) = \alpha_n(\alpha_{kn}(a_k)\alpha_{ln}(b_l)).$$

Using that $\alpha_c = \alpha_d \circ \alpha_{cd}$ and that α_{cd} is a ring homomorphism for all $c, d \in I$, we may find an upper bound $m, n \leq p$ to reduce to showing that

$$\alpha_p(\alpha_{ip}(a_i)\alpha_{jp}(b_j)) = \alpha_p(\alpha_{kp}(a_k)\alpha_{lp}(b_l)).$$

Notice that similar to above, we can find an upper bound $i, k \leq e$ so that $\alpha_e(\alpha_{ie}(a_i)) = \alpha_e(\alpha_{ke}(a_k))$, which implies that $\alpha_e(\alpha_{ie}(a_i) - \alpha_{ke}(a_k)) = 0$. Then there exists an index f such that $\alpha_{if}(a_i) = \alpha_{kf}(a_k)$. Similarly, we can find an index g such that $\alpha_{jg}(b_j) = \alpha_{lg}(b_l)$. Using the reduction above and finding an upper bound of f and g, the result follows immediately.

This turns A into a commutative ring with identity $\alpha_i(1)$ (for any i, since $\alpha_i(1) = \alpha_j(1)$ (since ring maps preserve the identity by definition)). It is clear that

$$\alpha_i(a_1a_2) = \alpha_i(a_1)\alpha_i(a_2)$$

by definition of multiplication (with upper bound i). Therefore, the maps $\alpha_i : A_i \mapsto A$ are ring homomorphisms.

Now assume that $A = 0$. Then the identity $\alpha_i(1) = 0$, which implies there exists $i \leq j$ such that $\alpha_{ij}(1) = 0$. Since α_{ij} is a ring homomorphism, $0 \in A_j$ is the identity, which implies that $A_j = 0$.

22. Let (A_i, α_{ij}) be a directed system of rings over I, $\mathfrak{N}_i \subseteq A_i$ be the nilradical of each ring, and $\mathfrak{N} \subseteq A$ be the nilradical of the limit. It is clear that $\lim \mathfrak{N}_i \subseteq \mathfrak{N}$ since if we write an element in the form $\alpha_i(a_i)$, it is clear that it is nilpotent and so is in $\mathfrak{N}$. Conversely, for $(a_i)_{i\in I} \in \mathfrak{N}$, we may write $(a_i)_{i\in I} = \alpha_i(a_i)$ for some index $i \in I$ and $a_i \in A_i$. Since this is nilpotent, for some $n > 0$, we have

$$0 = ((a_i)_{i\in I})^n = (\alpha_i(a_i))^n = \alpha_i(a_i^n).$$

Then there exists $i \leq j$ such that $0 = \alpha_{ij}(a_i^n) = (\alpha_{ij}(a_i))^n$. Therefore, $\alpha_{ij}(a_i) \in \mathfrak{N}_j$ and $\alpha_i(a_i) = \alpha_j(\alpha_{ij}(a_i)) \in \lim \mathfrak{N}_i$. Therefore, $\mathfrak{N} = \lim \mathfrak{N}_i$.

Assume A_i is an integral domain for each $i \in I$. Let $\alpha_i(a_i)\alpha_j(a_j) = 0$ for some $i, j \in I$ and $a_i \in A_i$ and $a_j \in A_j$. By definition of multiplication in A, for $i, j \leq k$,

$$0 = \alpha_i(a_i)\alpha_j(a_j) = \alpha_k(\alpha_{ik}(a_i)\alpha_{jk}(a_j)).$$

Therefore, there exists an index $l \in I$ such that

$$0 = \alpha_{kl}(\alpha_{ik}(a_i)\alpha_{jk}(a_j)) = \alpha_{il}(a_i)\alpha_{jl}(a_j).$$

Since A_l is an integral domain, without loss of generality, we may assume that $\alpha_{il}(a_i) = 0$. This then implies that

$$\alpha_i(a_i) = \alpha_l(\alpha_{il}(a_i)) = 0.$$

Therefore, A is an integral domain as well.

23. Let $(B_\lambda)_{\lambda\in\Lambda}$ be a family of A-algebras with respective maps $f_\lambda : A \mapsto B_\lambda$. For a finite subset $J \subseteq \Lambda$, let B_J denote the tensor product of B_λ for $\lambda \in J$. Then $(B_J, i_{JJ'})_{J\subseteq\Lambda}$ is a directed system of rings (where $i_{JJ'}$ is the inclusion map $i_{JJ'} : B_J \mapsto B_{J'}$). Let $B = \lim B_J$ be the direct limit of this system (it is the tensor product of the family $(B_\lambda)_{\lambda\in\Lambda}$). From the preceding problems, B is a ring and there are maps $i_J : B_J \mapsto B$ that are ring homomorphisms. Define an A-algebra structure on B as follows. Any element of B can be written in the form $i_J((\otimes b_\lambda)_{\lambda\in J})$ for some index J. For $a \in A$, let a act on $i_J((\otimes b_\lambda)_{\lambda\in J})$ as follows.

$$ai_J((\otimes b_\lambda)_{\lambda\in J}) = i_J(a(\otimes b_\lambda)_{\lambda\in J})$$

That is, a acts on an element of B by considering them elements of B_J and acting on B_J in the normal way as the tensor product of A-algebras. This is well-defined since for $i_J((\otimes b_\lambda)_{\lambda\in J})$ and $i_{J'}((\otimes b_\lambda)_{\lambda\in J'})$, we may do the normal thing and find an upper bound $J, J' \subseteq K$, in which case, the action of A coincides and so is well-defined. Now, it is clear that the maps $i_J : B_J \mapsto B$ are A-algebra homomorphisms by definition of the action.

24. $(i) \implies (ii)$ Let M be a flat A-module. Consider a free (projective) resolution

$$\ldots \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to N \to 0.$$

Since M is flat and the tensor functor is right exact, tensoring this exact sequence with M results in another exact sequence,

$$\ldots \to P_n \otimes M \to P_{n-1} \otimes M \to \ldots \to P_1 \otimes M \to P_0 \otimes M \to N \otimes M \to 0.$$

Since the torsion modules $\mathrm{Tor})_i(M, N)$ are equal to the homology modules of this sequence, they are all zero since this sequence is exact.

$(ii) \implies (iii)$ This is immediate.

$(iii) \implies (i)$ Let M be such that for all A-modules N, $\mathrm{Tor}_1(M, N) = 0$. Consider a short exact sequence

$$0 \to N' \mapsto N \to N'' \to 0.$$

Tensoring this sequence with modules from a free (projective) resolution of M, we get a complex of short exact sequences. By the snake lemma, there is the induced long exact sequence

$$\ldots \to \mathrm{Tor}_{i+1}(M, N'') \to \mathrm{Tor}_i(M, N') \to \mathrm{Tor}_i(M, N) \to \mathrm{Tor}_i(M, N'') \to \ldots.$$

Since $\mathrm{Tor}_0(M, N) = M \otimes N$, we then have an exact sequence

$$\mathrm{Tor}_1(M, N'') \to M \otimes N' \to M \otimes N \to M \otimes N'' \to 0.$$

Since $\mathrm{Tor}_1(M, N'') = 0$, it follows that the sequence obtained by tensoring the original exact sequence with M is exact. Since N, N', N'' were arbitrary, it follows that M is flat.

25. Let

$$0 \to N' \to N \to N'' \to 0$$

be an exact sequence with N'' flat. From the Tor exact sequence, the following sequence is exact (since we can take the tensor product on either side, the Tor exact sequence can come from either $\mathrm{Tor}(M, -)$ or $\mathrm{Tor}(-, M)$).

$$\mathrm{Tor}_2(N'', M) \to \mathrm{Tor}_1(N', M) \to \mathrm{Tor}_1(N, M) \to \mathrm{Tor}_1(N'', M).$$

Since N'' is flat, $\mathrm{Tor}_i(N'', M) = 0$ for all A-modules M. The above then becomes

$$0 \to \mathrm{Tor}_1(N', M) \to \mathrm{Tor}_1(N, M) \to 0$$

for any A-module M. From this, if N is flat, $\mathrm{Tor}_1(N, M) = 0$ and so from above, $\mathrm{Tor}_1(N', M) = 0$ so that N' is flat. Similarly, if N' is flat, then N is flat.

26. Let N be a A-module. From the previous problems, if N is flat, then $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ for all finitely generated ideals $\mathfrak{a}$ of A (since $\mathrm{Tor}_1(M, N) = 0$ for all A-modules M).

Assume now that $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$ for all finitely generated ideals $\mathfrak{a}$ of A. It suffices to show that $\mathrm{Tor}_1(M, N) = 0$ for all finitely generated A-modules M. To see this, take an injective map $f : M' \mapsto M$, this a short exact sequence

$$0 \to M' \to M \to M'' = M/\mathrm{Im}\, f \to 0.$$

From the Tor exact sequence, we have

$$\mathrm{Tor}_1(M'', N) \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$$

is exact. Since $\mathrm{Tor}_1(M'', N) = 0$, the induced map $f \otimes \mathrm{Id} : M' \otimes N \mapsto M \otimes N$ is injective. Since M and M' were arbitrary, this would imply that N is flat.

It further suffices to show that $\mathrm{Tor}_1(M, N) = 0$ for all cyclic A-modules M. To see this, let M be a finitely generated A-module generated by $(x_i)_{i=1}^n$ and let $M_j = (x_i)_{i=1}^j$. Without loss of generality, we may assume that $x_{i+1} \notin M_i$ by changing our generating set. Clearly, $[x_{i+1}]$ generates M_{i+1}/M_i so this module is cyclic. Consider the exact sequence

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0.$$

The Tor exact sequence induces the exact sequence

$$\mathrm{Tor}_1(M_{i-1}, N) \to \mathrm{Tor}_1(M_i, N) \to \mathrm{Tor}_1(M_i/M_{i-1}, N).$$

Assuming that $\mathrm{Tor}_1(M_i/M_{i-1}, N) = 0$, we can use induction to show that $\mathrm{Tor}_1(M_i, N) = 0$ (since clearly, $\mathrm{Tor}_1(M_1, N) = 0$ in this case). Letting $i = n$, we have that $\mathrm{Tor}_1(M, N) = 0$. Since M was an arbitrary finitely generated A-module, from the above, it follows that N is flat.

Before anything else, a small result. Let (M_i, μ_{ij}) be the collection of finitely generated submodules of M ordered under inclusion with maps $\mu_{ij} : M_i \mapsto M_j$ inclusion maps. Assume the maps $\phi_i : M_i \mapsto N$ are injective and satisfy $\phi_i = \phi_j \circ \mu_{ij}$. There is then an induced map $\phi : M = \lim M_i \mapsto N$. Any element of M can be written as $\mu_i(m_i)$ for some $i \in I$ and $m_i \in M_i$. If $0 = \phi(\mu_i(m_i)) = \phi_i(\mu_i(m_i))$, then $\mu_i(m_i) = 0$ so that ϕ is injective.

Now let $\mathfrak{a}$ be an ideal of A such that $A/\mathfrak{a}$ is cyclic. Consider the directed system $(\mathfrak{a}_i, \mu_{ij})$ of finitely generated submodules (sub-ideals) of $\mathfrak{a}$ under inclusion as above and the new direct system $(\mathfrak{a}_i \otimes N, \mu_{ij} \otimes \mathrm{Id})$. The maps $\mathfrak{a}_i \otimes N \hookrightarrow A \otimes N$ are injective since the sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ induces the following from the Tor exact sequence.

$$\mathrm{Tor}_1(A/\mathfrak{a}) \to \mathfrak{a}_i \otimes N \to A \otimes N$$

Therefore, the induced map $\mathfrak{a} \otimes N = \lim(\mathfrak{a}_i \otimes N) \mapsto A \otimes N$ is injective as well. The exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ then induces an exact sequence

$$\mathrm{Tor}_1(A/\mathfrak{a}, N) \to \mathfrak{a} \otimes N \to A \otimes N.$$

Since the latter map is injective, we must have that $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$. Therefore, $\mathrm{Tor}_1(M, N) = 0$ for all cyclic A-modules M, which implies that N is flat.

27. $(i) \implies (ii)$ Let A be absolutely flat and (x) be a principal ideal of A. Consider the diagram

$$\begin{tikzcd}
(x) \otimes A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & (x) \otimes A/(x) \arrow[d, "\alpha"] \\
A \arrow[r, "\pi"'] & A/(x)
\end{tikzcd}$$

where $\gamma(x' \otimes a) = x'a$, $\beta = \mathrm{Id} \otimes \pi$, and α is defined as follows. Consider the injective function $i : (x) \to A$. Since $A/(x)$ is a flat A-module, the function $\alpha = i \otimes \mathrm{Id} : (x) \otimes A/(x) \to A \otimes A/(x) = A/(x)$ is injective as well. It is easy to see that this diagram commutes since π is a ring homomorphism and so respects multiplication (or just write it all out, the compositions are equal to zero). Since the composition $\pi \circ \gamma = 0$, the composition $\alpha \circ \beta = 0$ as well. Since α is injective, this implies that $\beta = 0$. Since π is surjective, $\beta = \mathrm{Id} \circ \pi$ is surjective and equal to zero. Therefore, $(x) \otimes A/(x) = 0$ identically. Again consider the exact sequence $0 \to (x) \to A \to A/(x) \to 0$. Tensoring with (x), we get the exact sequence

$$0 \to (x) \otimes (x) \to (x) \otimes A \to (x) \otimes A/(x) \to 0.$$

Since $(x) \otimes A/(x) = 0$, this is equivalent to the exact sequence

$$0 \to (x) \otimes (x) \to (x) \otimes A \to 0.$$

Therefore, the induced map $\phi = \mathrm{Id} \circ i : (x) \otimes (x) \mapsto (x) \otimes A$ is an isomorphism. Consider the composition $(x) \otimes (x) \to (x) \otimes A \to (x)$. Each map is an isomorphism, so the resulting map $(x) \otimes (x) \to (x)$ is an isomorphism, but this map is defined exactly by $\phi(x_1 \otimes x_2) = x_1 x_2 \in (x^2)$. In particular, this map is surjective, which implies that $x \in (x^2)$ so that $(x) = (x^2)$.

$(ii) \implies (iii)$ Let $\mathfrak{a} = (a_1, \ldots, a_n)$ be a finitely generated ideal of A. Note that for each $1 \leq i \leq n$, there exists b_i such that $a_i = b_i a_i^2$. From this, we have a new generating set $\mathfrak{a} = (b_1 a_1, \ldots, b_n a_n)$, where each element of the generating set is idempotent. Write $e_i = b_i a_i$ so that $\mathfrak{a} = (e_1, \ldots, e_n)$ (and each e_i is idempotent. Notice that we may decrease the number of generators as follows, $\mathfrak{a} = (e_1, \ldots, e_{n-2}, e_{n-1} + e_n - e_{n-1}e_n)$. It is clear that this is still a generating set because each e_i is idempotent. We may further assume that this new element $e_{n-1} + e_n - e_{n-1}e_n$ is idempotent from the above process. Therefore, $\mathfrak{a} = (e)$ is principal and generated by an idempotent. Since $A = (1) = (e) + (1-e)$ where $(e) \cap (1-e) = \{0\}$, we may write $A = \mathfrak{a} \oplus (1-e)$.

$(iii) \implies (i)$ Let A be such that every finitely generated ideal of A is a direct summand of A for some decomposition of A and let N be an arbitrary A-module. We need only show that N is flat. From the previous problem, this amounts to showing that for all finitely generated ideals $\mathfrak{a}$ of A, $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$. For any such finitely generated ideal, the quotient $A/\mathfrak{a}$ is isomorphic to the other summand. That is, $A = \mathfrak{a} \oplus A/\mathfrak{a}$. Consider part of the Tor exact sequence corresponding to $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$.

$$\mathrm{Tor}_1(A/\mathfrak{a}, N) \to \mathfrak{a} \otimes N \to A \otimes N = (\mathfrak{a} \oplus A/\mathfrak{a}) \otimes N = (\mathfrak{a} \otimes N) \oplus (A/\mathfrak{a} \otimes N)$$

The latter map is exactly the inclusion map $\mathfrak{a} \otimes N \hookrightarrow (\mathfrak{a} \otimes N) \oplus (A/\mathfrak{a} \otimes N)$. Since this is injective and the above sequence is exact, we have that $\mathrm{Tor}_1(A/\mathfrak{a}, N) = 0$. Therefore, N is flat and so A is absolutely flat.

28. In a Boolean ring, every element is idempotent so that every principal ideal is idempotent. If A is a ring such that for every $x \in A$ there exists $n > 1$ such that $x^n = x$, then $x = x^{n-2}x^2 \in (x^2)$ implies that $(x) = (x^2)$ so that every principal ideal is idempotent and therefore, A is absolutely flat. If A is absolutely flat, $\phi : A \mapsto B$ is surjective, and $x \in B$, there exists $y \in A$ such that $\phi(y) = x$ and $a \in A$ such that $y = ay^2$. Then $x = \phi(y) = \phi(a)\phi(y)^2 = \phi(a)x^2$ so that $(x) = (x^2)$ and so A is absolutely flat. If A is a local ring that is absolutely flat, then for every $x \in A$, $x = ax^2$ for some $a \in A$. Therefore, ax is idempotent, which implies that either $ax = 0$ or $ax = 1$. If $ax = 0$, then $x = ax^2 = 0$. Otherwise, x is invertible. Therefore, A is a field.

    Let A be absolutely flat and $x \in A$ be a non-unit. Since $x(1 - ax) = 0$ for some $a \in A$ and $ax \neq 1$, we have that x is a zero-divisor.

Chapter 3

1. For any A-module M, if there exists $s \in S$ such that $sM = 0$, then $S^{-1}M = 0$ since for any $m/r \in S^{-1}M$,

$$\frac{m}{r} = \frac{ms}{rs} = \frac{0}{rs} = 0.$$

    Conversely, if $M = (x_1, \ldots, x_n)$ is finitely generated and $S^{-1}M = 0$, then $x_i/1 = 0$ implies there exists $s_i \in S$ such that $x_i s_i = 0$. Let $s = \prod_{i=1}^n s_i$. We have

$$x_i s = (x_i s_i) \prod_{j \neq i} s_j = 0.$$

    Since these elements generate M, we have that $sM = 0$.

2. Let $\mathfrak{a}$ be an ideal of A and $S = 1 + \mathfrak{a}$. For any maximal ideal $\mathfrak{m}$ of $S^{-1}A$, $\mathfrak{m} = \mathfrak{b}^e = S^{-1}\mathfrak{b}$ for some ideal $\mathfrak{b}$ of A. Since this ideal is not equal to all of $S^{-1}A$, $\mathfrak{b} \cap S = \mathfrak{b} \cap (1 + \mathfrak{a}) = \emptyset$. This implies that $(\mathfrak{a} + \mathfrak{b}) \cap (1 + \mathfrak{a}) = \emptyset$ so that $S^{-1}\mathfrak{a} + S^{-1}\mathfrak{b} = S^{-1}(\mathfrak{a} + \mathfrak{b}) \neq (1)$. By maximality of $S^{-1}\mathfrak{b} = \mathfrak{m}$, we then necessarily have that $S^{-1}\mathfrak{a} \subseteq \mathfrak{m}$. Since $\mathfrak{m}$ was arbitrary, this implies that $S^{-1}\mathfrak{a} \subseteq \mathfrak{J}$.

    Let M be a finitely generated A-module and $\mathfrak{a}$ an ideal of A such that $\mathfrak{a}M = M$. Let $S = 1 + \mathfrak{a}$ as above and notice

$$S^{-1}M = S^{-1}(\mathfrak{a}M) = (S^{-1}\mathfrak{a})(S^{-1}M).$$

    From the above, $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical so that from Nakayama's lemma, we have that $S^{-1}M = 0$. From the first problem, this implies there exists $1 + a \in S = 1 + \mathfrak{a}$ such that $(1 + a)M = 0$. In this case, $1 + a \equiv 1 \bmod \mathfrak{a}$ as desired.

3. Let S and T be two multiplicatively closed subsets of A, ST be the product set (which is also multiplicative), and U be the image of T under the inclusion map $i : A \mapsto S^{-1}A$. Consider the map $\phi : S^{-1}A \mapsto (ST)^{-1}A$ defined by

$$\phi(a/s) = a/(s1).$$

It is very easy to check that this map is a ring homomorphism. For $t/1 \in U$, we have that $\phi(t/1) = t/1$ is a unit since $1/t \in (ST)^{-1}A$. If

$$\phi(a/s) = a/(s1) = 0,$$

then there exists $s't' \in ST$ such that $as't' = 0$. From this, we have

$$(a/s)(t'/1) = at'/s = 0,$$

since $as't' = 0$. Therefore, there exists $t'/1 \in U$ such that $(a/s)(t'/1) = 0$. Finally, for $a/(st) \in (ST)^{-1}A$, we may write

$$a/(st) = (a/s)(t/1)^{-1} = \phi(a/s)(\phi(t/1))^{-1}.$$

From this, the induced map $\widetilde{\phi} : U^{-1}(S^{-1}A) \mapsto (ST)^{-1}A$ defined by

$$\widetilde{\phi}((a/s)/(t/1)) = \phi(a/s)(\phi(t/1))^{-1} = a/st$$

is an isomorphism.

4. Let $f : A \mapsto B$ be a ring homomorphism, $S \subseteq A$ a multiplicatively closed subset, and $T = f(S)$ (which is also multiplicatively closed). First, B is an A-module with scalar multiplication given by $a \cdot b = f(a)b$. Then $S^{-1}B$ is an $S^{-1}A$-module with multiplication given by $(a/s) \cdot (b/s') = (f(a)b)/(ss')$. Similarly, $T^{-1}B$ is an $S^{-1}A$-module with multiplication given by $(a/s) \cdot (b/f(s')) = (f(a)b)/f(ss')$. Define a map $\phi : S^{-1}B \mapsto T^{-1}B$ defined by

$$\phi(b/s) = b/f(s).$$

This is well-defined since for $b/s = b'/s' \in S^{-1}B$, there exists $s \in S$ such that $(bs' - b's)s = (bf(s') - b'f(s))f(s) = 0$. It is easily verified to be an $S^{-1}A$-linear map. If $\phi(b/s) = 0$, then there exists $f(s') \in T$ such that $bf(s') = bs' = 0$. But this implies that $b/s = 0$ so ϕ is injective. It is clear that this map is surjective (by finding a common denominator). Therefore, ϕ is an isomorphism.

5. Let A be a ring such that for every prime ideal $\mathfrak{p}$, the nilradical of $A_\mathfrak{p}$ is trivial. Since $S^{-1}(r(\mathfrak{a})) = r(S^{-1}\mathfrak{a})$ and the nilradical is literally the radical of the zero ideal, we have that the nilradical of $A_\mathfrak{p}$ is the localization of the nilradical of A. That is, $\mathfrak{N}_\mathfrak{p}$ is the nilradical of $A_\mathfrak{p}$. Since $\mathfrak{N}_\mathfrak{p} = 0$ for all prime ideals $\mathfrak{p}$, $\mathfrak{N} = 0$ (considering it as an A-module).

   Consider the ring $A = \mathbb{Z}/6\mathbb{Z}$. This ring has exactly two prime ideals, (2) and (3). Notice

$$A_{(2)} = \{m/n : m \in A,\ n \in \{1, 3, 5\}\}.$$

   It is easy to see that $m/n = 0$ iff $m \in \{0, 2, 4\}$. If $m_1/n_1, m_2/n_2 \neq 0$ are such that $m_1m_2/n_1n_2 = 0$, then $m_1m_2 \in \{0, 2, 4\}$, but $m_1, m_2 \in \{1, 3, 5\}$. This is not possible. Therefore, $A_{(2)}$ is an integral domain. Similarly,

$$A_{(3)} = \{m/n : m \in A,\ n \in \{1, 2, 4, 5\}\}.$$

   $m/n = 0$ iff $m \in \{0, 3\}$. If $m_1/n_1, m_2/n_2 \neq 0$ are such that their product is zero, then $m_1m_2 \in \{0, 3\}$, but $m \in \{1, 2, 4, 5\}$. This is also impossible. Therefore, $A_{(3)}$ is an integral domain. Since $A_\mathfrak{p}$ is an integral domain for every prime ideal $\mathfrak{p}$ of A, but A isn't an integral domain, the result does not hold.

6. Let A be a nonzero ring and Σ the set of multiplicatively closed subsets of A such that $0 \notin S$ ordered under inclusion. By Zorn's lemma, to show there are maximal elements of Σ, it suffices to show any increasing chain has an upper bound. Let $S_1 \subseteq S_2 \subseteq \ldots$ and let $S = \cup S_i$. Since $1 \in S_i$ for all i, $1 \in S$. For $a, b \in S$, there is an index i sufficiently large so that $a, b \in S_i$. Then $ab \in S_i \subseteq S$ so that $ab \in S$. Therefore S is multiplicative and an upper bound of this increasing chain. By Zorn's lemma, maximal elements of Σ exist.

   Let $S \in \Sigma$ be maximal and $\mathfrak{p} = A \setminus S$. If $a + b = s \in S$, consider the multiplicative sets $S(a^n)_{n \geq 0}$ and $S(b^n)_{n \geq 0}$. If $0 \in S(a^n)_{n \geq 0} \cap S(b^n)_{n \geq 0}$, then there exists $s_1, s_2 \in S$ and $n, m \geq 0$ such that

$$s_1a^n = s_2b^m = 0.$$

We then have that

$$0 = s_1 s_2 (a+b)^{n+m} = s_1 s_2 s^{n+m} \in S.$$

This is a contradiction. Therefore, without loss of generality, $0 \notin S(a^n)_{n\geq 0}$. By maximality of S, this then implies that $S(a^n)_{n\geq 0} = S$ so that $a \in S$. That is, $a+b \in S$ implies either $a \in S$ or $b \in S$. The contrapositive of this is that $a, b \in \mathfrak{p}$ implies $a+b \in \mathfrak{p}$. Similarly, let $a \in \mathfrak{p}$ be such that there exists $r \in A$ such that $ra = s \in S$. If $0 \in S(a^n)_{n\geq 0}$ then there exists $s_1 \in S$ and $n \geq 0$ such that $s_1 a^n = 0$. Then

$$0 = s_1 r^n a^n = s_1 s^n \in S.$$

This is a contradiction. Therefore, again by maximality, $a \in S$. The contrapositive of this is that if $a \in \mathfrak{p}$, then $ra \in \mathfrak{p}$ for all $r \in A$. Combining these, we have that $\mathfrak{p}$ is an ideal of A. The contrapositive of the $x, y \in S \implies xy \in S$ exactly states that $\mathfrak{p}$ is a prime ideal. Clearly, if $\mathfrak{p}$ is not a minimal prime ideal, then $S = A \setminus \mathfrak{p}$ is not maximal. Therefore, $A \setminus S = \mathfrak{p}$ is a minimal prime ideal.

Conversely, assume that $A \setminus S = \mathfrak{p}$ is a minimal prime ideal of A. Then S is contained in some maximal $S' \in \Sigma$. From the above, $A \setminus S' = \mathfrak{q}$ for some minimal prime ideal $\mathfrak{q}$ of A. Since $S \subseteq S'$, $\mathfrak{q} \subseteq \mathfrak{p}$. By minimality of $\mathfrak{p}$, $\mathfrak{q} = \mathfrak{p}$ so that $S = S'$ and S is maximal.

7a. If $A \setminus S$ is a union of prime ideals, $\cup \mathfrak{p}_i$, then the requirement for saturation is given via contrapositives by

$$xy \in \bigcup \mathfrak{p}_i \iff x \in \bigcup \mathfrak{p}_i \text{ or } y \in \bigcup \mathfrak{p}_i.$$

Both directions are fairly immediate. If $xy \in \cup \mathfrak{p}_i$, then $xy \in \mathfrak{p}_i$ for some i, which implies, without loss of generality, that $x \in \mathfrak{p}_i \subseteq \cup \mathfrak{p}_i$. Conversely, if $x \in \cup \mathfrak{p}_i$, then $x \in \mathfrak{p}_i$ for some i. Since $\mathfrak{p}_i$ is an ideal, $xy \in \mathfrak{p}_i \subseteq \cup \mathfrak{p}_i$. Therefore, S is saturated.

If S is saturated and $x \notin S$, then $rx \notin S$ for all $r \in A$. This implies that $(x) \cap S \neq \emptyset$ so that $S^{-1}(x) \neq (1)$. This implies that $S^{-1}(x) \subseteq \mathfrak{p}$ for some prime ideal $\mathfrak{p}$. Then $(x) \subseteq (S^{-1}(x))^c \subseteq \mathfrak{p}^c$ where $\mathfrak{p}^c$ is prime. By the correspondence of ideals of A with $S^{-1}A$, $\mathfrak{p}^c \cap S = \emptyset$. Therefore, $\mathfrak{p}^c \subseteq A \setminus S$. Since every element of $A \setminus S$ is contained in some prime ideal which is contained in $A \setminus S$, we may write $A \setminus S$ as the union of these prime ideals.

7b. Let S be a multiplicatively closed subset of A. From the above, if $\{\mathfrak{p}_i\}$ is the set of prime ideals such that $\mathfrak{p}_i \cap S = \emptyset$, then

$$\overline{S} = A \setminus \left(\bigcup \mathfrak{p}_i \right) = \bigcap (A \setminus \mathfrak{p}_i)$$

is saturated and $S \subseteq \overline{S}$. If there is another saturated set S' such that $S \subseteq S'$, then we may write $A \setminus S' = \cup \mathfrak{q}_i$ for some prime ideals $\mathfrak{q}_i$. Since $S \subseteq S'$, $A \setminus S' \subseteq A \setminus S$ implies that $\mathfrak{q}_i \cap S = \emptyset$ for each i. Therefore, $A \setminus S' = \cup \mathfrak{q}_i \subseteq \cup \mathfrak{p}_i = A \setminus \overline{S}$. This then implies that $\overline{S} \subseteq S'$ so that $\overline{S}$ is minimal.

8. $(i) \implies (ii)$ If ϕ is bijective, there exists $a/s \in S^{-1}A$ such that $\phi(a/s) = 1/t$. Then $\phi(a/s)\phi(t/1) = 1$ implies that $(a/s)(t/1) = 1$ (by injectivity). Therefore, $t/1$ is a unit in $S^{-1}A$ for all $t \in T$.

$(ii) \implies (i)$ Let $a/t \in T{-}1A$. Since $t/1$ is a unit in $S^{-1}A$, there exists $a'/s \in S^{-1}A$ such that $(a'/s)(t/1) = 1$. Then $\phi(a'/s)\phi(t/1) = 1$ implies that $\phi(a'/s) = 1/t$. Therefore, $\phi(aa'/s) = \phi(a/1)\phi(a'/s) = a/t$. Therefore, ϕ is surjective. If $\phi(a/s) = 0$, there exists $t \in T$ such that $at = 0$. This implies that $(a/s)(t/1) = 0$. Since $t/1$ is a unit, $a/s = 0$ so ϕ is injective.

$(ii) \implies (iii)$ $(a/s)(t/1) = 1$ implies there exists $s' \in S$ such that $as't = ss' \in S$. Let $x = as'$.

$(iii) \implies (ii)$ If $xt = s$ for some $x \in A$ and $s \in S$, then $(x/s)(t/1) = 1$ so $t/1$ is a unit in $S^{-1}A$.

$(iii) \implies (iv)$ For $t \in T$, let $x \in A$ be such that $xt \in S \subseteq \overline{S}$. Then $t \in \overline{S}$ since $\overline{S}$ is saturated. Therefore, $T \subseteq \overline{S}$.

$(iv) \implies (ii)$ If $T \subseteq \overline{S}$ and $t \in T$, then $t/1 \in S^{-1}A$ is not in any prime ideal since

$$\emptyset = (A \setminus \overline{S}) \cap T = \left(\bigcup_{\substack{\mathfrak{p} \subseteq A \text{ prime} \\ \mathfrak{p} \cap S = \emptyset}} \mathfrak{p} \right) \cap T = \left(\bigcup_{\mathfrak{p} \subseteq S^{-1}A \text{ prime}} \mathfrak{p}^c \right) \cap T.$$

Therefore, $t/1$ is a unit.

$(iv) \implies (v)$ If $T \subseteq \overline{S}$, then $\overline{T} \subseteq \overline{S}$ and $A \setminus \overline{S} \subseteq A \setminus \overline{T}$. Writing these sets as their respective union of prime ideals, this shows that for a prime ideal $\mathfrak{p}$, $\mathfrak{p} \cap S = \emptyset \implies \mathfrak{p} \cap T = \emptyset$. The contrapositive of this is exactly $\mathfrak{p} \cap T \neq \emptyset \implies \mathfrak{p} \cap S \neq \emptyset$.

$(v) \implies (iv)$ From above, the contrapositive of (v) is $\mathfrak{p} \cap S = \emptyset \implies \mathfrak{p} \cap T = \emptyset$. Therefore, the set of prime ideals that do not intersect S is a subset of the set of prime ideals that do not intersect T. That is, $A \setminus \overline{S} \subseteq A \setminus \overline{T}$. This is equivalent to $\overline{T} \subseteq \overline{S}$. Then $T \subseteq \overline{T} \subseteq \overline{S}$.

9a. As shown in a previous problem, the set of zero-divisors is a union of prime ideals. Therefore, its complement S_0 of non-zero-divisors is a saturated multiplicative set. For any $s \in S_0$, $(s_0^n)_{n \geq 0}$ is a multiplicative set and hence, for any maximal multiplicative set $S = A \setminus \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, $0 \in S(s_0^n)_{n \geq 0}$ or $S(s_0^n)_{n \geq 0} = S$ so that $s_0 \in S = A \setminus \mathfrak{p}$. If $0 \in S(s_0^n)_{n \geq 0}$, there exists $a \in S = A \setminus \mathfrak{p}$ and $n \geq 0$ such that $0 = as_0^n = (as_0^{n-1})s_0$, which would imply that $s_0 \in D$, but $s_0 \in S$. So we must have that $s_0 \in A \setminus \mathfrak{p}$. Since $\mathfrak{p}$ was an arbitrary minimal prime ideal, we have that

$$S_0 \subseteq \bigcap_{\mathfrak{p} \text{ minimal}} (A \setminus \mathfrak{p}) = A \setminus \left(\bigcup_{\mathfrak{p} \text{ minimal}} \mathfrak{p} \right).$$

Taking complements, we have that every minimal prime ideal is contained in D.

Let $i : A \mapsto S_0^{-1}A$ be the inclusion map and $i(a) = a/1 = 0$. Then there exists $s \in S_0$ such that $as = 0$. Since s is not a zero-divisor, this immediately implies that $a = 0$ so that i is injective.

Assume that $S_0 \subset S$ then there exists some zero-divisor $\eta \in S$. Let $\mu \in A$ be such that $\eta\mu = 0$. We have

$$i(\mu) = \frac{\mu}{1} = \frac{\mu\eta}{\eta} = \frac{0}{\eta} = 0.$$

Therefore, $i : A \mapsto S^{-1}A$ is not injective. This shows that S_0 is maximal in this respect.

9b. It is easy to see that $a/s_0 = 0 \in S_0^{-1}A$ if and only if $a = 0$. Therefore, $a/s_0 \in S_0^{-1}A$ is a zero-divisor if and only if $a \in A$ is a zero-divisor. If $a \in A$ is not a zero-divisor, then $a \in S_0$ so that $a/s \in S_0^{-1}A$ is a unit (with inverse s/a).

9c. Let A be a ring such that every non-unit is a zero-divisor. Then S_0 is exactly the set of units. Any element of $S_0^{-1}A$ can be written as $a/s = as^{-1}/1$. The map $\phi(as^{-1}/1) = as^{-1}$ is an inverse to the inclusion map $i : A \mapsto S_0^{-1}A$ (it is easily seen to be a ring homomorphism). Therefore, $A \simeq S_0^{-1}A$ in this case.

10a. Let A be absolutely flat and S any multiplicative subset. Consider a principal ideal (x/s) of $S^{-1}A$. Since $x \in A$, $x = ax^2$ for some $a \in A$ (since $(x) = (x^2)$). Therefore,

$$\frac{x}{s} = \frac{ax^2}{s} = \left(\frac{as}{1}\right)\left(\frac{x}{s}\right)^2.$$

This implies that $(x/s) = ((x/s)^2)$. Since x/s was arbitrary, this implies that $S^{-1}A$ is absolutely flat.

10b. If A is absolutely flat and $\mathfrak{m}$ is a maximal ideal, then from the previous problem, $A_\mathfrak{m}$ is absolutely flat as well. Since $A_\mathfrak{m}$ is a local ring, from a previous problem, this implies that $A_\mathfrak{m}$ is a field.

Assume $A_\mathfrak{m}$ is a field for every maximal ideal $\mathfrak{m}$. For any A-module M, $M_\mathfrak{m}$ is a $A_\mathfrak{m}$ module. Since $A_\mathfrak{m}$ is a field, $M_\mathfrak{m}$ is isomorphic to a direct sum of copies of $A_\mathfrak{m}$ and is therefore flat by a previous problem.

11. $(i) \implies (ii)$ If $A/\mathfrak{N}$ is absolutely flat. Since $\mathfrak{N} \subseteq \mathfrak{p}$ for all prime ideals $\mathfrak{p}$, there is an induced map $\widetilde{\pi} : A/\mathfrak{N} \mapsto A/\mathfrak{p}$ (that is surjective). Since the image of an absolutely flat ring is absolutely flat, $A/\mathfrak{p}$ is absolutely flat. Therefore, every element of $A/\mathfrak{p}$ is a non-unit is a zero-divisor. Since $A/\mathfrak{p}$ is an integral domain, this then implies every nonzero element is a unit so that $A/\mathfrak{p}$ is a field and so $\mathfrak{p}$ is maximal.

$(ii) \implies (i)$ (This was taken from the internet) Notice first that if every prime ideal of A is maximal, every prime ideal of $A/\mathfrak{N}$ is maximal. Therefore, $A' = A/\mathfrak{N}$ is a ring with no nilpotents such that every prime ideal is maximal. Fix $x \in A'$ and define $S = \{x^n(1 + ax) : n \geq 0,\ a \in A'\}$. If $0 \notin S$, then we can compute $S^{-1}A'$ and find some prime ideal of it. Then it is of the form $S^{-1}\mathfrak{p}$ for some prime ideal $\mathfrak{p} \cap S = \emptyset$. Since either $x \in \mathfrak{p}$

or $1-ax \in \mathfrak{p}$ by maximality of $\mathfrak{p}$, we have a contradiction. Therefore, $0 \in S$. Therefore, there exists $n \geq 0$ and $a \in A'$ such that

$$x^n(1-ax) = 0.$$

Therefore, $x(1-ax)$ is nilpotent and equal to zero. This shows that $(x) = (x^2)$ so that A' is absolutely flat.

$(ii) \implies (iii)$ This is immediate. For every $\mathfrak{p} \in \operatorname{Spec}(A)$, we have

$$\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = \{\mathfrak{p}\}.$$

$(iii) \implies (ii)$ This is also immediate. For every $\mathfrak{p} \in \operatorname{Spec}(A)$,

$$\{\mathfrak{p}\} = \overline{\{\mathfrak{p}\}} = V(\mathfrak{p}).$$

Therefore, $\mathfrak{p}$ is maximal.

$(i) \implies (iv)$ Since $\operatorname{Spec}(A)$ is homeomorphic to $\operatorname{Spec}(A/\mathfrak{N})$, it suffices to show that $X = \operatorname{Spec}(A/\mathfrak{N})$ is Hausdorff. Let $\mathfrak{p}, \mathfrak{q} \in X$ be distinct. Since they are distinct, without loss of generality, there exists $x \in \mathfrak{p}$ and $x \notin \mathfrak{q}$. Then there exists $a \in A/\mathfrak{N}$ such that $x = ax^2$. This implies that $a \notin \mathfrak{q}$ since otherwise, $x \in \mathfrak{q}$. Therefore, we may replace x by ax and the above holds. That is, we may assume that x is idempotent. Notice that $x(1-x) = 0$ so that $1-x \in \mathfrak{q}$ (since $x \notin \mathfrak{q}$). Similarly, both cannot lie in the same prime ideal, since $x + (1-x) = 1$. Therefore, $\mathfrak{p} \in X_{1-x}$, $\mathfrak{q} \in X_x$ and $X_{1-x} \cap X_x = X_{x(1-x)} = \emptyset$. Therefore, X and $\operatorname{Spec}(A)$ are Hausdorff.

$(iv) \implies (iii)$ This is immediate by definition.

If the above hold, then $\operatorname{Spec}(A)$ is Hausdorff and quasi-compact and so is compact. To see that $\operatorname{Spec}(A)$ is totally disconnected, it suffices to show that $\operatorname{Spec}(A/\mathfrak{N})$ is totally disconnected. Using the notation from $(iii) \implies (iv)$, we have that $x(1-x) = 0$ so that every prime ideal contains either x or $(1-x)$, but not both. That is, every prime ideal is in either X_x or X_{1-x}. Then for any open set with at least two elements, we can find a separation of this set in this way. Therefore, the only connected sets are singletons.

12a. Let A be an integral domain and M an A-module. It is easy to check that $T(M)$ is a submodule of M (since A is an integral domain, $\operatorname{ann}(x) \cap \operatorname{ann}(y) \neq \{0\}$ for $\operatorname{ann}(x) \neq \{0\} \neq \operatorname{ann}(y)$). Consider the quotient $M/T(M)$ and let $x + T(M) \in M/T(M)$. If there exists $a \in A$ such that $a(x + T(M)) = ax + T(M) = 0$, then $ax \in T(M)$, which implies there exists $b \in M$ such that $b(ax) = (ba)x = 0$. Therefore, $x \in T(M)$ and so $M/T(M)$ has no torsion elements.

12b. Let $f : M \mapsto N$ be a module homomorphism. For $x \in T(M)$, there exists $a \in A$ such that $ax = 0$. Then $f(a)f(x) = f(ax) = 0$ implies that $f(x) \in T(N)$. Therefore, $f(T(M)) \subseteq T(N)$.

12c. Let

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

be exact and consider the induced sequence

$$0 \to T(M') \xrightarrow{\widetilde{f}} T(M) \xrightarrow{\widetilde{g}} T(M'').$$

Since $\widetilde{f}$ and $\widetilde{g}$ are restrictions of the original functions, we have that $\widetilde{f}$ is injective and $\widetilde{g} \circ \widetilde{f} = 0$ so that $\operatorname{Im} \widetilde{f} \subseteq \ker \widetilde{g}$. Let $m \in \ker \widetilde{g}$. Considered as an element of M, $m \in \ker g$ so that $m \in \operatorname{Im} f$. That is, $m = f(m')$ for some $m' \in M'$. Let $a \in A$ be such that $am = 0$. Then

$$0 = am = af(m') = f(am').$$

By injectivity of f, we have that $am' = 0$ so that $m' \in T(M')$ and so $m \in \operatorname{Im} \widetilde{f}$. Therefore, the induced sequence is exact.

12d. Let $\phi : M \mapsto k \otimes_A M$ be the map $\phi(x) = 1 \otimes x$, where $k = (A \setminus \{0\})^{-1}A$. There is an isomorphism $k \otimes_A M \mapsto (A \setminus \{0\})^{-1}M$ defined by $a/s \otimes m = am/s$. Then the kernel of the composition is exactly the kernel of the original map since the latter is an isomorphism. The composition is given exactly by $m \mapsto m/1$. Therefore, $m \in \ker \phi$ iff there exists $s \in A \setminus \{0\}$ such that $sm = 0$. That is, if $m \in T(M)$. Conversely, it is easy to see that $T(M) \subseteq \ker \phi$. Therefore, $\ker \phi = T(M)$.

13. First, it is easy to consider both $S^{-1}(T(M))$ and $T(S^{-1}M)$ as A-submodules of $S^{-1}M$. For $m/s \in T(S^{-1}M)$, there exists $a \in A$ such that $(a/1)(m/s) = 0$. That is, there exists $s' \in S$ such that $s'am = 0$ and so $m \in T(M)$ so that $m/s \in S^{-1}(T(M))$. Conversely, for $m/s \in S^{-1}(T(M))$, there exists $a \in A$ such that $am = 0$ so that $(a/1)(m/s) = 0$. Therefore, $m/s \in T(S^{-1}M)$. Combining inclusions, we have that $S^{-1}(T(M)) = T(S^{-1}M)$.

    $(i) \implies (ii)$ This is immediate from the above. If $T(M) = \{0\}$, then $T(M_{\mathfrak{p}}) = (T(M))_{\mathfrak{p}} = 0$.

    $(ii) \implies (iii)$ This is even more immediate.

    $(iii) \implies (i)$ If $M_{\mathfrak{m}}$ is torsion-free for every maximal ideal $\mathfrak{m}$ of A, then

    $$(T(M))_{\mathfrak{m}} = T(M_{\mathfrak{m}}) = 0$$

    for all maximal ideals $\mathfrak{m}$ of A. Since being equal to zero is a local property, we have that $T(M) = 0$.

14. Let M be an A-module and $\mathfrak{a}$ an ideal of A such that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{a} \subseteq \mathfrak{m}$. For such maximal ideals, we have

    $$(M/\mathfrak{a}M)_{\mathfrak{m}} \simeq M_{\mathfrak{m}}/(\mathfrak{a}M)_{\mathfrak{m}} = 0.$$

    For a maximal ideal $\mathfrak{m}$ such that $\mathfrak{a} \not\subseteq \mathfrak{m}$, there exists $a \in \mathfrak{a}$ such that $a \notin \mathfrak{m}$. For any $(m + \mathfrak{a}M)/s \in (M/\mathfrak{a}M)_{\mathfrak{m}}$, we have

    $$\frac{m + \mathfrak{a}M}{s} = \frac{a}{a}\frac{m + \mathfrak{a}M}{s} = \frac{am + \mathfrak{a}M}{as} = 0.$$

    Therefore, $(M/\mathfrak{a}M)_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$. This implies that $M/\mathfrak{a}M = 0$ and so $M = \mathfrak{a}M$.

15. Let A be a ring, $F = A^n$, $(e_1, \ldots, e_n)$ be the canonical basis of F, and $(x_1, \ldots, x_n)$ be a generating set of F. Define a map $\phi : F \mapsto F$ defined by $\phi(e_i) = x_i$ and extend linearly so that ϕ is surjective. To show that ϕ is injective, it suffices to consider the case that A is a local ring, since then $\phi_{\mathfrak{m}} : F_{\mathfrak{m}} \mapsto F_{\mathfrak{m}}$ is injective for every maximal ideal $\mathfrak{m}$ of A and hence, ϕ itself is injective. Let $N = \ker \phi$, $\mathfrak{m}$ be the unique maximal ideal of A, and $k = A/\mathfrak{m}$. Consider the exact sequence

    $$0 \to N \xrightarrow{i} F \xrightarrow{\phi} F \to 0.$$

    Since the tensor functor is right exact,

    $$k \otimes N \xrightarrow{\mathrm{Id} \otimes i} k \otimes F \xrightarrow{\mathrm{Id} \otimes \phi} k \otimes F \to 0$$

    is exact as well. These maps then may be considered as k-module homomorphisms so that the latter map is a surjective map between vector spaces of dimension k and is therefore injective. This implies that $k \otimes N = 0$. By previous problems, N is finitely-generated and $k \otimes N \simeq N/\mathfrak{m}N = 0$. Since A is local, $\mathfrak{m} \subseteq \mathfrak{J}$ (in fact, equal to) and so by Nakayama's lemma, we have that $N = 0$. Therefore, ϕ is injective and hence, an isomorphism.

    If $k < n$ and $x_1, \ldots, x_k \in F$, then extend this set by adjoining $e_1, \ldots$ if necessary to get a generating set $(e_{i_1}, \ldots, e_{i_{n-k}}, x_1, \ldots, x_k)$ with n elements. Then $\phi : F \mapsto F$ defined above is an isomorphism. Notice that $\phi(e_1) = e_{i_1}$ and ϕ is injective. Therefore, there is no element

    $$\phi\left(\sum_{i > n-k} a_i e_i\right) = \sum_{i=1}^{k} a_i x_i = e_{i_1}.$$

    Therefore, the x_i do not span F.

16. $(i) \implies (ii)$ this exactly proposition 3.16 from the text.

    $(ii) \implies (iii)$ For a maximal ideal $\mathfrak{m} \subseteq A$, write $\mathfrak{m} = \mathfrak{p}^c$ for some $\mathfrak{p} \subseteq B$ prime. Then

    $$\mathfrak{m}^e = \mathfrak{p}^{ce} \subseteq \mathfrak{p} \subset (1).$$

    $(iii) \implies (iv)$ For M a nonzero A-module, fix any $x \neq 0$ in M and let $M' = Ax$. Since the inclusion $i : M' \mapsto M$ is injective, the resulting A-module homomorphism $\mathrm{Id} \otimes i : B \otimes_A M' \mapsto B \otimes_A M$ is injective. This map can easily be regarded as a B-module homomorphism and so is a B-module homomorphism $i' : M'_B \mapsto M_B$. Since i' is injective, it suffices to show that $M'_B \neq 0$. Note that $M' \simeq A/\mathfrak{a}$ where $\mathfrak{a} = \operatorname{ann}(x) \neq (1)$. Therefore,

$$M'_B = B \otimes_A A/\mathfrak{a} \simeq B/\mathfrak{a}B = B/\mathfrak{a}^e.$$

Since $\mathfrak{a} \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, $\mathfrak{a}^e \subseteq \mathfrak{m}^e \subset (1)$. Therefore, $\mathfrak{a}^e \neq B$ so that $M'_B \neq 0$. This implies that $M_B \neq 0$.

$(iv) \implies (v)$ Let $\phi : M \mapsto M_B$ be the inclusion map $i(m) = 1 \otimes m$ (considering M_B as an A-module). Consider the exact sequence

$$0 \to \ker \phi \xrightarrow{i} M \xrightarrow{\phi} M_B \to 0.$$

Since B is flat, the induced sequence of B-modules is exact.

$$0 \to (\ker \phi)_B \xrightarrow{i'} M_B \xrightarrow{\phi'} (M_B)_B \to 0$$

From a previous problem, the inclusion map $M \mapsto M_B$ is injective. Since B is flat, the induced map $M_B \mapsto (M_B)_B$ is injective as well. This implies that $(\ker \phi)_B = 0$. Using (iv), this implies that $\ker \phi = 0$ so that ϕ is injective.

$(v) \implies (i)$ Let $\mathfrak{a}$ be an ideal of A and $M = A/\mathfrak{a}$. Then the map $\phi : M \mapsto M_B$ is injective. That is, the map

$$A/\mathfrak{a} \mapsto B \otimes_A A/\mathfrak{a} \simeq B/\mathfrak{a}B = B/\mathfrak{a}^e$$

is injective. Let $a \in \mathfrak{a}^{ec}$ and consider the composition $f : A \mapsto A/\mathfrak{a} \mapsto B/\mathfrak{a}^e$. Then $f(a) \in \mathfrak{a}^{ece} = \mathfrak{a}^e$ and so $f(a) = 0$ so that $a \in \mathfrak{a}$. Therefore, $\mathfrak{a}^{ec} = \mathfrak{a}$. Since $\mathfrak{a}$ was arbitrary, the result follows.

17. A ring homomorphism $f : A \mapsto B$ is flat if B with the corresponding A-algebra structure is flat (resp. faithfully flat). Let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be ring homomorphisms such that $g \circ f$ is flat and g is faithfully flat. Let M and N be A-modules and $\phi : M \mapsto N$ be an A-module homomorphism. Since M_B and N_B are B-modules and C is a faithfully flat B-module, we have the inclusion map $M_B \mapsto (M_B)_C$ is injective. That is, the map

$$B \otimes_A M \to C \otimes_B (B \otimes_A M) = (C \otimes_B B) \otimes_A M = C \otimes_A M$$

is injective. This map is clearly given by $g \otimes \mathrm{Id}_M$. Similarly, the map $g \otimes \mathrm{Id}_N : B \otimes_A N \mapsto C \otimes_A N$ is injective. Similarly, since C is a flat A-module, the map $\mathrm{Id}_C \otimes \phi : C \otimes_A M \mapsto C \otimes_A N$ is injective. Finally, we have a map $\mathrm{Id}_B \otimes \phi : B \otimes_A M \mapsto B \otimes_A N$. These maps clearly satisfy

$$(\mathrm{Id}_C \otimes \phi) \circ (g \otimes \mathrm{Id}_M) = (g \otimes \mathrm{Id}_N) \circ (\mathrm{Id}_B \otimes \phi).$$

Since the right hand side is the composition of injective maps, it is injective. Therefore, the right hand side is injective, which implies that $\mathrm{Id}_B \otimes \phi$ is injective. Therefore, B is a flat A-module and so the map f is flat.

18. If $f : A \mapsto B$ is a flat map, $\mathfrak{q} \subseteq B$ prime and $\mathfrak{p} = \mathfrak{q}^c \subseteq A$ prime. Then B is a flat A-module so that $B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module. Therefore, the map $f_\mathfrak{p} : A_\mathfrak{p} \mapsto B_\mathfrak{p}$ is flat. From problem 12, $B_\mathfrak{p} \simeq (f(A \setminus \mathfrak{p}))^{-1}B$ where $f(A \setminus \mathfrak{p}) \subseteq A \setminus \mathfrak{q}$ so that we may consider $B_\mathfrak{p} \subseteq B_\mathfrak{q}$. Since $\mathfrak{q}$ generates a prime ideal in $B_\mathfrak{q}$, it generates a prime (maximal) ideal in $B_\mathfrak{p}$ (consider the inclusion map between them). Localizing at this ideal, we get exactly $B_\mathfrak{q}$ so that $B_\mathfrak{q}$ is a localization of $B_\mathfrak{p}$. From this, the map $i : B_\mathfrak{p} \mapsto B_\mathfrak{q}$ is flat ($B_\mathfrak{q}$ is flat as a $B_\mathfrak{p}$-module since it is a localization). It is easy to see that the composition of flat maps is flat. Therefore, $B_\mathfrak{q}$ is a flat $A_\mathfrak{p}$-module. The only maximal ideal of $A_\mathfrak{p}$ is generated by $\mathfrak{p}$. We have that the extension of this ideal in $B_\mathfrak{q}$ is contained in the maximal ideal generated by $\mathfrak{q}$ and therefore, not equal to (1). This implies that $B_\mathfrak{q}$ is a faithfully flat $A_\mathfrak{p}$-module so that the map $\operatorname{Spec}(B) \mapsto \operatorname{Spec}(A)$ is surjective.

19a. This is obvious from the fact that $M = 0$ if and only if $M_\mathfrak{p} = 0$ for every prime ideal $\mathfrak{p}$ of A.

19b. Notice for an ideal $\mathfrak{a}$ of A and a prime ideal $\mathfrak{p}$,

$$(A/\mathfrak{a})_\mathfrak{p} = A_\mathfrak{p} \otimes_A (A/\mathfrak{a}) \simeq A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p} = A_\mathfrak{p}/\mathfrak{a}^e.$$

Therefore, if $\mathfrak{a} \subseteq \mathfrak{p}$, then $\mathfrak{a}^e \subseteq \mathfrak{p}^e \subset (1)$. Therefore, the quotient above is nonzero and so $\mathfrak{p} \in \mathrm{supp}(A/\mathfrak{a})$. Conversely, if $\mathfrak{p} \in \mathrm{supp}(A/\mathfrak{a})$, then the quotient above is nonzero so that $\mathfrak{a}^e \subset (1)$. Since $\mathfrak{a}^e$ is contained in some maximal ideal and there is only one, $\mathfrak{a}^e \subseteq \mathfrak{p}^e$. Then $\mathfrak{a} \subseteq \mathfrak{a}^{ec} \subseteq \mathfrak{p}^{ec} = \mathfrak{p}$ by the prime ideal correspondence with localizations. Therefore, $\mathfrak{p} \in V(\mathfrak{a})$. Therefore, $V(\mathfrak{a}) = \mathrm{supp}(A/\mathfrak{a})$.

19c. Let

$$0 \to M' \to M \to M'' \to 0$$

be exact. Then the induced sequence

$$0 \to M'_\mathfrak{p} \to M_\mathfrak{p} \to M''_\mathfrak{p} \to 0$$

is exact as well. From this it is clear that $M_\mathfrak{p} \neq 0$ if and only if $M'_\mathfrak{p} \neq 0$ or $M''_\mathfrak{p} \neq 0$ (that is, if they are both zero, so is M and vice versa). Therefore, $\mathrm{supp}(M) = \mathrm{supp}(M') \cup \mathrm{supp}(M'')$.

19d. Let $M = \sum M_i$. If $M_\mathfrak{p} \neq 0$, then there exists some $m \in M$ and $s \in A \setminus \mathfrak{p}$ such that $m/s \neq 0$. We may write $m = m_{i_1} + \ldots + m_{i_n}$ for some $m_{i_j} \in M_{i_j}$. Then at least one of m_{i_j}/s is nonzero since otherwise, $m/s = 0$. Therefore, $(M_{i_j})_\mathfrak{p} \neq \emptyset$. Conversely, if $(M_{i_j})_\mathfrak{p} \neq 0$ for some i_j, then since the inclusion map $M_{i_j} \hookrightarrow M$ is injective, so is the induced map $(M_{i_j})_\mathfrak{p} \hookrightarrow M_\mathfrak{p}$. Therefore, $M_\mathfrak{p} \neq 0$. Combining these inclusions, we have that $\mathrm{supp}(\sum M_i) = \cup \mathrm{supp}(M_i)$.

19e. If $\mathfrak{p}$ is such that $\mathrm{ann}(M) \not\subseteq \mathfrak{p}$, then there exists $a \in \mathrm{ann}(M)$ such that $a \notin \mathfrak{p}$. Then in $M_\mathfrak{p}$,

$$\frac{m}{s} = \frac{am}{as} = 0.$$

Therefore, $M_\mathfrak{p} = 0$. Conversely, if $M_\mathfrak{p} = 0$, then $1/1 = 0$ in $M_\mathfrak{p}$. That is, there exists $a \in A \setminus \mathfrak{p}$ such that $a1 = 0$. Then for all $m \in M$, $am = a1m = 0$. Therefore, $a \in \mathrm{ann}(M)$. That is, $\mathrm{ann}(M) \not\subseteq \mathfrak{p}$. Taking contrapositives, we have that $M_\mathfrak{p} \neq 0$ if and only if $\mathrm{ann}(M) \subseteq \mathfrak{p}$. That is, $\mathrm{supp}(M) = V(\mathrm{ann}(M))$.

19f. Let M and N be finitely generated. Let $\mathfrak{p}$ be such that

$$(M \otimes_A N)_\mathfrak{p} = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p} = 0.$$

From a previous problem, if and only if $M_\mathfrak{p} = 0$ or $N_\mathfrak{p} = 0$. Taking contrapositives, we have that $(M \otimes N)_\mathfrak{p} \neq 0$ if and only if $M_\mathfrak{p} \neq 0$ and $N_\mathfrak{p} \neq 0$. That is, $\mathrm{supp}(M \otimes N) = \mathrm{supp}(M) \cap \mathrm{supp}(N)$.

19g. Let M be finitely generated and $\mathfrak{a}$ a proper ideal of A. Since $M/\mathfrak{a}M = (A/\mathfrak{a}) \otimes M$, it is clear that $\mathrm{supp}(M/\mathfrak{a}M) \subseteq \mathrm{supp}(A/\mathfrak{a}) \cap \mathrm{supp}(M) = V(\mathfrak{a}) \cap \mathrm{supp}(M)$. Conversely, If $\mathfrak{p} \in V(\mathfrak{a}) \cap \mathrm{supp}(M)$, then $M_\mathfrak{p} \neq 0$ and $(\mathfrak{a}M)_\mathfrak{p} \neq M_\mathfrak{p}$ since $1 \notin (\mathfrak{a})_\mathfrak{p}$. To see this, simply write $1 = am/s$ for $s \in A \setminus \mathfrak{p}$. Then there exists $t \in A \setminus \mathfrak{p}$ such that $t(s - am) = 0$, or $st = atm \in \mathfrak{p}$, but $s, t \notin \mathfrak{p}$. Now notice

$$(M/\mathfrak{a}M)_\mathfrak{p} = M_\mathfrak{p}/(\mathfrak{a}M)_\mathfrak{p} \neq 0.$$

Therefore, the other inclusion holds as well. Using that M is finitely generated, $\mathrm{supp}(M) = V(\mathrm{ann}(M))$. Then,

$$\mathrm{supp}(M/\mathfrak{a}M) = V(\mathfrak{a}) \cap V(\mathrm{ann}(M)) = V(\mathfrak{a} \cup \mathrm{ann}(M)) = V(\mathfrak{a} + \mathrm{ann}(M)).$$

19h. Let $f : A \mapsto B$ be a ring homomorphism and M a finitely generated A-module. Using $S^{-1}M = S^{-1}A \otimes_A M$, for $\mathfrak{q} \subseteq B$ prime, we have

$$(B \otimes_A M)_\mathfrak{q} = B_\mathfrak{q} \otimes_B (B \otimes_A M) = (B_\mathfrak{q} \otimes_B B) \otimes_A M = B_\mathfrak{q} \otimes_A M.$$

If $f(\mathrm{ann}(M)) \not\subseteq \mathfrak{q}$, then there exists $a \in A$ such that $f(a) \in \mathrm{ann}(M)$ and $f(a) \notin \mathfrak{q}$. Then for any element of $B_\mathfrak{q} \otimes M$,

$$\frac{b}{s} \otimes m = \frac{f(a)b}{f(a)s} \otimes m = \left(a \cdot \frac{b}{f(a)s}\right) \otimes m = \frac{b}{f(a)s} \otimes am = 0.$$

Therefore, $B_\mathfrak{q} \otimes M = 0$. That is, $\operatorname{supp}(B \otimes M) \subseteq V(\operatorname{ann}(M)^e)$. For the converse, let $m_1, \ldots, m_n$ be a generating set for M and notice

$$\operatorname{supp}(B \otimes M) = \operatorname{supp}\left(\sum (B \otimes Am_i)\right) = \bigcup \operatorname{supp}(B \otimes Am_i).$$

Similarly,

$$V(\operatorname{ann}(M)) = (f^*)^{-1}(\operatorname{supp}(M)) = (f^*)^{-1}\left(\bigcup \operatorname{supp}(Am_i)\right) = \bigcup (f^*)^{-1}(\operatorname{supp}(Am_i)) = \bigcup V(\operatorname{ann}(Am_i)).$$

Therefore, it suffices to consider the case that M is cyclic and so equal to $Am_i = A/\mathfrak{a}$, where $\mathfrak{a} = \operatorname{ann}(m_i)$. We see

$$(B \otimes_A M)_\mathfrak{q} = (B \otimes_A A/\mathfrak{a})_\mathfrak{q} = (B/\mathfrak{a}B)_\mathfrak{q} = B_\mathfrak{q}/(\mathfrak{a}B)_\mathfrak{q}.$$

Now if $(B \otimes_A M)_\mathfrak{q} = 0$, then $(\mathfrak{a}B)_\mathfrak{q} = B_\mathfrak{q}$ so that there exists $b \in B$ and $s \in B \setminus \mathfrak{q}$ such that $(a/1)(b/s) = 1$. Therefore, there exists $t \in B \setminus \mathfrak{q}$ such that

$$t(as - b) = 0 \implies ast = bt \notin \mathfrak{q}.$$

Therefore, $ast \in \mathfrak{a}$ and $ast \notin \mathfrak{q}$. The contrapositive is if $\mathfrak{a} \subseteq \mathfrak{q}$, then $(B \otimes_A M)_\mathfrak{q} \neq 0$. Since $\mathfrak{a} = \operatorname{ann}(m_i)$, the result follows.

20a. This is exactly the equivalence $(i) \iff (ii)$ from problem 16.

20b. Let $f : A \mapsto B$ be a ring homomorphism and $f^* : \operatorname{Spec}(B) \mapsto \operatorname{Spec}(A)$ be the induced map. If every prime ideal of B is an extended ideal, consider the map (not necessarily continuous or anything) $\phi : \operatorname{Im} f \mapsto \operatorname{Spec}(B)$ by $f(\mathfrak{p}) = \mathfrak{p}^e$. This map actually sends elements of $\operatorname{Im} f$ to prime ideals of B since for any prime ideal $\mathfrak{q}$ of B, $\mathfrak{q} = \mathfrak{a}^e$ for some ideal $\mathfrak{a}$ of A. Then $f^*(\mathfrak{q}) = \mathfrak{q}^c$, and so $\phi(\mathfrak{q}^c) = \mathfrak{q}^{ce} = \mathfrak{a}^{ece} = \mathfrak{a}^e = \mathfrak{q} \in \operatorname{Spec}(B)$. Note that this computation also shows that ϕ is a (point-wise) left-inverse of f^*. Therefore, f^* is injective.

21a. Let A be a ring, $S \subseteq A$ a multiplicative subset, $\phi : A \mapsto S^{-1}A$ be the inclusion, and $\phi^* : Y \mapsto X$ be the induced map where $Y = \operatorname{Spec}(S^{-1}A)$ and $X = \operatorname{Spec}(A)$. It is clear that $\operatorname{Im} \phi^* = \{\mathfrak{p} \in X : \mathfrak{p} \cap S = \emptyset\}$ and that $S^{-1} : \operatorname{Im} \phi^* \mapsto Y$ is a two-sided inverse. All that remains to be shown is that the map S^{-1} is continuous on $\operatorname{Im} \phi^*$ in the subspace topology. One way of doing this is to show simply that ϕ^* is an closed map. This is a lot to write.

$$\begin{aligned}
\phi^*(V(\mathfrak{b})) = \phi^*(V(r(\mathfrak{b}))) = \phi^*\left(V\left(\bigcap_{\substack{\mathfrak{q} \in Y \\ \mathfrak{b} \subseteq \mathfrak{q}}} \mathfrak{q}\right)\right) \\
&= \left\{\phi^{-1}(\mathfrak{x}) : \bigcap_{\substack{\mathfrak{q} \in Y \\ \mathfrak{b} \subseteq \mathfrak{q}}} \mathfrak{q} \subseteq \mathfrak{x}\right\} \\
&= \left\{\phi^{-1}(\mathfrak{x}) : \bigcap_{\substack{\mathfrak{q} \in Y \\ \mathfrak{b} \subseteq \mathfrak{q}}} \phi^{-1}(\mathfrak{q}) \subseteq \phi^{-1}(\mathfrak{x})\right\} \\
&= \left\{\mathfrak{x} \in X : \bigcap_{\substack{\mathfrak{q} \in Y \\ \mathfrak{b} \subseteq \mathfrak{q}}} \phi^{-1}(\mathfrak{q}) \subseteq \mathfrak{x}\right\} \bigcap \operatorname{Im} \phi^* \\
&= V\left(\phi^{-1}\left(\bigcap_{\substack{\mathfrak{q} \in Y \\ \mathfrak{b} \subseteq \mathfrak{q}}} \mathfrak{q}\right)\right) \bigcap \operatorname{Im} \phi^* \\
&= V(\phi^{-1}(r(\mathfrak{b}))) \cap \operatorname{Im} \phi^* \\
&= V(r(\phi^{-1}(\mathfrak{b}))) \cap \operatorname{Im} \phi^* \\
&= V(\mathfrak{b}^c) \cap \operatorname{Im} \phi^*.
\end{aligned}$$

This set is closed in $\operatorname{Im}\phi^*$ and completes the proof. Note the use that ϕ is bijective is used in the third line.

If $f \in A$ and $S = (f^n)_{n\geq 0}$, the image of $\operatorname{Spec}(A_f)$ is X_f since the image of $\operatorname{Spec}(A_f)$ is the set of all prime ideals that do not contain f, by what was shown above.

21b. Let $f : A \mapsto B$ be a ring homomorphism, $f^* : \operatorname{Spec}(B) \mapsto \operatorname{Spec}(A)$ its induced map, $i_1 : A \mapsto S^{-1}A$ and $i_2 : B \mapsto S^{-1}B$ be inclusion maps, $S^{-1}f : S^{-1}A \mapsto S^{-1}B$ another induced map, and $(S^{-1}f)^* : \operatorname{Spec}(S^{-1}B) \mapsto \operatorname{Spec}(S^{-1}A)$ its induced map. We have a commutative diagram.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_1"'] & B \arrow[d, "i_2"] \\ S^{-1}A \arrow[r, "S^{-1}f"'] & S^{-1}B \end{tikzcd}$$

Applying the Spec functor, we get a new commutative diagram with arrows reversed.

$$\begin{tikzcd} \operatorname{Spec}(A) & \operatorname{Spec}(B) \arrow[l, "f^*"'] \\ \operatorname{Spec}(S^{-1}A) \arrow[u, "i_1^*"] & \operatorname{Spec}(S^{-1}B) \arrow[l, "(S^{-1}f)^*"] \arrow[u, "i_2^*"'] \end{tikzcd}$$

Since the maps i_j^* are embeddings, we may consider $\operatorname{Spec}(S^{-1}A)$ as its image in $\operatorname{Spec}(A)$ and $\operatorname{Spec}(S^{-1}B)$ as its image in $\operatorname{Spec}(B)$. With this identification, it is then clear that $(S^{-1}f)^*$ is the restriction of f^* to $\operatorname{Spec}(S^{-1}B)$. If $f^*(\mathfrak{q}) \in \operatorname{Spec}(S^{-1}A) = i_1^*(\operatorname{Spec}(S^{-1}A)$, then $f^*(\mathfrak{q}) \cap S = \emptyset$. This is equivalent to $\mathfrak{q} \cap f(S) = \emptyset$ so that $S^{-1}\mathfrak{q} \in \operatorname{Spec}(S^{-1}B)$ that maps to $f^*(\mathfrak{q})$ since the diagram above commutes. Therefore, $\operatorname{Spec}(S^{-1}B) = (f^*)^{-1}(\operatorname{Spec}(S^{-1}A))$.

21c. Let $\mathfrak{a}$ be an ideal of A, $\mathfrak{b} = \mathfrak{a}^e$ be its extension in B, $\overline{f} : A/\mathfrak{a} \mapsto B/\mathfrak{b}$ be the map induced by f and $\overline{f}^* : \operatorname{Spec}(B/\mathfrak{b}) \mapsto \operatorname{Spec}(A/\mathfrak{a})$ be its induced map. Considering the quotient maps $\pi_1 : A \mapsto A/\mathfrak{a}$ and $\pi_2 : B \mapsto B/\mathfrak{b}$, we have a diagram as above. Since π_i is surjective, π_i^* is a homeomoprhism onto $\ker \pi_i$ and so we may identify (as above) $\operatorname{Spec}(A/\mathfrak{a})$ with its image under π_1^*, $V(\mathfrak{a}) \subseteq \operatorname{Spec}(A)$ and similarly, $\operatorname{Spec}(B/\mathfrak{b})$ with $V(\mathfrak{b}) \subseteq \operatorname{Spec}(B)$. With this identification, it is clear that $\overline{f}^*$ is the restriction of f^* to $V(\mathfrak{b})$.

21d. Let $\mathfrak{p} \in \operatorname{Spec}(A)$ and consider $f^* : \operatorname{Spec}(B_\mathfrak{p}) \mapsto \operatorname{Spec}(A_\mathfrak{p})$ from the second part. In this space, the preimage of the restriction f^* coincides with the preimage of f^* itself, so we focus attention here only. Consider the subspace $V(\mathfrak{p})$ in $\operatorname{Spec}(A/\mathfrak{p})$. It is clearly the singleton $\{\mathfrak{p}\}$. The preimage of this subspace is exactly $V(\mathfrak{p}^e)$ (if $\mathfrak{p} \subseteq \mathfrak{q}^c$, then $\mathfrak{p}^e \subseteq \mathfrak{q}^{ce} \subseteq \mathfrak{q}$ so the preimage is contained in this set and clearly, this set maps into our set as well), which can then be identified by $\operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}^e) = \operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p})$. We have

$$(f^*)^{-1}(\mathfrak{p}) = \operatorname{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) = \operatorname{Spec}(A/\mathfrak{p} \otimes A_\mathfrak{p} \otimes B) = \operatorname{Spec}(k(\mathfrak{p}) \otimes B).$$

22. Let A be a ring and $\mathfrak{p}$ a prime ideal of A. The canonical image of $\operatorname{Spec}(A_\mathfrak{p})$ is the set of prime ideals $\mathfrak{q}$ such that $\mathfrak{q} \subseteq \mathfrak{p}$ (that is, they do not intersect $A \setminus \mathfrak{p}$). Clearly, if $U \subseteq \operatorname{Spec}(A)$ is such that $\mathfrak{p} \in U$, then $\operatorname{Spec}(A_\mathfrak{p}) \subseteq U$ since if $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q} \notin U$, then since $\operatorname{Spec}(A) \setminus U$ is closed,

$$\mathfrak{p} \in V(\mathfrak{q}) = \overline{\{\mathfrak{q}\}} \subseteq \operatorname{Spec}(A) \setminus U.$$

From this, $\operatorname{Spec}(A_\mathfrak{p}) \subseteq \cap_{\mathfrak{p}\in U} U$. Conversely, if $\mathfrak{q}$ is in every open set that contains $\mathfrak{p}$, then $\mathfrak{p} \in \overline{\{\mathfrak{q}\}} = V(\mathfrak{q})$ so that $\mathfrak{q} \subseteq \mathfrak{p}$ and therefore, $\mathfrak{q} \in \operatorname{Spec}(A_\mathfrak{p})$.

23a. Let A be a ring, $X = \operatorname{Spec}(A)$, and U be a basic open set ($U = X_f$ for some $f \in A$). If $U = X_f = X_g$, consider the rings A_f and A_g. Since $X_f = X_g$, then $r((f)) = r((g))$ and so there exists $n, m > 0$ and $a, b \in A$ such that

$$f^n = ag, \; g^m = bf.$$

Now let $i_f : A \mapsto A_f$ and $i_g : A \mapsto A_g$. Since

$$\frac{f}{1}\frac{b}{g^m} = 1 \in A_g,$$

$i_g(f^k)$ is a unit for each element of $(f^n)_{n\geq 0}$ and so there is an induced map $\phi : A_f \mapsto A_g$ such that $i_g = \phi \circ i_f$. Similarly, $i_f(g^k)$ is a unit for every element of $(g^n)_{n\geq 0}$ and so there is an induced map $\varphi : A_g \mapsto A_f$ such that $i_f = \varphi \circ i_g$. Finally, since $i_f(f^k)$ is a unit for every element of $(f^n)_{n\geq 0}$ there is a unique map $\mathrm{Id} : A_f \mapsto A_f$ such that $i_f \circ \mathrm{Id} \circ i_f$. However,

$$i_f = \varphi \circ i_g = \varphi \circ \phi \circ i_f \implies \varphi \circ \phi = \mathrm{Id}.$$

Similarly, by replacing A_f with A_g, $\phi \circ \varphi = \mathrm{Id}$ and so $A_f \simeq A_g$. It is clear that with any choice of equation used to define these maps, you get the same maps. Therefore, the assignment $U \mapsto A(U) = A_f$ is well-defined.

23b. Let $U = X_f$ and $U' = X_g$ be such that $U' \subseteq U$. Then $V(r(f)) \subseteq V(r(g))$ which implies $r(g) \subseteq r(f)$. That is, $g \in r(f)$ and so there exists $n > 0$ and $u \in A$ such that $g^n = uf$. Define $\rho : A(U) \mapsto A(U')$ by

$$\rho(a/f^m) = au^m/g^{nm}.$$

This map is well-defined since if $a/f^m = b/f^l$, then there exists f^i such that $f^i(af^l - bf^m) = af^{l+i} - bf^{m+i} = 0$. Multiplying through by u^{l+m+i}, we get that

$$au^m g^{n(l+i)} - bu^l g^{n(m+i)} = au^{l+m+i} f^{l+i} - bu^{l+m+i} f^{m+i} = 0.$$

That is, $au^{nm}/g^m = bu^l/g^{nl}$. It is easy to check that this map is a ring homomorphism. It is easy to see that for any other choice of equation $g^m = bf$, that we obtain the same map. To see that this map depends only on U and U', if we take $A(U) = A_f = A_{f'}$ and $A(U') = A_g = A_{g'}$, then any choice of defining functions makes the box diagram commute (which is a mess of exponents to check, but not hard). With this, the map ρ is well-defined up to the isomorphism classes $A(U)$ and $A(U')$ and hence, depends only on U and U'.

23c. If $U = U'$, where $U = X_f$ and $U' = X_g$, then from the first part, $A_f \simeq A_g$ so that choosing both $A(U) = A_f$ and $A(U') = A_f$ as a representative, the restriction homomorphism is the identity map (or, the equivalence class of the identity map).

23d. This problem is a notation chase. It is similar to proving that the diagram from the second part commutes. Choose any defining equation for the restriction homomorphism (they all define the same map) and show that the compositions are equal.

23e. Let $\mathfrak{p} \in \mathrm{Spec}(A) = X$. If $\mathfrak{p} \in U = X_f$, then $f \notin \mathfrak{p}$ so that $(f^n)_{n\geq 0} \subseteq A \setminus \mathfrak{p}$. That is, $A_f \subseteq A_\mathfrak{p}$. Therefore, we have injective maps $i_f : A_f \mapsto A_\mathfrak{p}$ satisfying $i_f = i_g \circ \mu_{fg}$ where $\mu_{fg} : A_f \mapsto A_g$ (in the case that $X_f \subseteq X_g$. Plug and chug to check). Therefore, there is an induced map $\phi : \lim A(U) \mapsto A_\mathfrak{p}$. This map is injective because each of the inclusion maps i_f is injective. To see that it is surjective, let $a/s \in A_\mathfrak{p}$. Then $s \in A \setminus \mathfrak{p}$ so that $\mathfrak{p} \in X_s (= A(U)$ for some $\mathfrak{p} \in U)$ and

$$a/s = \mu_s(a/s) = \phi(a/s),$$

considering $a/s \in \lim A(U)$. Therefore, ϕ is bijective and hence, an isomorphism.

*24.

25. Let $f : A \mapsto B$ and $g : A \mapsto C$ be ring homomorphisms and define $h : A \mapsto B \otimes_A C$ to be the map

$$h(x) = x(1 \otimes_A 1) = f(x) \otimes_A 1 = 1 \otimes_A g(x).$$

For $\mathfrak{p} \in \mathrm{Spec}(A)$, we have (from a previous problem)

$$(h^*)^{-1}(\mathfrak{p}) = \mathrm{Spec}(k \otimes_A (B \otimes_A C)) = \mathrm{Spec}((k \otimes_A B) \otimes_k (k \otimes_A C)),$$

where $k = k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Here, $k \otimes_A B$ and $k \otimes_A C$ are k-algebras and so of course k-vector spaces so their tensor product is zero if and only if both are nonzero vector spaces. From this, $\mathfrak{p}$ is in the image of h^* if and only if the tensor space $(k \otimes_A B) \otimes_k (k \otimes_A C) \neq 0$ (if a ring has no prime ideal, it is the zero ring and vice versa). Again, since they are k-vector spaces, this tensor product is nonzero if and only if both are nonzero. That is,

$$\mathfrak{p} \in h^*(\operatorname{Spec}(B \otimes_A C) \iff k \otimes_A B \neq 0 \neq k \otimes_A C.$$

Notice that $(f^*)^{-1}(\mathfrak{p}) = \operatorname{Spec}(k \otimes_A B)$ so that $\mathfrak{p} \in f^*(\operatorname{Spec}(B)$ if and only if $k \otimes_A B \neq 0$ and similarly, $\mathfrak{p} \in g^*(\operatorname{Spec}(C)$ if and only if $k \otimes_A C \neq 0$. The above implications then give us

$$\mathfrak{p} \in h^*(\operatorname{Spec}(B \otimes_A C) \iff \mathfrak{p} \in f^*(\operatorname{Spec}(B)) \text{ and } \mathfrak{p} \in g^*(\operatorname{Spec}(C)).$$

That is,

$$h^*(\operatorname{Spec}(B \otimes_A C) = f^*(\operatorname{Spec}(B)) \cap g^*(\operatorname{Spec}(C)).$$

26. Let $(B_\alpha, g_{\alpha\beta})$ be a directed system of rings over a directed set I and $B = \lim B_\alpha$. For each $\alpha \in I$, let $f_\alpha : A \mapsto B_\alpha$ be a ring homomorphism such that for $\alpha \leq \beta$, $g_{\alpha\beta} \circ f_\alpha = f_\beta$. Then the maps f_α induce a map $f : A \mapsto B$. For $\mathfrak{p} \in \operatorname{Spec}(A)$, notice

$$(f^*)^{-1}(\mathfrak{p}) = \operatorname{Spec}(k \otimes_A B) = \operatorname{Spec}\left(\varinjlim(k \otimes_A B_\alpha)\right),$$

since the tensor product commutes with direct limits (where $k = k(\mathfrak{p})$). The spectrum of a ring is empty if and only if the ring is the zero ring. Therefore,

$$\mathfrak{p} \in f^*(\operatorname{Spec}(B)) \iff \varinjlim(k \otimes_A B_\alpha) \neq 0.$$

From a previous problem, the direct limit of rings nonzero if and only if each individual ring is nonzero. That is,

$$\mathfrak{p} \in f^*(\operatorname{Spec}(B)) \iff \forall \alpha \in I,\ k \otimes_A B_\alpha \neq 0.$$

Again, these rings are nonzero if and only if $\mathfrak{p} \in f_\alpha^*(\operatorname{Spec}(B_\alpha))$. Therefore,

$$\mathfrak{p} \in f^*(\operatorname{Spec}(B)) \iff \forall \alpha \in I,\ \mathfrak{p} \in f_\alpha^*(\operatorname{Spec}(B_\alpha)).$$

Finally, we then have

$$f^*(\operatorname{Spec}(B)) = \bigcap_{\alpha \in I} f_\alpha^*(\operatorname{Spec}(B_\alpha)).$$

27a. Let $f_\alpha : A \mapsto B_\alpha$ be a family of A-algebras indexed by I, $(B_J, i_{JJ'})$ be the directed system of tensor products indexed by the directed set I' of finite subsets of I, and let $B = \lim B_J$ be the tensor product of this family of A-algebras. For each $J \in I'$, there is an induced map $f_J : A \mapsto B_J$ defined by $f_J(x) = x(1 \otimes \ldots \otimes 1)$ satisfying $i_{JJ'} \circ f_J = f_{J'}$. Therefore, there is an induced map $f : A \mapsto B$. From the previous problem, we have

$$f^*(\operatorname{Spec}(B)) = \bigcap_{J \in I'} f_J^*(\operatorname{Spec}(B_J)).$$

Now from the problem before that, we can extend the result to the finite case. In this case, the map f_J is exactly the map h in the problem, so we have

$$f_J^*(\operatorname{Spec}(B_J)) = \bigcap_{\alpha \in J} f_\alpha^*(B_\alpha).$$

Therefore,

$$f^*(\operatorname{Spec}(B)) = \bigcap_{J \in I'} f_J^*(\operatorname{Spec}(B_J)) = \bigcap_{J \in I'} \bigcap_{\alpha \in J} f_\alpha^*(\operatorname{Spec}(B_\alpha)) = \bigcap_{\alpha \in I} f_\alpha^*(\operatorname{Spec}(B_\alpha)).$$

27b. Let $f_\alpha : A \mapsto B_\alpha$, $1 \leq i \leq n$, be a finite collection of A-algebras, $B = \prod B_\alpha$ and $f : A \mapsto B$ be the map $f(x) = (f_\alpha(x))$. For $\mathfrak{p} \in \operatorname{Spec}(A)$, we know (where $k = k(\mathfrak{p})$)

$$(f^*)^{-1}(\mathfrak{p}) = \operatorname{Spec}(k \otimes B).$$

Therefore, $\mathfrak{p} \in f^*(\operatorname{Spec}(B))$ if and only if $k \otimes_A B \neq 0$.

It will now be shown that $k \otimes B = 0$ if and only if $k \otimes B_i$ for all i. Assuming that $k \otimes B = 0$, consider the projection maps $\pi_i : B \mapsto B_i$. Since the tensor functor is right exact (preserves surjective maps), the induced map $\operatorname{Id} \circ \pi_i : k \otimes B \mapsto k \otimes B_i$ is surjective. However, $k \otimes B = 0$ then implies $k \otimes B_i = 0$ for i. Conversely, if $k \otimes B_i = 0$ for each index i, consider the right exact sequence

$$B_1 \to B \to \prod_{j>1} B_j \to 0.$$

Again, applying the tensor functor to this sequence, we obtain the right exact sequence

$$k \otimes B_1 \to k \otimes B \to k \otimes \left(\prod_{j>1} B_j \right) \to 0.$$

Since $k \otimes B_1 = 0$, we have that the induced map $k \otimes B \mapsto k \otimes (\prod_{j>1} B_j)$ is injective. Continuing considering these sequences and applying the tensor functor, we continue to get a sequence of injective maps

$$k \otimes B \to k \otimes \left(\prod_{j>1} B_j \right) \to k \otimes \left(\prod_{j>2} B_j \right) \to \ldots \to k \otimes B_n = 0.$$

Since the composition is injective, this then implies that $k \otimes B = 0$. Taking the contrapositive, we have that $k \otimes B \neq 0$ if and only if $k \otimes B_1 \neq 0$ or $k \otimes B_2 \neq 0$.

In the notation from earlier, this says exactly that

$$\mathfrak{p} \in f^*(\operatorname{Spec}(B)) \iff \exists i,\ \mathfrak{p} \in f_i^*(\operatorname{Spec}(B_i)).$$

That is,

$$f^*(\operatorname{Spec}(B)) = \bigcup_{1 \leq i \leq n} f_i^*(\operatorname{Spec}(B_i)).$$

27c. From the above, the sets of the form $f^*(\operatorname{Spec}(B))$ where $f : A \mapsto B$ is a ring homomorphism satisfy the axioms determining closed sets of a topology on $X = \operatorname{Spec}(A)$. The topology is known as the constructible topology on X and X in this topology is denoted X_C. For any closed set $V(\mathfrak{a})$ of the Zariski topology, the projection map $f : A \mapsto A/\mathfrak{a}$ has image exactly $V(\mathfrak{a})$ so that every closed (resp. open) set is closed (resp. open) in the constructible topology. Therefore, the constructible topology is finer than the Zariski topology.

27d. As above, let X_C be $X = \operatorname{Spec}(A)$ in the constructible topology and let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X_C. This is equivalent to $\cap_{\alpha \in I} C_\alpha = \emptyset$, where $C_\alpha = X \setminus U_\alpha$. Since the C_α are closed, so we may write $C_\alpha = f_\alpha^*(\operatorname{Spec}(B_\alpha))$ where $f : A \mapsto B_\alpha$ is some ring homomorphism. From the first part, this closed set is equal to the spectrum of the tensor product of this family of A-algebras. Since the spectrum is empty, this implies that the tensor product is the zero ring. Since the tensor product is the direct limit of rings and is equal to zero, this implies there exists some finite subset $J \subseteq I$ (since these index the direct limit defining the tensor product) such that $B_J = 0$. Writing $f_J : A \mapsto B_J$, we see

$$\bigcap_{\alpha \in J} C_\alpha = \bigcap_{\alpha \in J} = f_\alpha^*(\operatorname{Spec}(B_\alpha)) = f_J^*(\operatorname{Spec}(B_J)) = \emptyset.$$

Therefore, the complements of the sets C_α for $\alpha \in J$ have union equal to all of X_C. Therefore, this finite open cover has a finite subcover. That is, X_C is quasi-compact.

28a. Fix $g \in A$. If g is nilpotent, then $X_g = \emptyset$ is open and closed. Assume now that g is not nilpotent. Since X_g is open in the Zariski topology, X_g is open in the constructible topology. Since g is not nilpotent, the set $(g^n)_{n \geq 0}$ is a multiplicatively closed subset of A not containing 0 wit localization A_g. There is an induced map $f : A \mapsto A_g$. The image $f^*(\operatorname{Spec}(A_g)) = X_g$ (this was done in problem 21). Therefore, X_g is closed as well.

28b. Let $X_{C'}$ denote X in the coarsest topology on X in which the sets X_g are both open and closed. If $\mathfrak{p}, \mathfrak{q} \in X$ are distinct, then without loss of generality, $\mathfrak{p} \not\subseteq \mathfrak{q}$. That is, there exists $f \in \mathfrak{p}$ and $f \notin \mathfrak{q}$. That is, $\mathfrak{p} \notin X_f$ and $\mathfrak{q} \in X_f$. In other words, $\mathfrak{p} \in X_{C'} \setminus X_f$ and $\mathfrak{q} \in X_f$. These sets are both open and disjoint. Therefore, $X_{C'}$ is Hausdorff.

28c. Let $f : X_C \mapsto X_{C'}$ be the identity map. Clearly, it is bijective and continuous since the constructible topology is finer than C'. To see that f is a homeomorphism, it will be shown that f is a closed map. Let $C \subseteq X_C$ be closed. Since X_C is quasi-compact, this implies that C is quasi-compact. Then its image $f(C)$ is quasi-compact. Since Y is Hausdorff, the usual justification shows that $f(C)$ is closed. Therefore, f is a homeomorphism.

28d. X_C is Hausdorff since the sets X_f are open and closed and therefore, X_C is compact. For any set S of two or more elements, we can find a separation of this set of the form X_f and $X_C \setminus X_f$ (as in showing $X_{C'}$ is Hausdorff). Therefore, singletons are the only connected subsets of X_C and so X_C is totally disconnected.

29. Let $f : A \mapsto B$ be a ring homomorphism and consider $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ in the constructible topology. To show that $f^* : \mathrm{Spec}(B) \mapsto \mathrm{Spec}(A)$ is continuous, notice that the sets of the form X_g form a basis for the constructible topology since they form a basis for C' (from the previous problem). Therefore, it suffices to show that $(f^*)^{-1}(X_g)$ is open in $\mathrm{Spec}(B)$ in the constructible topology. Note that $(f^*)^{-1}(X_g) = Y_{f(g)}$ is open in $\mathrm{Spec}(B)$ in the constructible topology. Therefore, f^* is continuous. The verification that f^* is closed is the same as showing that the map $f : X_C \mapsto X_{C'}$ is closed (since $\mathrm{Spec}(B)$ is quasi-compact in the constructible topology and $\mathrm{Spec}(A)$ is Hausdorff).

30. If the Zariski and constructible topologies coincide on $\mathrm{Spec}(A)$, then by $\mathrm{Spec}(A)$ is Hausdorff in the Zariski topology. From a previous problem, this occurs if and only if $A/\mathfrak{N}$ is absolutely flat. Conversely, if $A/\mathfrak{N}$ is absolutely flat, then $\mathrm{Spec}(A)$ is Hausdorff in the Zariski topology. Then the identity map $f : \mathrm{Spec}(A)_C \mapsto \mathrm{Spec}(A)$ is a continuous bijection (since the Zariski topology is coarser than the constructible topology). Note that $\mathrm{Spec}(A)_C$ is (quasi-)compact and $\mathrm{Spec}(A)$ is Hausdorff in this case, so f is in fact a homeomorphism. That is, the two topologies coincide.

Chapter 4

1. Let A be a ring and $\mathfrak{a}$ be an ideal of A. The irreducible components of $\mathrm{Spec}(A/\mathfrak{a})$ are exactly the minimal ideals of $\mathrm{Spec}(A/\mathfrak{a})$. These are in a bijective correspondence with the prime ideals $r(\mathfrak{q}_i) = \mathfrak{p}_i$, which is finite and therefore, there are finitely many irreducible components of $\mathrm{Spec}(A/\mathfrak{a})$.

2. If $r(\mathfrak{a}) = \mathfrak{a}$, then $\mathfrak{a}$ is the intersection of all prime ideals that contain $\mathfrak{a}$. We then have

$$\mathfrak{a} = \bigcap_i \mathfrak{p}_i$$

for some prime ideals $\mathfrak{p}_i$. We may assume that this intersection is minimal in the sense that no factor is repeated and there are no proper inclusions since they may be condensed. Then it is clear that $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for each pair of indices $1 \leq i, j \leq n$ since then we could condense the intersection further.

3. Let A be absolutely flat and $\mathfrak{q}$ be a primary ideal of A. For $x \in A$, we may write $x(1 - ax) = 0$ for some $a \in A$. If $1 - ax \notin \mathfrak{q}$, then $x^n \in \mathfrak{q}$. Notice this implies

$$x = ax^2 = a^2x^3 = \ldots = a^{n-1}x^n \in \mathfrak{q}.$$

Therefore, either $x \in \mathfrak{q}$ or $1 - ax \in \mathfrak{q}$. That is, in $A/\mathfrak{q}$, either $\overline{x} = 0$ or $\overline{x}$ is a unit. Therefore, $A/\mathfrak{q}$ is a field and so $\mathfrak{q}$ is maximal.

4. Let $\mathfrak{m} = (2, t)$ and $\mathfrak{a} = (4, t)$ be ideals of $\mathbb{Z}[t]$. It is easy to see that $\mathbb{Z}[t]/\mathfrak{m} \simeq \mathbb{Z}/2\mathbb{Z}$ so that $\mathfrak{m}$ is maximal. Similarly, $\mathbb{Z}[t]/\mathfrak{a} \simeq \mathbb{Z}/4\mathbb{Z}$, whose only zero-divisors is 2, which is nilpotent. Therefore, $\mathfrak{a}$ is primary. Notice that $\mathfrak{a} \subset \mathfrak{m}$ is a proper containment since $2 \notin \mathfrak{a}$. Notice that $t \in \mathfrak{a}$, but $t \notin \mathfrak{m}^k$ for $k > 1$ (by observation of generators for $\mathfrak{m}^k$ for $k > 1$). Therefore, $\mathfrak{a}$ is not equal to any power of $\mathfrak{m}$. Notice

$$r(\mathfrak{a}) = r((4) + (t)) = r(r((4)) + r((t)) = r((2) + (t)) = r(\mathfrak{m}) = \mathfrak{m}.$$

Therefore, $\mathfrak{a}$ is $\mathfrak{m}$-primary.

5. Let k be a field and consider the polynomial ring $k[x, y, z]$, prime ideals $\mathfrak{p}_1 = (x, y)$, $\mathfrak{p}_2 = (x, z)$, maximal ideal $\mathfrak{m} = (x, y, z)$, and $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$. Clearly, $\mathfrak{a} \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$ and $\mathfrak{a} \subseteq \mathfrak{m}^2$ so that $\mathfrak{a} \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Conversely, by comparing generators, we have the opposite inclusion so that $\mathfrak{a} = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$. Since each of these ideals is primary, this is a primary decomposition of $\mathfrak{a}$. $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime, so these are automatically associated primes of $\mathfrak{a}$. Similarly, $r(\mathfrak{m}^2) = \mathfrak{m}$ so that this is another associated prime of $\mathfrak{a}$. Then it is clear that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are minimal so these are the isolated components and $\mathfrak{m}$ is contained in both of these and so is the only embedded component.

6. Let X be an infinite compact Hausdorff space, and $C(X)$ be the ring of real-valued continuous functions on X. It will be shown that 0 does not have a primary decomposition (since X is infinite). Notice first that 0 has a primary decomposition if and only if has a decomposition of the form

$$0 = \bigcap_{i=1}^{n} \mathfrak{p}_i$$

for some primes $\mathfrak{p}_i$. The "if" direction is obvious. For the "only if" direction, take a primary decomposition and take the radical of both sides. Use the fact that the nilradical $\mathfrak{N}_{C(X)} = 0$ since $\mathbb{R}$ is a field to get a decomposition of the form above (since radicals of primary ideals are prime). Assume such a decomposition exists.

For $x \in X$, recall that $\mathfrak{m}_x = \{f \in C(X) : f(x) = 0\}$ is prime (since the quotient is $\mathbb{R}$). Therefore,

$$\bigcap_{i=1}^{n} \mathfrak{p}_i = 0 \subseteq \mathfrak{m}_x.$$

From chapter 1, since this intersection is finite, this implies that $\mathfrak{p}_i \subseteq \mathfrak{m}_x$ for some $1 \le i \le n$. Since X is infinite, there is necessarily some $1 \le i \le n$ such that $\mathfrak{p}_i \subseteq \mathfrak{m}_x \cap \mathfrak{m}_y$ for some $x, y \in X$. Since X is Hausdorff, we may take disjoint neighborhoods U_x of x and U_y of y such that $U_x \cap U_y = \emptyset$. By Urysohn's lemma, there exists a function $f \in C(X)$ such that $f(x) = 1$ and $\mathrm{supp}(f) \subseteq U_x$. Similarly, there exists $g \in C(X)$ such that $g(y) = 1$ and $\mathrm{supp}(g) \subseteq U_y$. Then $fg = 0 \in \mathfrak{p}_i$, but $f \notin \mathfrak{m}_x$ and $g \notin \mathfrak{m}_y$ so that $f \notin \mathfrak{p}_i$ and $g \notin \mathfrak{p}_i$. This contradicts that $\mathfrak{p}_i$ is prime. Therefore, no such decomposition exists.

7a. Let A be a ring and $\mathfrak{a}$ be an ideal. Notice that $\mathfrak{a}^e$ (the extension of $\mathfrak{a}$ to $A[x]$) necessarily contains $\mathfrak{a}[x]$ since it contains all monomials with coefficients from $\mathfrak{a}$. Conversely, any ideal of $A[x]$ that contains $\mathfrak{a}$ necessarily contains all monomials with coefficients from $\mathfrak{a}$ and so necessarily contains $\mathfrak{a}[x]$. Therefore, $\mathfrak{a}^e = \mathfrak{a}[x]$.

7b. Let $\mathfrak{p}$ be a prime ideal of A. Notice that

$$A[x]/\mathfrak{p}[x] \simeq (A/\mathfrak{p})[x].$$

Since the latter is the polynomial ring over an integral domain, it is necessarily an integral domain as well (from a previous problem in the first chapter).

7c. Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal. We again have

$$A[x]/\mathfrak{q}[x] \simeq (A/\mathfrak{q})[x].$$

If $f \in (A/\mathfrak{q})[x]$ is a nonzero zero divisor, then from the first chapter again, there exists $a + \mathfrak{q} \in A/\mathfrak{q}$ such that $(a + \mathfrak{q})f(x) \equiv 0$. That is, each coefficient is a zero-divisor of $A/\mathfrak{q}$ and so is nilpotent. From the same problem from the first chapter, this then implies that f is nilpotent. Therefore, $\mathfrak{q}[x]$ is a primary ideal. To see that this ideal is $\mathfrak{p}[x]$-primary, again consider the quotient $A[x]/\mathfrak{q}[x] \simeq (A/\mathfrak{q})[x]$. The radical $r(\mathfrak{q}[x])$ is equal to the contraction of the nilradical of this quotient. From chapter 1, we know the nilradical of a polynomial ring is the set of polynomials with nilpotent coefficients. That is, $\mathfrak{N}_{A[x]} = \mathfrak{N}_A[x]$. Since the nilradical of $A/\mathfrak{q}$ is $\mathfrak{p}/\mathfrak{q}$ (since $\mathfrak{q}$ is primary) we have

$$\mathfrak{N}_{(A/\mathfrak{q})[x]} = (\mathfrak{p}/\mathfrak{q})[x] \simeq \mathfrak{p}[x]/\mathfrak{q}[x].$$

By the bijective correspondence of prime ideals of $A[x]$ with $(A/\mathfrak{q})[x]$, this then implies that the contraction of this nilradical is exactly $\mathfrak{p}[x]$. Therefore, $r(\mathfrak{q}[x]) = \mathfrak{p}[x]$ and $\mathfrak{q}[x]$ is $\mathfrak{p}[x]$-primary.

7d. Let $\mathfrak{a}$ be an ideal of A with minimal primary decomposition

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i.$$

It is clear by checking inclusions that

$$\mathfrak{a}[x] = \left(\bigcap_{i=1}^{n} \mathfrak{q}_i \right) [x] = \bigcap_{i=1}^{n} \mathfrak{q}_i[x].$$

From the previous problem, each $\mathfrak{q}_i[x]$ is primary, so this is a primary decomposition of $\mathfrak{a}$. To see that this decomposition is minimal, notice that since $r(\mathfrak{q}_i) = \mathfrak{p}_i \neq \mathfrak{p}_j = r(\mathfrak{q}_j)$ for $i \neq j$, we have

$$r(\mathfrak{q}_i[x]) = \mathfrak{p}_i[x] \neq \mathfrak{p}_j[x] = r(\mathfrak{q}_j[x])$$

from the previous problem. Therefore, no associated prime ideal of $\mathfrak{a}[x]$ is repeated. Similarly, since $\cap_{j\neq i}\mathfrak{q}_j \not\subseteq \mathfrak{q}_i$, we have

$$\bigcap_{j\neq i} \mathfrak{q}_j[x] = \left(\bigcap_{j\neq i} \mathfrak{q}_j \right) [x] \not\subseteq \mathfrak{p}_i[x].$$

Therefore, the primary decomposition above is minimal.

7e. If $\mathfrak{p}$ is a minimal prime ideal associated to $\mathfrak{a}$, then from the previous problem, $\mathfrak{p}[x]$ is associated to $\mathfrak{a}[x]$ and is of course minimal.

8. Let k be a field, $k[x_1, \ldots, x_n]$ be its polynomial ring, and $\mathfrak{p}_i = (x_1, \ldots, x_i)$ for $1 \leq i \leq n$. By looking at the quotients, it is clear that each $\mathfrak{p}_i$ is prime. Since $\mathfrak{p}_n$ is maximal, it is clear that the powers of $\mathfrak{p}_n$ are primary. For $1 \leq i < n$ and $k > 1$, we may write

$$k[x_1, \ldots, x_n]/\mathfrak{p}_i^k \simeq (k[x_1, \ldots, x_i]/\mathfrak{p}_i^k)[x_{i+1}, \ldots, x_n].$$

From chapter 1, if an element $g(x_1, \ldots, x_n) \in (k[x_1, \ldots, x_i]/\mathfrak{p}_i^k)[x_{i+1}, \ldots, x_n]$ is a zero-divisor, there exists $f(x_1, \ldots, x_i) \in k[x_1, \ldots, x_i]/\mathfrak{p}_i^k$ such that $f(x_1, \ldots, x_i)g(x_1, \ldots, x_n) = 0$. That is, the coefficients of g are zero divisors in $k[x_1, \ldots, x_i]/\mathfrak{p}_i^k$. However, $\mathfrak{p}_i$ is maximal in $k[x_1, \ldots, x_i]$ as above so that $\mathfrak{p}_i^k$ is primary and the only zero divisors are nilpotent. Therefore, the coefficients of g are all nilpotent so that g is nilpotent. Therefore, $\mathfrak{p}_i^k$ is primary.

9. Let A be a ring and $D(A)$ be the set of prime ideals $\mathfrak{p}$ for which there exists $a \in A$ such that $\mathfrak{p}$ is minimal in the set of primes containing $(0 : a)$. If $x \in A$ is a zero-divisor, then there exists $y \in A$ such that $xy = 0$. That is, $x \in (0 : y)$. Let $\mathfrak{p}$ be minimal in the set of prime ideals containing $(0 : y)$. Then $x \in (0 : y) \subseteq \mathfrak{p}$ for $\mathfrak{p} \in D(A)$. Conversely, if $x \in \mathfrak{p}$ with $\mathfrak{p} \in D(A)$, there exists $a \in A$ such that $(0 : a) \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is minimal in the set of these ideals. Then $\mathfrak{p}/(0 : a)$ is minimal in $A/(0 : a)$ so that $S = A/(0 : a) \setminus \mathfrak{p}/(0 : a)$ is a maximal multiplicative subset not containing $0 + (0 : a)$. Since the set $S(x^n + (0 : a))_{n\geq 0}$ is multiplicative and contains S, we necessarily have $0 + (0 : a) \in S(x^n + (0 : a))_{n\geq 0}$ That is, there exists $b + (0 : a) \in S$ such that

$$0 + (0 : a) = (b + (0 : a))(x^n + (0 : a)) = bx^n + (0 : a) \implies bx^n \in (0 : a) \implies abx^n = 0.$$

Therefore, x is a zero-divisor.

Let S be a multiplicative subset of A and let $i : A \mapsto S^{-1}A$ be the inclusion map. If $\mathfrak{p} \in D(A) \cap i^*(\mathrm{Spec}(S^{-1}A))$, then $\mathfrak{p} = \mathfrak{q}^c$ for some prime ideal $\mathfrak{q}$ of $S^{-1}A$. For $a \in A$ satisfying the requirement that $\mathfrak{p} \in D(A)$, we have $(0 : a) = \mathrm{ann}(a) = \mathrm{ann}((a))$ so that

$$(0 : a/1) = \mathrm{ann}(S^{-1}(a)) = S^{-1}(\mathrm{ann}(a)) \subseteq S^{-1}\mathfrak{p} = \mathfrak{q}$$

by the prime ideal correspondence for localization. To see that $\mathfrak{q}$ is minimal, assume there is a prime ideal $\mathfrak{q}'$ such that $(0 : a/1) \subset \mathfrak{q}' \subset \mathfrak{q}$. Then

$$(0 : a) \subseteq (\mathrm{ann}(a))^{ec} = (0 : a/1)^c \subset \mathfrak{q}'^c \subset \mathfrak{q}^c = \mathfrak{p}.$$

Therefore, $\mathfrak{q}$ is minimal so that $\mathfrak{q} \in D(S^{-1}A)$. Therefore, $\mathfrak{p} = \mathfrak{q}^c \in i^*(D(S^{-1}A))$. Conversely, let $\mathfrak{p} = \mathfrak{q}^c$ for some $\mathfrak{q} \in D(S^{-1}A)$, (that is, $\mathfrak{p} \in i^*(D(S^{-1}A))$). Clearly, $\mathfrak{p} \cap S = \emptyset$ so that $\mathfrak{p} \in i^*(\text{Spec}(S^{-1}A))$. Let $a/s \in S^{-1}A$ satisfy the requirement that $\mathfrak{q} \in D(S^{-1}A)$. Then it is easy to see that we may assume $s = 1$. Then

$$(0 : a) \subseteq (0 : a)^{ec} = (0 : a/1)^c \subseteq \mathfrak{q}^c = \mathfrak{p}.$$

If $(0 : a) \subset \mathfrak{p}' \subset \mathfrak{p}$, extending gives a contradiction to the minimality of $\mathfrak{q}$. Therefore, $\mathfrak{p} \in D(A)$. Combining all these inclusions, we have

$$i^*(D(S^{-1}A)) = D(A) \cap i^*(\text{Spec}(S^{-1}A)).$$

10a. Let $\mathfrak{p}$ be a prime ideal of a ring A and $S_\mathfrak{p}(0) = \ker \phi$ where $\phi : A \mapsto A_\mathfrak{p}$ is the inclusion map. Clearly, if $a \in S_\mathfrak{p}(0)$, then $a/1 = 0 \in A_\mathfrak{p}$ implies there exists $s \in A \setminus \mathfrak{p}$ such that $as = 0$. Since $0 \in \mathfrak{p}$ and $s \notin \mathfrak{p}$, we have that $a \in \mathfrak{p}$. Therefore, $S_\mathfrak{p}(0) \subseteq \mathfrak{p}$.

10b. Assume that $r(S_\mathfrak{p}(0)) = \mathfrak{p}$. Then for each $x \in \mathfrak{p}$, there exists $n > 0$ such that $x^n/1 = 0$ in $A_\mathfrak{p}$. That is, there exists $s \in A \setminus \mathfrak{p}$ such that $x^n s = 0$. From this, S cannot be extended to a larger multiplicative set not containing zero. Indeed, if $x \in A \setminus S = \mathfrak{p}$, there exists $n > 0$ and $s \in S$ such that $x^n s = 0$ so that $x \in S$ would imply $0 \in S$. Therefore, S is a maximal multiplicative subset so that $A \setminus S = \mathfrak{p}$ is a minimal prime of A.

Conversely, if $\mathfrak{p}$ is a minimal prime of A, then $S = A \setminus \mathfrak{p}$ is a maximal multiplicative subset of A. Then for all $x \in A \setminus S = \mathfrak{p}$, the multiplicative set $S(x^n)_{n \geq 0}$ necessarily contains 0. Therefore, there exists $s \in S$ and $n > 0$ such that $x^n s = 0$. That is, $x^n/1 = 0$ in $A_\mathfrak{p}$ so that $x^n \in S_\mathfrak{p}(0)$ and $x \in r(S_\mathfrak{p}(0))$.

10c. If $\mathfrak{p}' \subseteq \mathfrak{p}$, we may consider $A_\mathfrak{p} \subseteq A_{\mathfrak{p}'}$ so we have a composition of inclusions $A \hookrightarrow A_\mathfrak{p} \hookrightarrow A_{\mathfrak{p}'}$. It is clear $S_\mathfrak{p}(0)$ is the kernel of the first map and $S_{\mathfrak{p}'}(0)$ is the kernel of the composition. From this, it is clear that $S_\mathfrak{p}(0) \subseteq S_{\mathfrak{p}'}(0)$.

10d. If $x \neq 0$, then $(0 : x) \neq (1)$ and so there are prime ideals that contain it. By Zorn's lemma, there exists minimal elements $\mathfrak{p} \in D(A)$. Then $(0 : x) \subseteq \mathfrak{p}$ implies for every $s \in A \setminus \mathfrak{p}$, $xs \neq 0$. That is, $x/1 \neq 0$ in $A_\mathfrak{p}$ so $x \notin S_\mathfrak{p}(0)$. The contrapositive of this is if $x \in S_\mathfrak{p}(0)$ for all $\mathfrak{p} \in D(A)$, then $x = 0$. The result follows.

11. Let $\mathfrak{p}$ be a minimal ideal of A. From a previous problem, this implies that $r(S_\mathfrak{p}(0)) = \mathfrak{p}$. If $xy \in S_\mathfrak{p}(0)$ and $x \notin S_\mathfrak{p}(0)$, then there exists $s \in A \setminus \mathfrak{p}$ such that $sxy = 0$. Since $x \notin S_\mathfrak{p}(0)$, for all $t \in A \setminus \mathfrak{p}$, $tx \neq 0$ so that $\text{ann}(x) \subseteq \mathfrak{p}$. Since $sxy = 0$, we then have $sy = 0$. $s \notin \mathfrak{p}$ then implies that $y \in \mathfrak{p} = r(S_\mathfrak{p}(0))$. Therefore, $S_\mathfrak{p}(0)$ is primary. Let $\mathfrak{q}$ be any $\mathfrak{p}$-primary ideal. For $a \in S_\mathfrak{p}(0)$, $a/1 = 0$ implies there exists $s \in A \setminus \mathfrak{p}$ such that $as = 0$. Since $0 \in \mathfrak{q}$ and $s \notin \mathfrak{p} = r(\mathfrak{q})$, this immediately implies that $a \in \mathfrak{q}$. Therefore, $S_\mathfrak{p}(0) \subseteq \mathfrak{q}$ and so $S_\mathfrak{p}(0)$ is the minimal $\mathfrak{p}$-primary ideal.

Let $\mathfrak{a}$ be the intersection of $S_\mathfrak{p}(0)$ indexed over the minimal prime ideals of A. Then

$$\mathfrak{a} \subseteq r(\mathfrak{a}) \subseteq \bigcap_{\mathfrak{p} \text{ minimal}} r(S_\mathfrak{p}(0)) = \bigcap_{\mathfrak{p} \text{ minimal}} \mathfrak{p} = \mathfrak{N}.$$

Finally, assume that 0 is decomposable so there are finitely many primes associated to zero. If $\mathfrak{a} = 0$, then the decomposition of $\mathfrak{a}$ can be reduced to a minimal decomposition of 0. From this decomposition and minimality of each prime, it is clear that each prime associated to 0 is isolated. Conversely, assume each prime associated to 0 is isolated. Then each associated prime to 0 is necessarily minimal. Writing a primary decomposition $0 = \cap \mathfrak{q}_\mathfrak{p}$ indexed by the minimal primes, $S_\mathfrak{p}(0) \subseteq \mathfrak{q}_\mathfrak{p}$ so that $\mathfrak{a} \subseteq 0$ so that $\mathfrak{a} = 0$.

12a. Since both contractions and S^{-1} commute with finite intersections, we have

$$S(\mathfrak{a}) \cap S(\mathfrak{b}) = (S^{-1}\mathfrak{a})^c \cap (S^{-1}\mathfrak{b})^c = (S^{-1}\mathfrak{a} \cap S^{-1}\mathfrak{b})^c = (S^{-1}(\mathfrak{a} \cap \mathfrak{b}))^c = S(\mathfrak{a} \cap \mathfrak{b}).$$

12b. Similar to the above, contractions and S^{-1} commute with radicals so we have

$$S(r(\mathfrak{a})) = (S^{-1}(r(\mathfrak{a})))^c = (r(S^{-1}\mathfrak{a}))^c = r((S^{-1}\mathfrak{a})^c) = r(S(\mathfrak{a})).$$

12c. If $S(\mathfrak{a}) = (1)$, then $S^{-1}\mathfrak{a} = (S^{-1}\mathfrak{a})^{ce} = S(\mathfrak{a})^e = (1)^e = (1)$ so that $\mathfrak{a} \cap S \neq \emptyset$. Conversely, if $\mathfrak{a} \cap S \neq \emptyset$, then $S^{-1}\mathfrak{a} = (1)$ so that $S(\mathfrak{a}) = (S^{-1}\mathfrak{a})^c = (1)$.

12d. If S_1 and S_2 are multiplicative subsets of A, then S_1S_2 is a multiplicative subset of A. Notice that $x \in S(\mathfrak{a})$ if and only if there exists some $s \in S$ such that $xs \in \mathfrak{a}$. Therefore, if $x \in (S_1S_2)(\mathfrak{a})$, then there exists $s_1s_2 \in S_1S_2$ such that $s_1s_2x \in \mathfrak{a}$. Then $s_2(s_1x) \in \mathfrak{a}$ implies that $s_1x \in S_2(\mathfrak{a})$ and this implies that $x \in S_1(S_2(\mathfrak{a}))$. Conversely, if $x \in S_1(S_2(\mathfrak{a}))$, then there exists $s_1 \in S_1$ such that $xs_1 \in S_2(\mathfrak{a})$. Then there exists $s_2 \in S_2$ such that $xs_1s_2 \in \mathfrak{a}$. Therefore, $x \in (S_1S_2)(\mathfrak{a})$. Combining inclusions, we get the result.

Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal and $S \subseteq A$ be multiplicative. If $S \cap \mathfrak{p} \neq \emptyset$, then there exists $p \in \mathfrak{p} \cap S$. Since $r(\mathfrak{q}) = \mathfrak{p}$, $p^n \in \mathfrak{q}$ for some $n > 0$. Since S is multiplicative, then $p^n \in \mathfrak{q} \cap S$ so that $\mathfrak{q} \cap S \neq \emptyset$. This then implies that $S(\mathfrak{q}) = (1)$. If $S \cap \mathfrak{p} = \emptyset$, then for $x \in S(\mathfrak{q})$, there exists $s \in S$ such that $xs \in \mathfrak{q}$, but $s \notin \mathfrak{p}$ so that $x \in \mathfrak{q}$. That is, $S(\mathfrak{q}) \subseteq \mathfrak{q}$. Since $\mathfrak{q} \subseteq S(\mathfrak{q}$ by definition, we then have that $S(\mathfrak{q}) = \mathfrak{q}$.

If $\mathfrak{a}$ has a primary decomposition, we may write

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

for some primary ideals $\mathfrak{q}_i$. From this, for any multiplicative subset S of A,

$$S(\mathfrak{a}) = \bigcap_{i=1}^{n} S(\mathfrak{q}_i).$$

Here, for each index i there are two possibilities. If $\mathfrak{p}_i \cap S \neq \emptyset$, then $S(\mathfrak{q}_i) = (1)$. If $\mathfrak{p}_i \cap S = \emptyset$, then $S(\mathfrak{q}_i) = \mathfrak{q}_i$. Therefore, there are at most 2^n possibilities for $S(\mathfrak{a})$.

13a. Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Let $S_{\mathfrak{p}} = A \setminus \mathfrak{p}$ and define

$$\mathfrak{p}^{(n)} = S_{\mathfrak{p}}(\mathfrak{p}^n).$$

Clearly, by the prime ideal correspondence from localization, we have

$$r(\mathfrak{p}^{(n)}) = r(S_{\mathfrak{p}}(\mathfrak{p}^n)) = S_{\mathfrak{p}}(r(\mathfrak{p}^n)) = S_{\mathfrak{p}}(\mathfrak{p}) = \mathfrak{p}.$$

To see that $\mathfrak{p}^{(n)}$ is primary, let $xy \in \mathfrak{p}^{(n)}$ and assume $y \notin \mathfrak{p}$. Then there exists $s \in S_{\mathfrak{p}} = A \setminus \mathfrak{p}$ such that $sxy \in \mathfrak{p}^n$. Since $sy \in S_{\mathfrak{p}} = A \setminus \mathfrak{p}$, this then implies that $x \in S_{\mathfrak{p}}(\mathfrak{p}^n) = \mathfrak{p}^{(n)}$. Therefore, $\mathfrak{p}^{(n)}$ is primary.

13b. Assume $\mathfrak{p}^n$ has a (minimal) primary decomposition

$$\mathfrak{p}^n = \bigcap_{i=1}^{m} \mathfrak{q}_i,$$

where $r(\mathfrak{q}_i) = \mathfrak{p}_i$. Taking radicals, we then have

$$\mathfrak{p} = \bigcap_{i=1}^{m} \mathfrak{p}_i.$$

From the first chapter, this implies that $\mathfrak{p} = \mathfrak{p}_j$ for some $1 \leq j \leq m$. That is, $\mathfrak{p}$ is an associated ideal of $\mathfrak{p}^n$. From the above, we also have that $\mathfrak{p}$ is an isolated ideal of $\mathfrak{p}^n$. Notice that $\mathfrak{p}^n \subseteq \mathfrak{p}^{(n)}$ so that

$$\mathfrak{p}^n = \mathfrak{p}^n \cap \mathfrak{p}^{(n)} = (\mathfrak{p}^{(n)} \cap \mathfrak{q}_j) \cap \left(\bigcap_{i \neq j} \mathfrak{q}_i \right).$$

From this, we get a new primary decomposition. By invariance of the isolated components in a minimal decomposition, we have that $\mathfrak{q}_j = \mathfrak{q}_j \cap \mathfrak{p}^{(n)}$ so that $\mathfrak{q}_j \subseteq \mathfrak{p}^{(n)}$. Conversely, for $x \in \mathfrak{p}^{(n)}$, there exists $s \in A \setminus \mathfrak{p}$ such that $sx \in \mathfrak{p}^n \subseteq \mathfrak{q}_j$. Since $s \notin \mathfrak{p} = r(\mathfrak{q}_j)$, this implies that $x \in \mathfrak{q}_j$ and so $\mathfrak{p}^{(n)} \subseteq \mathfrak{q}_j$. Therefore, $\mathfrak{q}_j = \mathfrak{p}^{(n)}$ as desired.

13c. Assume that $\mathfrak{p}^{(n)}\mathfrak{p}^{(m)}$ has a (minimal) primary decomposition

$$\mathfrak{p}^{(n)}\mathfrak{p}^{(m)} = \bigcap_{i=1}^{l} \mathfrak{q}_i.$$

Similar to above, we have

$$\mathfrak{p} = r(\mathfrak{p}^{(n)}) \cap r(\mathfrak{p}^{(m)}) = r(\mathfrak{p}^{(n)}\mathfrak{p}^{(m)}) = \bigcap_{i=1}^{l} \mathfrak{p}_i.$$

Therefore, $\mathfrak{p} = \mathfrak{p}_j$ for some $1 \leq j \leq l$. Again, we see that $\mathfrak{p}$ is an isolated ideal of $\mathfrak{p}^{(n)}\mathfrak{p}^{(m)}$. Notice that $\mathfrak{p}^{(n)}\mathfrak{p}^{(m)} \subseteq \mathfrak{p}^{(n+m)}$, which is straightforward to check. By intersecting with the primary decomposition as before, we get from invariance of isolated components of minimal decompositions that $\mathfrak{q}_j \subseteq \mathfrak{p}^{(n+m)}$. Conversely, if $x \in \mathfrak{p}^{(n+m)}$, there exists $s \in A \setminus \mathfrak{p}$ such that $sx \in \mathfrak{p}^{n+m} \subseteq \mathfrak{q}_j$. Then $s \notin \mathfrak{p} = r(\mathfrak{q}_j)$ implies that $x \in \mathfrak{q}_j$ so $\mathfrak{p}^{(n+m)} \subseteq \mathfrak{q}_j$. Therefore, $\mathfrak{q}_j = \mathfrak{p}^{(n+m)}$ as desired.

13d. Clearly, if $\mathfrak{p}^n = \mathfrak{p}^{(n)}$, then $\mathfrak{p}^n$ is $\mathfrak{p}$-primary. Conversely, if $\mathfrak{p}^n$ is $\mathfrak{p}$-primary, we already know that $\mathfrak{p}^n \subseteq \mathfrak{p}^{(n)}$ so let $x \in \mathfrak{p}^{(n)}$. Then there exists $s \in A \setminus \mathfrak{p}$ such that $xs \in \mathfrak{p}^n$. $s \notin \mathfrak{p} = r(\mathfrak{p}^n)$ implies that $x \in \mathfrak{p}^n$ so that the opposite inclusion holds as well. Therefore, $\mathfrak{p}^n = \mathfrak{p}^{(n)}$.

14. Let $\mathfrak{a}$ be decomposable and $\mathfrak{p}$ maximal in the set of $(\mathfrak{a} : x)$ for $x \notin \mathfrak{a}$. Clearly, $\mathfrak{p}$ is an ideal. To see that it is prime, let $yz \in \mathfrak{p}$ and assume that $y \notin \mathfrak{p}$. For $x \in A$ such that $\mathfrak{p} = (\mathfrak{a} : x)$, we see

$$\mathfrak{p} = (\mathfrak{a} : x) \subseteq ((\mathfrak{a} : x) : y) = (\mathfrak{a} : xy).$$

(Note that this implicitly uses that $y \notin \mathfrak{p} = (\mathfrak{a} : x)$) By maximality, we then necessarily have that $\mathfrak{p} = (\mathfrak{a} : xy)$. Since $yz \in \mathfrak{p} = (\mathfrak{a} : x)$, we have that $xyz \in \mathfrak{p}$, but this implies that $z \in (\mathfrak{a} : xy) = \mathfrak{p}$. Therefore, $\mathfrak{p}$ is prime.

Now let $\mathfrak{a}$ have the (minimal) primary decomposition

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i.$$

Clearly, for all $x \notin \mathfrak{a}$,

$$(\mathfrak{a} : x) = \bigcap_{i=1}^{n} (\mathfrak{q}_i : x).$$

Therefore, if $\mathfrak{p} = (\mathfrak{a} : x)$ for some $x \notin \mathfrak{a}$, then from chapter 1, we know that $\mathfrak{p} = (\mathfrak{q}_j : x)$ for some $1 \leq j \leq n$. In particular, this implies that $x \notin \mathfrak{q}_j$. Taking the radical of both sides, we have that $\mathfrak{p} = r(\mathfrak{q}_j : x) = \mathfrak{p}_j$. Therefore, $\mathfrak{p}$ is associated to $\mathfrak{a}$.

15. Let $\mathfrak{a}$ be a decomposable ideal of a ring A, Σ be a subset of isolated prime ideals of $\mathfrak{a}$, and let $f \in A$ have the property that for all prime ideals $\mathfrak{p}$ associated with $\mathfrak{a}$, $f \in \mathfrak{p} \iff \mathfrak{p} \notin \Sigma$. Write $S_f = (f^n)_{n \geq 0}$ and let

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

be a minimal primary decomposition of $\mathfrak{a}$. We see

$$S_f(\mathfrak{a}) = \bigcap_{i=1}^{n} S_f(\mathfrak{q}_i).$$

Note that because of the property that f has, if $f \in \mathfrak{p}_i$, then $f^n \in \mathfrak{q}_i$ implies that $S_f(\mathfrak{q}_i) = (1)$ and if $f \notin \mathfrak{p}_i$, then $\mathfrak{p}_i \in \Sigma$. Using this in the above, the intersection is then over all $\mathfrak{p}$-primary ideals of the intersection with $\mathfrak{p} \in \Sigma$. That is, over all $\mathfrak{q}_i$ such that $r(\mathfrak{q}_i) \in \Sigma$.

$$S_f(\mathfrak{a}) = \bigcap_{r(\mathfrak{q}_i) \in \Sigma} S_f(\mathfrak{q}_i)$$

We always have that $\mathfrak{a} \subseteq S(\mathfrak{a})$. If $\mathfrak{p}_i = r(\mathfrak{q}_i) \in \Sigma$, then $f \notin \mathfrak{p}_i$. For $x \in S_f(\mathfrak{q}_i)$, there exists $f^n \in S_f$ such that $xf^n \in \mathfrak{q}_i$. Since $f \notin \mathfrak{p}_i$, $f^n \notin \mathfrak{p}_i$ so that we immediately have that $x \in \mathfrak{q}$ by primality. Therefore, $S_f(\mathfrak{q}_i) \subseteq \mathfrak{q}_i$ and we have $S_f(\mathfrak{q}_i) = \mathfrak{q}_i$. Therefore,

$$S_f(\mathfrak{a}) = \bigcap_{r(\mathfrak{q}_i) \in \Sigma} S_f(\mathfrak{q}_i) = \bigcap_{r(\mathfrak{q}_i) \in \Sigma} \mathfrak{q}_i = \mathfrak{q}_\Sigma.$$

16. Let A be a ring such that every ideal has a primary decomposition. For any ideal $\mathfrak{i}$ of $S^{-1}A$, it is the extension of some ideal $\mathfrak{a}$ of A. $\mathfrak{a}$ then has a primary decomposition

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i.$$

Since S^{-1} commutes with finite intersections, we have

$$\mathfrak{i} = S^{-1}\mathfrak{a} = \bigcap_{i=1}^{n} S^{-1}\mathfrak{q}_i.$$

From this point, if $S \cap \mathfrak{p}_i \neq \emptyset$ (where $\mathfrak{p}_i = r(\mathfrak{q}_i)$), then $S^{-1}\mathfrak{q}_i = (1)$ and so we may remove it from this intersection. Otherwise, $S \cap \mathfrak{p}_i = \emptyset$, from which it is straightforward to check that $S^{-1}\mathfrak{q}_i$ is primary.

17. Let A be a ring with the property $(L1)$ and $\mathfrak{a}$ be a proper ideal of A. Let $\mathfrak{p}_1$ be a minimal element of the set of prime ideals containing $\mathfrak{a}$. Let $\mathfrak{q}_1 = S_{\mathfrak{p}_1}(\mathfrak{a})$. Clearly, $r(\mathfrak{q}_1) = S_{\mathfrak{p}_1}(r(\mathfrak{a}))$. Every prime ideal $\mathfrak{p}$ containing $\mathfrak{a}$ either intersects $A \setminus \mathfrak{p}_1$ (in which case, $S_{\mathfrak{p}_1}(\mathfrak{p}) = (1)$) or is contained in $\mathfrak{p}_1$ (and hence, equal by minimality of $\mathfrak{p}_1$. Then $S_{\mathfrak{p}_1}(\mathfrak{p}) = \mathfrak{p}_1$). Therefore, $r(\mathfrak{q}_1) = \mathfrak{p}_1$. Now if $xy \in \mathfrak{q}_1 = S_{\mathfrak{p}_1}(\mathfrak{a})$, there exists $s \in A \setminus \mathfrak{p}_1$ such that $sxy \in \mathfrak{a}$. If $x \notin \mathfrak{p}_1$, then $sx \notin \mathfrak{p}_1$. This then implies that $y \in S_{\mathfrak{p}_1}(\mathfrak{a}) = \mathfrak{q}$. Therefore, $\mathfrak{q}_1$ is $\mathfrak{p}_1$-primary. Since A satisfies $(L1)$, there exists some $x \notin \mathfrak{p}_1$ such that

$$\mathfrak{q}_1 = S_{\mathfrak{p}_1}(\mathfrak{a}) = (\mathfrak{a} : x).$$

Clearly, $\mathfrak{a} \subseteq \mathfrak{q}_1 \cap (\mathfrak{a} + (x))$. Conversely, let $a + bx \in \mathfrak{q}_1 \cap (\mathfrak{a} + (x))$. Then $ax + bx^2 \in \mathfrak{a}$ implies that $bx^2 \in \mathfrak{a}$. Since $x \notin \mathfrak{p}_1$, $x^2 \notin \mathfrak{p}_1 = r(\mathfrak{q}_1)$. Since $bx^2 \in \mathfrak{a} \subseteq S_{\mathfrak{p}_1}(\mathfrak{a}) = \mathfrak{q}_1$, this then implies that $b \in \mathfrak{q}_1$ so that $bx \in \mathfrak{a}$. Therefore, $a + bx \in \mathfrak{a}$ and $\mathfrak{a} = \mathfrak{q}_1 \cap (\mathfrak{a} + (x))$. By Zorn's lemma, the set of ideals $\mathfrak{b}$ such that $\mathfrak{a} + (x) \subseteq \mathfrak{b}$ and $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{b}$ has a maximal element. Denote this element by $\mathfrak{a}_1$ (note that $x \in \mathfrak{a} + (x) \subseteq \mathfrak{b}$, therefore, $\mathfrak{a}_1 \not\subseteq \mathfrak{p}_1$). Inductively applying this procedure, for any $n > 0$, we can find primary ideals $\mathfrak{q}_i$ for $1 \leq i \leq n$ and $\mathfrak{a}_n$ such that $\mathfrak{a} \subseteq \mathfrak{a}_n$ (really, $\mathfrak{a}_{n-1} \subset \mathfrak{a}_n$) and

$$\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n \cap \mathfrak{a}_n.$$

If at any point $\mathfrak{a}_n = (1)$, then we have a primary decomposition of $\mathfrak{a}$.

The claim is that for all ordinals, there is a representation of $\mathfrak{a}$ of the above form. For the ordinal 0, the representation $\mathfrak{a} = \mathfrak{a}$ suffices. For an ordinal $\mu \in \mathrm{Ord}$, if μ has a predecessor η, then following the procedure above with $\mathfrak{a} = \mathfrak{a}_\eta$, we can get a desired representation. If $\mu \in \mathrm{Ord}$ is a limit ordinal and we can achieve

$$\mathfrak{a} = \left(\bigcap_{\alpha < \eta} \mathfrak{q}_\alpha \right) \cap \mathfrak{a}_\eta$$

for every ordinal $\eta < \mu$, then let

$$\mathfrak{a}_\mu = \bigcup_{\alpha < \mu} \mathfrak{a}_\alpha.$$

This is an ideal since the ideals $\mathfrak{a}_\alpha$ are ascending. Then following the procedure from earlier, let $\mathfrak{p}_\mu$ be a minimal prime ideal containing $\mathfrak{a}_\mu$ and $\mathfrak{q}_\mu = S_{\mathfrak{p}_\mu}(\mathfrak{a})$ so that $\mathfrak{q}_\mu$ is $\mathfrak{p}_\mu$-primary. Notice that

$$\left(\bigcap_{\alpha < \mu} \mathfrak{q}_\alpha \right) \cap \mathfrak{a}_\mu = \bigcup_{\beta < \mu} \left(\left(\bigcap_{\alpha < \mu} \mathfrak{q}_\alpha \right) \cap \mathfrak{a}_\beta \right) = \bigcup_{\beta < \mu} \left(\left(\bigcap_{\beta \leq \alpha < \mu} \mathfrak{q}_\alpha \right) \cap \left(\bigcap_{\alpha < \beta} \mathfrak{q}_\alpha \right) \cap \mathfrak{a}_\beta \right)$$
$$= \bigcup_{\beta < \mu} \left(\left(\bigcap_{\beta \leq \alpha < \mu} \mathfrak{q}_\alpha \right) \cap \mathfrak{a} \right) = \bigcup_{\beta < \mu} \mathfrak{a} = \mathfrak{a}.$$

Therefore, by transfinite induction, there is always such a decomposition. For sufficiently large ordinals, the cardinality of $\mathfrak{a}_\mu$ will be equal to the cardinality of A and hence, $\mathfrak{a}_\mu = (1)$ for sufficiently large ordinals (in the induction, the new ideal $\mathfrak{a}_\mu$ always has greater cardinality than all previous. In particular, this implies that the cardinality of $\mathfrak{a}_\mu$ is greater than or equal to μ). Therefore, such a representation of $\mathfrak{a}$ as an intersection of primary ideals always exists.

18. $(i) \implies (ii)$ Let A be a ring such that every ideal has a primary decomposition. For an ideal $\mathfrak{a}$ of A, we may write a primary decomposition

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i.$$

Consider a descending chain of multiplicative subsets $S_1 \supseteq S_2 \supseteq \ldots$. For any j, we have

$$S_j(\mathfrak{a}) = \bigcap_{i=1}^{n} S_j(\mathfrak{q}_i) = \bigcap_{S_j \cap \mathfrak{p}_i = \emptyset} \mathfrak{q}_i.$$

Let n_j be the cardinality of $\{i \in \{1, \ldots, n\} : S_j \cap \mathfrak{p}_i = \emptyset\}$. That is, n_j is the number of terms in the intersection above. Since the multiplicative subsets are descending, n_j is a decreasing sequence. Any decreasing sequence of natural numbers is eventually constant. Therefore, there is some N such that $j \geq N$ implies $n_j = n_N$. Since the sets $\{i \in \{1, \ldots, n\} : S_j \cap \mathfrak{p}_i = \emptyset\}$ all have the same cardinality for $j \geq N$ and they are related under inclusion, they are all equal. Since the terms of the intersection are also the same, we have $S_j(\mathfrak{a}) = S_N(\mathfrak{a})$ for $j \geq N$. Therefore, A satisfies $(L2)$.

Now let $\mathfrak{a}$ be a proper ideal and $\mathfrak{p}$ be a prime ideal. We may write a minimal primary decomposition for $\mathfrak{a}$,

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i,$$

for some primary ideals $\mathfrak{q}_i$. As above,

$$S_{\mathfrak{p}}(\mathfrak{a}) = \bigcap_{\mathfrak{p}_i \subseteq \mathfrak{p}} \mathfrak{q}_i.$$

For those ideals for which $\mathfrak{q}_i \subseteq \mathfrak{p}_i \not\subseteq \mathfrak{p}$, there exists $x_i \in \mathfrak{q}_i$ for which $x_i \notin \mathfrak{p}$. Consider $x = \prod_{\mathfrak{p}_i \not\subseteq \mathfrak{p}} x_i$. Then $x \in \cap_{\mathfrak{p}_i \not\subseteq \mathfrak{p}} \mathfrak{q}$ and $x \notin \mathfrak{p}$. In particular, for prime ideals such that $\mathfrak{p}_i \subseteq \mathfrak{p}$, $x \notin \mathfrak{p}_i$. Therefore,

$$(\mathfrak{a} : x) = \left(\bigcap_{\mathfrak{p}_i \not\subseteq \mathfrak{p}} (\mathfrak{q}_i : x) \right) \cap \left(\bigcap_{\mathfrak{p}_i \subseteq \mathfrak{p}} (\mathfrak{q}_i : x) \right) = (1) \cap \left(\bigcap_{\mathfrak{p}_i \subseteq \mathfrak{p}} \mathfrak{q}_i \right) = S_{\mathfrak{p}}(\mathfrak{a}).$$

From this, A satisfies $(L1)$.

$(ii) \implies (i)$ Let $\mathfrak{a}$ be an ideal of A. Following the construction from the previous problem (that is, using $(L1)$), for any $n > 0$, we may write

$$\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n \cap \mathfrak{a}_n,$$

where $\mathfrak{q}_i = S_{\mathfrak{p}_i}(\mathfrak{a})$ for some prime ideals $\mathfrak{p}_i$. Write $S_n = S_{\mathfrak{p}_1} \cap \ldots \cap S_{\mathfrak{p}_n}$. Notice that $\mathfrak{a}_n \cap S_n \neq \emptyset$ since by construction $\mathfrak{a}_j \not\subseteq \mathfrak{p}_i$ for $i \leq j$ (from chapter 1, if $\mathfrak{a} \subseteq \cup \mathfrak{p}_i$ for a finite union, then $\mathfrak{a} \subseteq \mathfrak{p}_i$ for some i). Therefore, $S_n(\mathfrak{a}_n) = (1)$. Since $S_n \subseteq S_{\mathfrak{p}_i}$ for each i, $S_n(\mathfrak{q}_i) = \mathfrak{q}_i$ for each i and

$$S_n(\mathfrak{a}) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n.$$

The sequence $S_1 \supseteq S_2 \supseteq \ldots$ is a descending chain of multiplicative subsets. By $(L2)$, there is some N such that $m > N$ implies $S_m(\mathfrak{a}) = S_N(\mathfrak{a})$. If $\mathfrak{a}_N = (1)$, we are done since then

$$\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_N \cap \mathfrak{a}_N = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_N.$$

If $\mathfrak{a}_N \neq (1)$, then let $\mathfrak{p}_{N+1}$ be a minimal containing $\mathfrak{a}_N$ so that $\mathfrak{q}_{N+1} = S_{\mathfrak{p}_{N+1}}(\mathfrak{a}_N)$ is $\mathfrak{p}_{N+1}$-primary (continue the process essentially). Then we know $S_N(\mathfrak{a}) = S_{N+1}(\mathfrak{a})$ so that $\mathfrak{q}_{N+1} \subseteq \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_N$ and

$$\mathfrak{a}_N \subseteq S_{\mathfrak{p}_{N+1}}(\mathfrak{a}_N) = \mathfrak{q}_{N+1} \subseteq \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_N.$$

Therefore,

$$\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_N \cap \mathfrak{a}_N = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_N.$$

In either case, $\mathfrak{a}$ has a primary decomposition.

19. The first statement that every $\mathfrak{p}$-primary ideal contains $S_{\mathfrak{p}}(0)$ was done in problem 10. Let A be a ring such that the intersection of all $\mathfrak{p}$-primary ideals is exactly $S_{\mathfrak{p}}(0)$. It will be shown by induction that if $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are distinct primes, none of which are minimal in A, then there exists an ideal $\mathfrak{a}$ of A whose associated primes are $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. The base case, $n = 1$ is trivial since we may take $\mathfrak{a} = \mathfrak{p}_1$. Assume now that the result holds for a fixed $n - 1$. Then there exists an ideal $\mathfrak{b}$ with decomposition

$$\mathfrak{b} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_{n-1}.$$

where each $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. If $\mathfrak{b} \subseteq S_{\mathfrak{p}_n}(0)$, let $\mathfrak{p}$ be a minimal prime of A contained in $\mathfrak{p}_n$. Then $S_{\mathfrak{p}_n}(0) \subseteq S_{\mathfrak{p}}(0)$ so that $\mathfrak{b} \subseteq S_{\mathfrak{p}}(0)$. Taking radicals, we have

$$\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_{n-1} \subseteq r(S_{\mathfrak{p}}(0)) \subseteq \mathfrak{p}.$$

Therefore, $\mathfrak{p}_i \subseteq \mathfrak{p}$ for some index $1 \leq i \leq n-1$. Since then $\mathfrak{p}_i \subseteq \mathfrak{p} \subseteq \mathfrak{p}_n$, by minimality of $\mathfrak{p}$, we must have $\mathfrak{p} = \mathfrak{p}_i$. This is a contradiction since no $\mathfrak{p}_i$ is minimal, but $\mathfrak{p}$ is assumed to be minimal. Therefore, $\mathfrak{b} \not\subseteq S_{\mathfrak{p}_n}(0)$. Since $S_{\mathfrak{p}_n}(0)$ is the intersection of all $\mathfrak{p}_n$-primary ideals, this then implies there is some $\mathfrak{p}_n$-primary ideal $\mathfrak{q}_n$ such that $\mathfrak{b} \not\subseteq \mathfrak{q}_n$. Finally, let

$$\mathfrak{a} = \mathfrak{b} \cap \mathfrak{q}_n = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n.$$

Clearly, $\mathfrak{a}$ is decomposable and it is easy to see that the associated prime ideals are exactly $\mathfrak{p}_i$ for $1 \leq i \leq n$. Therefore, the proof follows by induction.

20. Let M be a fixed A-module and N a submodule of M. Define

$$r_M(N) = \{x \in A : \exists q > 0,\ x^q M \subseteq N\}$$

to be the radical of N in M. It is very clear by writing out the definitions that $r_M(N) = r(M : N) = r(\operatorname{ann}(M/N))$ so that $r_M(N)$ is an ideal.

First, an obvious statement is that if $N \subseteq N'$ are submodules of M, then $r_M(N) \subseteq r_M(N')$, which follows by writing out the definitions.

Another obvious statement if for a submodule N, $r_M(N^k) = r_M(N)$. $\subseteq$ follows from the statement above and the other inclusion is easy to check pointwise.

If N, N' are two submodules of M, then both NN' and $N \cap N'$ are submodules of M. In particular, $(N \cap N')^2 \subseteq NN' \subseteq N \cap N'$. From the above, this gives

$$r_M(N \cap N') = r_M((N \cap N')^2) \subseteq r_M(NN') \subseteq r_M(N \cap N').$$

Therefore, $r_M(NN') = r_M(N \cap N')$. It is clear that $r_M(N \cap N') \subseteq r_M(N) \cap r_M(N')$. The other inclusion is simple to check pointwise. Therefore,

$$r_M(NN') = r_M(N \cap N') = r_M(N) \cap r_M(N').$$

If $r_M(N) = (1)$, then $M \subseteq N$ (since $1 \in r_M(N)$). Therefore, $M = N$. Conversely, if $M = N$, then it is clear that $r_M(N) = (1)$. Finally, if $x \in r(r_M(N) + r_M(N'))$, then $x^q \in r_M(N) + r_M(N')$ for some $q > 0$. That is, $x^q = a + b$ where $a^s M \subseteq N$ and $b^t M \subseteq N'$ for s, t sufficiently large. Then $x^{qst} = (a+b)^{st}$ is such that $x^{qst} M \subseteq N + N'$. That is to say, $x \in r(N + N')$. Therefore, $r(r_M(N) + r_M(N')) \subseteq r(N + N')$.

From the last bit, it is clear that if $r_M(N) + r_M(N') = (1)$, then $N + N' = M$.

21. For $x \in A$, define $\phi_x : M \mapsto M$ to be the map $\phi_x(m) = xm$. $x \in A$ is a zero-divisor in M if ϕ_x is not injective and $x \in A$ in nilpotent in M if ϕ_x is nilpotent. A submodule Q of M is primary in M if $Q \neq M$ and every zero-divisor of M/Q is nilpotent. Notice that if Q is primary in M

Assume Q is primary in M, $xy \in (Q : M)$ and $x \notin (Q : M)$. Then there exists $m \in M$ such that $xm \notin Q$ (but $xm \in M$ of course). Then $xym \in Q$ implies that the map $\phi_y : M/Q \mapsto M/Q$ is not injective since $\overline{xm} \neq 0$ but $\phi_y(\overline{xm}) = 0$. Therefore, y is nilpotent. That is, for some $q > 0$, $\phi_y^q = \phi_{y^q}$ is the zero map. This is equivalent to $y^q \in \operatorname{ann}(M/Q)$ so that $y \in r(\operatorname{ann}(M/Q) = r_M(Q) = r(Q : M)$. Therefore, $(Q : M)$ is $(r_M(Q)$-)primary. This also shows that $r_M(Q)$ is a prime ideal of A.

It is easy to see the equivalent statements $x \in A$ is nilpotent in M/Q if and only if $x^q \in \operatorname{ann}(M/Q)$ if and only if $x \in r(\operatorname{ann}(M/Q)) = r_M(Q)$. Similarly, $x \in A$ is a zero-divisor of M/Q if and only if there exists $m \in M \setminus Q$

such that $xm \in Q$. Therefore, Q is primary in M if and only if for all $x \in A$ such that there exists $m \in M \setminus Q$ such that $xm \in Q$, then $x \in r_M(Q)$.

(4.3) Let $Q_1, \ldots, Q_n$ be $\mathfrak{p}$-primary submodules of M. Notice

$$r_M\left(\bigcap_{i=1}^n Q_i\right) = r\left(\bigcap_{i=1}^n Q_i : M\right) = \bigcap_{i=1}^n r(Q_i : M) = \bigcap_{i=1}^n r_M(Q_i) = \bigcap_{i=1}^n \mathfrak{p} = \mathfrak{p}.$$

Now let $x \in A$ be such that there exists $m \in M \setminus \bigcap_{i=1}^n Q_i$ such that $xm \in \bigcap_{i=1}^n Q_i$. Then there exists some index i such that $m \in M \setminus Q_i$, but $xm \in Q_i$. Since Q_i is primary in M, this implies that $x \in r_M(Q_i) = \mathfrak{p}$. Therefore, $\cap_{i=1}^n Q_i$ is $\mathfrak{p}$-primary in M.

(4.4) As usual, let $(Q : m) = \{x \in A : xm \in Q\}$. From this, (i) is obvious since if $m \in Q$, $xm \in Q$ for all $x \in A$. For (ii), assume $m \notin Q$ and assume $xy \in (Q : m)$ so that $xym \in Q$ and $x \notin (Q : m)$. Since $xm \notin Q$ and Q is primary, this implies that $y \in r_M(Q) = \mathfrak{p}$. From this, it suffices to show that $r(Q : m) = \mathfrak{p}$. Notice that if $x \in \mathfrak{p} = r_M(Q)$, then for some $q > 0$, $x^q M \subseteq Q$. In particular, $x^q m \in Q$. Therefore, $x \in r(Q : m)$. Conversely, if $x \in (Q : m)$, then $xm \in Q$ implies that $x \in r_M(Q) = \mathfrak{p}$ (since $m \notin Q$). Taking radicals of the latter inclusion, we get that $r(Q : m) = \mathfrak{p}$ so that from the above, $(Q : m)$ is $\mathfrak{p}$-primary in M.

22. A primary decomposition of a submodules N in M is a representation of the form

$$N = Q_1 \cap \ldots \cap Q_n,$$

where the submodules Q_i are primary in M. This decomposition is minimal if all the $\mathfrak{p}_i = r_M(Q_i)$ are distinct and for all j, $\cap_{i \neq j} Q_i \not\subseteq Q_j$.

Let N be a decomposable submodule of M with minimal primary decomposition

$$N = \bigcap_{i=1}^n Q_i,$$

with $\mathfrak{p}_i = r_M(Q_i)$. Notice for any $m \in M$,

$$(N : m) = \bigcap_{i=1}^n (Q_i : m)$$

so that

$$r(N : m) = \bigcap_{i=1}^n r(Q_i : m) = \bigcap_{m \notin Q_i} \mathfrak{p}_i.$$

If $r(N : m)$ is prime for some $m \in M$, then from chapter 1, we necessarily have $r(N : m) = \mathfrak{p}_i$ for some $1 \leq i \leq n$. Therefore, every prime ideal of the form $r(N : m)$ is one of the $\mathfrak{p}_i$. Conversely, for each j, we may take $m \in \cap_{i \neq j} Q_i$ so that $m \notin Q_j$. Then from the previous problem, we have exactly $r(N : m) = \mathfrak{p}_j$. This shows that the $\mathfrak{p}_i$ are independent of the decomposition and that they are exactly the prime ideals of the form $r(N : m)$ for $m \in M$. Notice that for $\overline{m} \in M/N$, $(0 : \overline{m}) = \{x \in A : xm \in N\} = (N : x)$. Therefore, $r(0 : \overline{m}) = r(N : x)$ so that if N is decomposable, its associated primes are those primes associated to 0 in M/N.

23. The proof of (4.7) is almost identical for submodules with the set of zero-divisors in N given by

$$D = \bigcup_{m \notin N} r(N : m).$$

For the proof of (4.8), for (i), if $m/s \in S^{-1}M$ and $x \in S \cap \mathfrak{p} = S \cap r_M(Q)$, then $x^q \in S$ and $x^q M \subseteq Q$ for some $q > 0$. Then $m/s = x^q m / x^q s \in S^{-1}Q$. Therefore, $S^{-1}Q = S^{-1}M$. For (ii), notice first that for a primary submodule Q of M, $S^{-1}Q$ is also primary (with prime radical $r_{S^{-1}M}(S^{-1}Q)) = r(S^{-1}Q : S^{-1}M) = S^{-1}(r(Q : M)) = S^{-1}\mathfrak{p}$). Similarly, the preimage of a primary module is primary (with radical equal to the preimage of the radical). If $S \cap \mathfrak{p} = \emptyset$, then $sm \in Q$ implies $m \in Q$ (since if $m \notin Q$, Q is primary and implies $s \in r_M(Q) = \mathfrak{q}$). From this (and chapter 3), the preimage of $S^{-1}Q$ is exactly Q (using the same verification that $\mathfrak{q}^{ec} = \cup_{s \in S}(\mathfrak{q} : s)$). Similarly, the correspondence holds in the other direction as well since every submodule of $S^{-1}M$ is of the form $S^{-1}N$ for some submodule N of M (which is easy to verify). Therefore, there is a bijective correspondence between primary submodules of M and primary submodules of $S^{-1}M$.

The proofs of (4.9), (4.10), and (4.11) are the exact same.

Chapter 5

1. Consider an integral ring homomorphism $f : A \mapsto B$. Note that we can write this map as the composition

$$A \xrightarrow{f} f(A) \xrightarrow{i} B,$$

where $i : f(A) \mapsto B$ is the inclusion map, which is injective and integral since f is integral. Then the map $f^* : \mathrm{Spec}(f(A)) \mapsto \mathrm{Spec}(A)$ is a homeomorphism onto $V(\ker f)$ (since $f : A \mapsto f(A)$ is surjective) and so is a closed map. Therefore, it suffices to show that $i^* : \mathrm{Spec}(B) \mapsto \mathrm{Spec}(f(A))$ is closed. That is, we can reduce to the case that $A \subseteq B$, B is integral over A and f is the inclusion map.

If $A \subseteq B$ and B is integral over A, then the result follows if we can show that

$$i^*(V(\mathfrak{b})) = \{\mathfrak{q} \cap A \in \mathrm{Spec}(A) : \mathfrak{b} \subseteq \mathfrak{q}\} = V(\mathfrak{b} \cap A),$$

where $i : A \mapsto B$ is the inclusion map and $\mathfrak{b}$ is an ideal of B. The inclusion $\subseteq$ is immediate. For the other inclusion, let $\mathfrak{b} \cap A \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Spec}(A)$. We know that $B/\mathfrak{b}$ is integral over $A/(\mathfrak{b} \cap A)$. From the text (5.10), for every prime ideal $\mathfrak{p}/(\mathfrak{b} \cap A)$ of $A/(\mathfrak{b} \cap A)$, there exists an ideal $\mathfrak{q}$ of $B/\mathfrak{b}$ such that $\mathfrak{p}/(\mathfrak{b} \cap A) = \mathfrak{q} \cap (A/(\mathfrak{b} \cap A))$. Then $\mathfrak{q}^c \in \mathrm{Spec}(B)$ is such that $\mathfrak{b} \subseteq \mathfrak{q}^c$ and $\mathfrak{q}^c \cap A = \mathfrak{p}$ (draw the box diagram, it commutes. $\mathfrak{q}^c \cap A$ is the contraction one way and is equal to the contraction the other way, which is obviously $\mathfrak{p}$). Therefore, $i^*(V(\mathfrak{b})) = V(\mathfrak{b} \cap A)$ and i^* is closed. This implies that $i^* : \mathrm{Spec}(B) \mapsto \mathrm{Spec}(f(A))$ is closed and so the composition $(i \circ f)^* : \mathrm{Spec}(B) \mapsto \mathrm{Spec}(A)$ is closed.

2. Let $A \subseteq B$ be rings where B is integral over A and $f : A \mapsto \Omega$ be a ring homomorphism into an algebraically closed field Ω. Since Ω is a field, $f(A)$ is an integral domain, which implies that $\mathfrak{p} = \ker f$ is prime in A. Therefore, there exists $\mathfrak{q}$ prime in B such that $\mathfrak{p} = \mathfrak{q} \cap A$. Then $A/\mathfrak{p}$ and $B/\mathfrak{q}$ are integral domains, we may consider $A/\mathfrak{p} \subseteq B/\mathfrak{q}$, and $B/\mathfrak{q}$ is integral over $A/\mathfrak{p}$. We may write $f : A \mapsto B$ as the composition

$$A \xrightarrow{\pi} A/\mathfrak{p} \xrightarrow{\overline{f}} \Omega.$$

From this, if $\overline{f}$ extends to a map $\overline{g} : B/\mathfrak{q} \mapsto \Omega$, then f extends to a map $g : B \mapsto \Omega$. Therefore, it suffices to consider the case that $A \subseteq B$ are integral domains, B is integral over A, and f is injective.

From here, localize at the zero ideal (in other words, consider the field of fractions). We have that $(A \setminus \{0\})^{-1}B$ is integral over frac(A). Since frac(A) is a field, $(A \setminus \{0\})^{-1}B$ is a field as well. Since frac(B) is the smallest field in which B embeds and $(A \setminus \{0\})^{-1}B \subseteq \mathrm{frac}(B)$, we necessarily have that $(A \setminus \{0\})^{-1}B = \mathrm{frac}(B)$. Therefore, frac$(B)$ is integral over frac(A). That is, frac(B) is an algebraic extension of frac(A). Considering the induced map $\widetilde{f} : \mathrm{frac}(A) \mapsto \Omega$ defined by $f(a_1/a_2) = f(a_1)/f(a_2)$ (remembering f is injective), there is necessarily a map $g : \mathrm{frac}(B) \mapsto \Omega$ that agrees on frac(A) (this follows from the fact that Ω is algebraically closed via Zorn's lemma). Consider the composition $g \circ i$, where $i : B \mapsto \mathrm{frac}(B)$ is the inclusion map. Clearly, this map agrees with f on A and so is an extension of f to B.

To prove the property above of algebraic closures, here is another proof of the problem, from which that property immediately follows. Let $A \subseteq B$ be rings such that B is integral over A and let $f : A \mapsto \Omega$ be a ring homomorphism into an algebraically closed field Ω. Let Σ be the set of pairs (R, g) where R is a subring of B containing A and $g : R \mapsto \Omega$ is a ring homomorphism that restricts to f on A. This set is nonempty since $(A, f) \in \Sigma$. Order Σ as follows. For (R_1, g) and (R_2, g'), say $(R_1, g) \leq (R_2, g')$ if $R_1 \subseteq R_2$ and $g = g'\big|_{R_1}$. For any increasing chain $(R_1, g_1) \leq (R_2, g_2) \leq \ldots$, there is clearly a well-defined function g on $R = \cup R_i$ defined by $g(x) = g_i(x)$ for $x \in R_i$. Therefore, by Zorn's lemma, there is some maximal element (R, g) of Σ. The claim now is that $R = B$, from which it follows that g is an extension of f to B.

Let $b \in B$. Since b is integral over A, b is integral over R, so there exists a monic polynomial of minimal degree $f \in R[x]$ such that $f(b) = 0$. Notice that if $g \in R[x]$ and $g(b) = 0$, then we may apply the Euclidean algorithm since f is monic to see that $f \mid g$ (since $\deg f$ is minimal). That is, $g \in (f)$. Conversely, every element of (f) has b as a root. Therefore, the map $m_b : R[x] \mapsto R[b]$ that sends $x \mapsto b$ has kernel exactly (f). That is, $R[b] \simeq R[x]/(f)$. Since Ω is algebraically closed, there exists $\zeta \in \Omega$ such that $f(\zeta) = 0$. Consider the composition

$$R[x] \xrightarrow{\widetilde{g}} \Omega[x] \xrightarrow{m_\zeta} \Omega.$$

Clearly, $f \in \ker(m_\zeta \circ \widetilde{g})$ so that there is an induced map $g' : R[b] \mapsto \Omega$. It is clear that restricted to R, this map agrees with g since it factors through the above composition. Therefore, $g'\big|_A = f$ so that $(R[b], g') \in \Sigma$. By maximality of (R, g), this then implies that $R[b] = R$ so that $b \in R$.

3. Let $f : B \mapsto C$ be an integral A-algebra homomorphism, D an arbitrary A-algebra, and $f \otimes \mathrm{Id} : B \otimes D \mapsto C \otimes D$ be the map f tensored with Id_D. For $c \otimes d \in C \otimes D$, there exists $b_i \in B$ such that

$$c^n + f(b_n)c^{n-1} + f(b_{n-1})c^{n-2} \ldots + f(b_n) = 0.$$

Consider the polynomial

$$x^n + (f(b_n) \otimes d)x^{n-1} + (f(b_{n-1}) \otimes d^2)x^{n-2} + \ldots + (f(b_n) \otimes d^n).$$

It is clear by plugging in $c \otimes d$ that $c \otimes d$ is a zero of this polynomial. Since the coefficients are in $f(B) \otimes D = (f \otimes \mathrm{Id})(B \otimes D)$ and $c \otimes d \in C \otimes D$ was arbitrary, $C \otimes D$ is integral over $(f \otimes \mathrm{Id})(B \otimes D)$ and the map $f \otimes \mathrm{Id}$ is integral.

4. In general, $B_{\mathfrak{n}}$ is not integral over $A_{\mathfrak{m}}$. To see this, consider $\mathbb{Q}[x^2 - 1] \subseteq \mathbb{Q}[x]$ with $\mathfrak{n} = (x - 1)$ so that $\mathfrak{m} = (x^2 - 1)$ (this is an integral extension since x is a root of $t^2 - x^2$ where $x^2 \in \mathbb{Q}[x^2 - 1]$. Then sums and products are integral as well). Then $A_{\mathfrak{m}} = A = \mathbb{Q}[x^2 - 1]$ and $1/(x + 1) \in B_{\mathfrak{n}}$. By the "rational root theorem" (using $\mathbb{Q}(x) = \mathrm{frac}(\mathbb{Q}[x])$ to find roots), $1/(x + 1)$ satisfies no monic polynomial in $\mathbb{Q}[x]$, let alone $\mathbb{Q}[x^2 - 1] = A_{\mathfrak{m}}$. Therefore, $1/(x + 1)$ is not integral over $A_{\mathfrak{m}}$ so that $B_{\mathfrak{n}}$ is not integral over $A_{\mathfrak{m}}$.

5a. Let $A \subseteq B$ be rings with B integral over A and $a \in A$ be a unit in B. Then there exists $b \in B$ such that $ab = 1$. Let a_i be such that

$$b^n + a_1 b^{n-1} + \ldots + a_n = 0.$$

Then multiplying through by a^n, we get

$$1 + a_1 a + \ldots + a_n a^n = 0 \implies a(-a_1 - \ldots - a_n a^{n-1}) = 1.$$

Therefore, a is a unit in A as well.

5b. Let $\mathfrak{J}_A, \mathfrak{J}_B$ be the Jacobson radicals of A and B respectively. If $a \in A \cap \mathfrak{J}_B$, then $1 - ya$ is a unit in B for every $y \in A$. From the above, this implies that $1 - ya$ is a unit in A for every $y \in A$. That is, $a \in \mathfrak{J}_A$. If $a \in \mathfrak{J}_A$, then for every maximal ideal $\mathfrak{m}$ of B, $A \cap \mathfrak{m} = \mathfrak{m}^c$ is a maximal ideal of A so that $a \in A \cap \mathfrak{m}$. Therefore,

$$a \in \bigcap_{\mathfrak{m} \text{ maximal}} (A \cap \mathfrak{m}) = A \cap \mathfrak{J}_B.$$

Therefore, we have $\mathfrak{J}_A = A \cap \mathfrak{J}_B$.

6. Let $B_1, \ldots, B_n$ be integral A-algebras and let $B = \prod_{i=1}^n B_i$. For each element $(b_1, \ldots, b_n) \in B$, there exists monic polynomials $f_i \in A[x]$ such that $f(b_i) = 0$. Consider the monic polynomial $f(x) = \prod_{i=1}^n f_i(x) \in A[x]$. Clearly,

$$f((b_1, \ldots, b_n)) = (f(b_1), \ldots, f(b_n)) = 0.$$

Therefore, each element of B is integral over A and hence, B is an integral A-algebra (technically, there should be some discussion about how the monic polynomials f_i are in $B_i[x]$ and the coefficients are elements of the image of A in B_i, but this is a notational issue alone).

7. Let $A \subseteq B$ be rings such that $B \setminus A$ is a multiplicative subset of B and C be the integral closure of A in B. Clearly, $A \subseteq C$. For $b \in C$, there exists a_i such that

$$b^n + a_1 b^{n-1} + \ldots + a_n = 0 \in A.$$

Therefore, we may find n minimal such that there exists coefficients a_i satisfying

$$b^n + a_1 b^{n-1} + \ldots + a_n \in A.$$

Since A is a ring, this implies

$$(b^{n-1} + a_1 b^{n-2} + \ldots + a_{n-1})b \in A.$$

Since $B \setminus A$ is multiplicative, this implies either $b \in A$ or $b^{n-1} + a_1 b^{n-2} + \ldots + a_{n-1} \in A$. The latter is impossible by minimality of n. Therefore, $b \in A$ and $C = A$ so that A is integrally closed in B.

8a. Let A be a subring of an integral domain B and let C be the integral closure of A in B. Assume $f, g \in B[x]$ are two monic polynomials such that $fg \in C[x]$. Consider the splitting field k of $\{f, g\}$, where we may write

$$f(x) = \prod_{i=1}^{n}(x - \xi_i), \;\; g(x) = \prod_{j=1}^{m}(x - \zeta_j).$$

Since each root ξ_i and ζ_j are roots of the monic polynomial fg with coefficients in C, they are integral over C in k. Notice that the coefficients of f and g are symmetric polynomials in the ξ_i and ζ_j respectively. Therefore, the coefficients of f and g are also integral over C in k. By transitivity, this implies the coefficients of f and g are integral over A in k. Since the coefficients are in B and they are integral over A, this implies that the coefficients lie in C. That is, $f, g \in C[x]$.

8b. Let $A \subseteq B$ be rings and let C be the integral closure of A in B. Assume $f, g \in B[x]$ are two monic polynomials such that $fg \in C[x]$. For any prime ideal $\mathfrak{q}$ of B, $C/(C \cap \mathfrak{q})$ is integral over $A/(A \cap \mathfrak{q})$. Since $\overline{f}\overline{g} \in (C/(C \cap \mathfrak{q}))[x]$, the previous problem implies that the coefficients of $\overline{f}$ and $\overline{g}$ lie in the integral closure of $A/(A \cap \mathfrak{q})$ in $B/\mathfrak{q}$. From this, it suffices to show that if $b \in B$ is such that $\overline{b} \in B/\mathfrak{q}$ is in the integral closure of $A/(A \cap \mathfrak{q})$ for all prime ideals $\mathfrak{q}$ of B, then b is in the integral closure of A in B (that is, $b \in C$).

It suffices to prove the contrapositive. Assume that b is not integral over A. That is, for all monic $f \in A[x]$, $f(b) \neq 0$. Then $S = \{f(b) : f \in A[x] \text{ monic}\}$ is a multiplicative subset of B and $0 \notin S$. Let $\mathfrak{q}$ be any ideal of B such that $S \cap \mathfrak{q} = \emptyset$ (take the contraction of a maximal ideal in $S^{-1}B$). Then consider $\overline{b} \in B/\mathfrak{q}$. If there exists coefficients $\overline{a_i} \in A/(A \cap \mathfrak{q})$ such that

$$\overline{b}^n + \overline{a_1}\overline{b}^{n-1} + \ldots + \overline{a_n} = 0,$$

then there are coefficients $a_i \in A$ such that

$$b^n + a_1 b^{n-1} + \ldots + a_n \in \mathfrak{q}.$$

This is a contradiction because the element above is an element of S, which is disjoint from $\mathfrak{q}$. Therefore, $\overline{b} \in B/\mathfrak{q}$ is not integral over $A/(A \cap \mathfrak{q})$. This shows that if $b \in B$ is such that $\overline{b} \in B/\mathfrak{q}$ is integral over $A/(A \cap \mathfrak{q})$ for all prime ideals $\mathfrak{q}$ of B, then b is integral over A. From the above, this then implies that the coefficients of f and g lie in C.

9. Let $A \subseteq B$ be rings and C be the integral closure of A in B. For $f \in C[x]$, the coefficients c_i are in C so that they are integral over $A[x]$. Then $A[x][c_1, \ldots, c_n]$ is a finitely-generated $A[x]$-module. Since $f \in A[x][c_1, \ldots, c_n] \subseteq C[x]$, we have that f is integral over $A[x]$ by the third condition for an element to be integral. Conversely, let $f \in B[x]$ be integral over $A[x]$. Then there exists g_i such that f satisfies

$$f^n + g_1 f^{n-1} + \ldots + g_n = 0.$$

Let $r > \max\{n, \deg f, \deg g_1, \ldots, \deg g_n\}$ be an integer. Notice that $f_1(x) = f(x) - x^r$ satisfies the equation

$$(f_1 + x^r)^m + g_1(f_1 + x^r)^{m-1} + \ldots + g_m = 0.$$

Expanding this out, we get an equation of the form

$$f_1^m + h_1 f_1^{m-1} + \ldots + h_m = 0,$$

where

$$h_m = (x^r)^m + g_1(x^r)^{m-1} + \ldots + g_m \in A[x].$$

We see

$$-f_1(f_1^{m-1} + h_1 f_1^{m-2} + \ldots + h_{m-1}) = h_m \in A[x] \subseteq C[x].$$

Since $-f_1$ is monic and the latter polynomial is monic by out assumption of r (with a possible negative sign), this implies that $-f_1 \in C[x]$. Since $x^r \in C[x]$, this also implies that $f \in C[x]$. Therefore, the algebraic closure of $A[x]$ in $B[x]$ is $C[x]$.

10a. $(i) \implies (ii)$ Let $f : A \mapsto B$ be a ring homomorphism such that the induced map $f^* : \operatorname{Spec}(B) \mapsto \operatorname{Spec}(A)$ is a closed map. Let $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ be prime ideals of $f(A)$ and let $\mathfrak{q}_1$ be an ideal of B such that $\mathfrak{q}_1 \cap f(A) = \mathfrak{p}_1$. Consider the closed subset $V(\mathfrak{q}_1)$ of $\operatorname{Spec}(B)$. Since f is closed, we necessarily have $f^*(V(\mathfrak{q}_1)) = V(\mathfrak{p}_1)$. From this, there exists $\mathfrak{q}_2 \in V(\mathfrak{q}_1)$ such that $f^*(\mathfrak{q}_2) = \mathfrak{q}_2 \cap f(A) = \mathfrak{p}_2$. That is, $\mathfrak{q}_2$ satisfies $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ and $\mathfrak{q}_1 \cap f(A) = \mathfrak{p}_1$, $\mathfrak{q}_2 \cap f(A) = \mathfrak{p}_2$. The going-up property then follows immediately by induction.

$(ii) \iff (iii)$ For a ring homomorphism $f : A \mapsto B$, fixed prime ideals $\mathfrak{q}$ of B and $\mathfrak{p} = \mathfrak{q}^c$ of A, we may write the map $f : A/\mathfrak{p} \mapsto B/\mathfrak{q}$ as the composition

$$A/\mathfrak{p} \xrightarrow{f} f(A)/(\mathfrak{q} \cap f(A)) \xrightarrow{i} B/\mathfrak{q}.$$

It takes a moment to show that the first map is an isomorphism. Therefore, we have maps

$$\operatorname{Spec}(B/\mathfrak{q}) \xrightarrow{i^*} \operatorname{Spec}(f(A)/(\mathfrak{q} \cap f(A))) \xrightarrow{f^*} \operatorname{Spec}(A/\mathfrak{p}),$$

where the latter is a homeomorphism. Almost by definition, f has the going-up property if and only if the first map is surjective. However, the first map is surjective if and only if the composition is surjective. This completes the proof.

Note that there is an equivalent notion for a map $f : A \mapsto B$ to have the going-up property as follows. For any prime ideals $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Spec}(A)$ such that $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ and there exists $\mathfrak{q}_1$ satisfying $f^*(\mathfrak{q}_1) = \mathfrak{p}_1$, then there exists $\mathfrak{q}_2$ such that $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ and $f^*(\mathfrak{q}_2) = \mathfrak{p}_2$. This equivalence follows immediately from the factorization $A \mapsto f(A) \mapsto B$ where the first map is a homeomorphism on spectra. Therefore, the above follow with this adjusted definition.

10b. $(i) \implies (iii)$ Let $f : A \mapsto B$ be a ring homomorphism such that the induced map $f^* : Y \mapsto X$ is an open map (where $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(B)$). For prime ideals $\mathfrak{q}$ of B and $\mathfrak{p}$ of A, there is a map $f : A/\mathfrak{p} \mapsto B/\mathfrak{q}$. Notice (see after the problem) that $B_\mathfrak{q} = \lim_{t \in B \setminus \mathfrak{q}} B_t$. From a previous problem, this implies

$$f^*(\operatorname{Spec}(B_\mathfrak{q})) = \bigcap_{t \in B \setminus \mathfrak{q}} f^*(i_t^*(\operatorname{Spec}(B_t))),$$

where $i_t : B \mapsto B_t$ is the inclusion map. From another problem, $i_t^*(\operatorname{Spec}(B_t)) = Y_t = Y \setminus V(t)$. Therefore, each of these sets is open in Y and contains $\mathfrak{q}$ (since $t \notin \mathfrak{q}$). Therefore, $f^*(Y_t)$ is an open set containing $\mathfrak{p}$. From this, each $f^*(Y_t)$ contains $\operatorname{Spec}(A_\mathfrak{p})$ (if not, take a prime ideal contained in $\mathfrak{p}$ not in $f^*(Y_t)$. Its closure is disjoint from $f^*(Y_t)$ since this set is open, but $\mathfrak{p}$ is in the closure. This is a contradiction since $\mathfrak{p} \in f^*(Y_t)$). Therefore, $\operatorname{Spec}(A_\mathfrak{p})$ is contained in the intersection and so $f^*(\operatorname{Spec}(B_\mathfrak{q})) = \operatorname{Spec}(A_\mathfrak{p})$ and f^* is surjective.

To show that $B_\mathfrak{q} = \lim_{t \in B \setminus \mathfrak{q}} B_t$, it will be shown that $B_\mathfrak{q}$ satisfies the universal property of the direct limit of the directed system (B_t, μ_{ts}) where the set of B_t is ordered under inclusion. Let $f_t : B_t \mapsto A$ be a sequence of maps such that $f_t = f_s \circ \mu_{ts}$ for all $t \leq s$. For $b/s \in B_\mathfrak{q}$ define $f : B_\mathfrak{q} \mapsto A$ by

$$f(b/s) = f_s(b/s).$$

This map is well-defined since for $b/s = b'/t$,

$$f_s(b/s) = f_{st}(bt/st) = f_{st}(b's/st) = f_t(b'/t).$$

It is easy to check that this is a ring homomorphism by finding an upper bound for indices involved. It is simple to check that f satisfies $f_t = f \circ \mu_t$ for each $t \in B \setminus \mathfrak{q}$. To show that this map is unique, assume there exists $g : B_\mathfrak{q} \mapsto A$ that also satisfies $f_t = g \circ \mu_t$ for each $t \in B \setminus \mathfrak{q}$. As usual, the range of the μ_t cover $B_\mathfrak{q}$ so that f and g necessarily agree pointwise. Therefore, $f = g$. From all of this, $B_\mathfrak{q} = \lim_{t \in B \setminus \mathfrak{q}} B_t$ and the result above follows.

$(ii) \iff (iii)$ (Assuming that f is injective) Note that the map $f : A_\mathfrak{p} \mapsto B_\mathfrak{q}$ can be factored as follows.

$$A_\mathfrak{p} \xrightarrow{f} (f(A \setminus \mathfrak{p}))^{-1} f(A) \xrightarrow{i} B_\mathfrak{q}$$

Since f is injective, $f : A \mapsto f(A)$ is an isomorphism so that $f : A_\mathfrak{p} \mapsto (f(A))_\mathfrak{p} \simeq (f(A \setminus \mathfrak{p}))^{-1} f(A)$ is an isomorphism. Notice that $f(A \setminus \mathfrak{p}) = f(A) \setminus (\mathfrak{q} \cap f(A))$ so that the above can be written as the composition

$$A_\mathfrak{p} \xrightarrow{f} f(A)_{\mathfrak{q} \cap f(A)} \xrightarrow{i} B_\mathfrak{q},$$

where the first map is an isomorphism. The induced maps on spectra give the compositions

$$\mathrm{Spec}(B_{\mathfrak{q}}) \xrightarrow{i^*} \mathrm{Spec}(f(A)_{\mathfrak{q}\cap f(A)}) \xrightarrow{f^*} \mathrm{Spec}(A_{\mathfrak{p}}),$$

where the latter map is a homeomorphism. f has the going-down property if and only if the first map is surjective, almost by definition. Note that the latter map is bijective so the composition is surjective if and only if the first map is surjective. That is, the map $\mathrm{Spec}(B_{\mathfrak{q}}) \mapsto \mathrm{Spec}(A_{\mathfrak{p}})$ is surjective if and only if f has the going-down property.

To show that the above does not hold in the case that f is not injective, consider the map $\phi : \mathbb{Z}[x] \mapsto \mathbb{Z}[x]/(2,x) \simeq \mathbb{Z}/2\mathbb{Z}$. This map is surjective and so satisfies the going-down property defined in the problem statement. However, if $\mathfrak{q} = (0)$ and $\mathfrak{p} = \mathfrak{q}^c = (2,x)$, then $\mathrm{Spec}((Z[x])_{\mathfrak{p}})$ is the set of all prime ideals contained in $\mathfrak{p} = (2,x)$, which contains (2) and (x) in particular. On the other hand, $(\mathbb{Z}/2\mathbb{Z})_{\mathfrak{q}} = \mathbb{Z}/2\mathbb{Z}$ has only one prime ideal, (0), so the induced map $\mathrm{Spec}((\mathbb{Z}/2\mathbb{Z})_{\mathfrak{q}}) \mapsto \mathrm{Spec}((Z[x])_{\mathfrak{p}})$ is not surjective.

On the other hand, let $A \subseteq B$ be any two rings not satisfying the going-down property (that is, the map $\mathrm{Spec}(B_{\mathfrak{q}}) \mapsto \mathrm{Spec}(A_{\mathfrak{p}})$ induced by the inclusion map is not surjective some prime $\mathfrak{q}$ and $\mathfrak{p} = \mathfrak{q}^c$). Then let $A' = A[x]/(x^2)$ and let $\phi : A' \mapsto B$ be evaluation at $x = 0$ (which is easily a well-defined ring homomorphism). Then the image is easily A so that ϕ does not have the going-down property. However, since the image of ϕ is A, $\phi^* : \mathrm{Spec}(A) \mapsto V((x^2)) = \mathrm{Spec}(A')$ is a homeomorphism. From this and the statement about the non-surjectivity of the map induced by the inclusion map, it follows that the map $\mathrm{Spec}(B_{\mathfrak{q}}) \mapsto \mathrm{Spec}(A'_{\mathfrak{p}})$ is not surjective for some prime ideal $\mathfrak{q}$ (take the same ideal used to show that $A \subseteq B$ does not satisfy the going-down property. Then the map $\mathrm{Spec}(A'_{\mathfrak{p}}) \mapsto \mathrm{Spec}(A_{\mathfrak{p}})$ is a bijection, so the map $\mathrm{Spec}(B_{\mathfrak{q}}) \mapsto \mathrm{Spec}(A'_{\mathfrak{p}})$ cannot be surjective).

As for the going-up scenario, there is an adjusted definition that in this case is slightly stronger. Say that a map $f : A \mapsto B$ has the going-down property if for every two prime ideals $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathrm{Spec}(A)$ such that $\mathfrak{p}_1 \supseteq \mathfrak{p}_2$ and there exists $\mathfrak{q}_1$ such that $f^*(\mathfrak{q}_1) = \mathfrak{p}_1$, then there exists $\mathfrak{q}_2$ such that $\mathfrak{q}_1 \supseteq \mathfrak{q}_2$ and $f^*(\mathfrak{q}_2) = \mathfrak{p}_2$. It will be shown that the above theorem holds with this new definition without the assumption of injectivity.

$(i) \implies (iii)$ This proof needs no adjustment because it does not use the going-down property.

$(ii) \iff (iii)$ This now follows almost immediately. $f^* : \mathrm{Spec}(B_{\mathfrak{q}}) \mapsto \mathrm{Spec}(A_{\mathfrak{p}})$ (where $\mathfrak{p} = f^*(\mathfrak{q})$) is surjective if and only if for every prime ideal $\mathfrak{p}' \subseteq \mathfrak{p}$ of A, there exists a prime ideal $\mathfrak{q}' \subseteq \mathfrak{q}$ such that $f^*(\mathfrak{q}') = \mathfrak{p}'$. That is, this map is surjective if and only if f has the (new) going-down property.

11. If $f : A \mapsto B$ is an injective flat ring homomorphism, then from a previous problem, for any prime ideals $\mathfrak{q}$ of B and $\mathfrak{p} = \mathfrak{q}^c$ of A, the induced map $f^* : \mathrm{Spec}(B_{\mathfrak{q}}) \mapsto \mathrm{Spec}(A_{\mathfrak{p}})$ is surjective. From the equivalences from the previous problem, it follows that f satisfies the going-down property.

    Clearly, using the new definition of going-up and going-down property for maps, any (not necessarily injective) flat ring homomorphism has the going-down property.

12. Let G be a finite group of automorphisms of a ring A and let A^G be the set of elements of A fixed by all elements of G. It is clear that A^G is a subring of A since the elements of G are ring homomorphisms. For $x \in A$, consider the polynomial

$$f(t) = \prod_{g\in G}(t - g(x)).$$

    Clearly this polynomial is monic and its coefficients are functions symmetric in the $g(x)$ for $g \in G$. Then multiplication by an element $h \in G$ is a permutation of the $g(x)$ and so the coefficients therefore remain unchanged. That is, the coefficients of $f(t)$ are in A^G. Clearly, the identity $\mathrm{Id} \in G$ so that $f(x) = 0$. That is, x is integral over A. Since $x \in A$ was arbitrary, A is integral over A^G.

    Let $S \subseteq A$ be multiplicative (not containing 0) such that $g(S) \subseteq S$ for every $g \in G$ and define $S^G = S \cap A^G$. Since $\ker g = \{0\}$ for each $g \in G$, $S \cap \ker g = \emptyset$ and so each g induces a ring homomorphism $g : S^{-1}A \mapsto S^{-1}A$ defined by

$$g(a/s) = g(a)/g(s).$$

Define a map $\phi : A^G \mapsto (S^{-1}A)^G$ defined by $\phi(a) = a/1$ (clearly, $a/1 \in (S^{-1}A)^G$ since $g(1) = 1$ for all $g \in G$). For $s \in S^G$, we have $1/s \in (S^{-1}A)^G$ so $\phi(s)$ is a unit for each $s \in S^G$. If $\phi(a) = a/1 = 0$, then there exists $s \in S$ such that $as = 0$. Then applying any element $g \in G$, we get $ag(s) = 0$. Summing these, we get $as' = 0$, where $s' = \sum_{g\in G} g(s) \in S^G$. Finally it is shown that every element of $(S^{-1}A)^G$ can be written in the form $\phi(a)\phi(s)^{-1}$. It is clear that every element of $(S^{-1}A)^G$ can be written in the form

$$\frac{a}{s} = \frac{a\prod_{g\neq \mathrm{Id}} g(s)}{\prod_{g\in G} g(s)},$$

where the denominator is in S^G. Therefore, it suffices to consider when $s \in S^G$ and $a/s \in (S^{-1}A)^G$. In this case, it is clear that $a/1 = (a/s)(s/1) \in (S^{-1}A)^G$. Therefore, for each $g \in G$, there exists $s_g \in S$ such that $s_g(a-g(a)) = 0$. Multiplying by $\prod_{h\neq \mathrm{Id}} h(s_g)$, we get $s'_g(a-g(a)) = 0$ for some $s'_g \in S^G$. Let $t = \prod_{g\in G} s'_g \in S^G$ so that $t(a - g(a)) = 0$ for all $g \in G$. We see for any $g \in G$,

$$g(ta) = tg(a) = ta.$$

Therefore, $ta \in A^G$. Then $a/s = at/st = \phi(at)\phi(st)^{-1}$ with $st \in S^G$. From all of this, it follows that $(S^G)^{-1}A^G \simeq (S^{-1}A)^G$.

13. Let G be a finite group of automorphisms of a ring A as above, let $\mathfrak{p}$ be a prime ideal of A^G, and let P be the set of prime ideals of A whose contraction is $\mathfrak{p}$ (that is, prime ideals $\mathfrak{q}$ of A such that $\mathfrak{q} \cap A^G = \mathfrak{p}$). Let $\mathfrak{q}, \mathfrak{q}' \in P$. For any $x \in \mathfrak{q}$, since $\mathrm{Id} \in G$,

$$\prod_{g\in G} g(x) \in \mathfrak{q} \cap A^G = \mathfrak{p} \subseteq \mathfrak{q}'.$$

Therefore, there exists $g \in G$ such that $g(x) \in \mathfrak{q}'$ and $x \in h^*(\mathfrak{q}')$ for $h = g^{-1}$. That is, $\mathfrak{q} \subseteq \cup_{g\in G} g^*(\mathfrak{q}')$. Since this union is finite, this implies $\mathfrak{q} \subseteq g^*(\mathfrak{q}')$ for some $g \in G$. But by theorem 5.9, this implies that $\mathfrak{q} = g^*(\mathfrak{q}')$. From this, it is clear that the action of G is transitive.

14. Let A be an integrally closed integral domain, k its field of fractions, and L a finite normal separable extension of k (it is Galois). Let G be the Galois group of L over k and let B be the integral closure of A in L. For $b \in B$, there is some monic polynomial $f \in A[x]$ that b satisfies. It is clear that $f(\sigma(b)) = \sigma(f(b)) = 0$ so that $\sigma(b) \in L$ satisfies the same polynomial and is integral over A. That is, $\sigma(b) \in B$ and $\sigma(B) \subseteq B$. Conversely, since σ is an automorphism, the same can be done with σ^{-1} to get that $\sigma^{-1}(B) \subseteq B$ so that $B \subseteq \sigma(B)$. This implies $\sigma(B) = B$ for all $\sigma \in G$. Since the extension $k \subseteq L$ is Galois, the only elements of L that are fixed by all elements of G are elements of k. From this, $B^G \subseteq k$. However, since A is integrally closed, the only integral elements of k over A are exactly the elements of A. Therefore, $B^G \subseteq A$. The other inclusion is obvious. Therefore, $B^G = A$.

*15.

16. Let k be an infinite field and A a finitely-generated k-algebra. The result will be shown by induction on n. The base case $n = 1$ is trivial since then $A = k[x_1]$ is integral over itself. Assume the result for a fixed $n-1$. Assume now A is generated by $x_1, \dots, x_n$ and that they are ordered so that $x_1, \dots, x_r$ are algebraically independent and $x_{r+1}, \dots, x_n$ are algebraic over $k[x_1, \dots, x_r]$ (take r maximal, then $x_{r+1}, \dots, x_n$ are necessarily algebraic over $k[x_1, \dots, x_r]$). If $r = n$, there is nothing to prove since $A = k[x_1, \dots, x_r]$ is integral over itself. If $r < n$, then x_n is algebraic over $k[x_1, \dots, x_{n-1}]$. That is, there exists a nonzero polynomial $f \in k[t_1, \dots, t_{n-1}, t_n]$ such that $f(x_1, \dots, x_{n-1}, x_n) = 0$. Let F be the homogeneous term of highest degree. Since k is infinite, there exists some $\lambda_1, \dots, \lambda_{n-1} \in k$ such that $F(\lambda_1, \dots, \lambda_{n-1}, 1) \neq 0$. Since k is a field, we may then normalize so that $F(\lambda_1, \dots, \lambda_{n-1}, 1) = 1$. Let $x'_i = x_i - \lambda_i x_n$. We have

$$F(x_1, \dots, x_n) = F(x'_1 + \lambda_1 x_n, \dots, x'_{n-1} + \lambda_{n-1}x_n, x_n).$$

Writing out F component wise and expanding this, we get a monic polynomial in x_n with coefficients in $k[x'_1, \dots, x'_{n-1}]$ and degree equal to the total degree of F. Since F was the monomial of maximal total degree of f, the polynomial

$$0 = f(x_1, \dots, x_{n-1}, x_n) = f(x'_1 + \lambda_1 x_n, \dots, x'_{n-1} + \lambda_{n-1}x_n, x_n)$$

as a polynomial in x_n with coefficients in $k[x'_1, \ldots, x'_{n-1}]$ is monic. Therefore, x_n is integral over $k[x'_1, \ldots, x'_{n-1}]$ and therefore, $A = k[x_1, \ldots, x_n]$ is integral over $k[x'_1, \ldots, x'_{n-1}]$ (since $x_i = x'_i + \lambda_i x_n$). By our inductive hypothesis, there exists algebraically independent over k, $y_1, \ldots, y_s \in k[x'_1, \ldots, x'_{n-1}]$, such that $k[x'_1, \ldots, x'_{n-1}]$ is integral over $k[y_1, \ldots, y_s]$. By transitivity, we then have A is integral over $k[y_1, \ldots, y_s]$. From this proof (that is, our choice of algebraically independent elements), it follows that we may choose the y_i to be linear combinations of the x_i (since they are in fact x'_i for $1 \leq i \leq r$).

Let k be algebraically closed and X be an algebraic variety of k^n with nonzero coordinate ring A (which is finitely generated from the surjective map $k[t_1, \ldots, t_n] \mapsto A$). From the previous problem, there are $y_1, \ldots, y_r \in A$ that are algebraically independent over k and A is integral over $k[y_1, \ldots, y_r]$. We may write $y_i = \sum_j a_{ij} x_j$. Then let $\varphi : k[y_1, \ldots, y_r] \mapsto A$ be the inclusion map. This is clearly a k-algebra homomorphism and so corresponds to a regular map $\phi : X \mapsto k^r$ where $\phi^* = \varphi$. It is clear from this that ϕ is defined by

$$\phi(t_1, \ldots, t_n) = \left(\sum_j a_{1j} t_j, \ldots, \sum_j a_{rj} t_j \right).$$

To see that ϕ is surjective, consider the inclusion map $i_a : \{0\} \mapsto k^r$ sending 0 to a. This induces a map $i_a^* : k[t_1, \ldots, t_r] \mapsto k$. Considering $k[t_1, \ldots, t_r]$ as $k[y_1, \ldots, y_r] \subseteq A$ (since the y_i are algebraically independent), since A is integral over $k[y_1, \ldots, y_r]$, there is an extension of this map to the k-algebra homomorphism $\theta : A \mapsto k$. Therefore, there is a map $\mu : \{0\} \mapsto X$. The image of this map then satisfies $\phi(\mu(0)) = a$ (draw the diagram, it commutes). Therefore, ϕ is surjective.

(Note: The proof of surjectivity did not feel very natrual. I looked some things up and this seemed to be the consensual argument.)

17. Let X be an affine algebraic variety in k^n where k is an algebraically closed field and let I be a defining ideal for X. If $I \neq (1)$, then let $A = k[t_1, \ldots, t_n]/I$ be the coordinate ring of X. It is clear that $A \neq 0$ so that from the above, there is a surjective map of X onto a linear subspace of k^n of dimension $r \geq 0$. This implies that $X \neq \emptyset$.

    Assuming k is algebraically closed, let $\mathfrak{m}$ be a maximal ideal of $k[t_1, \ldots, t_n]$. Consider the projection $\pi : k[t_1, \ldots, t_n] \mapsto k[t_1, \ldots, t_n]/\mathfrak{m}$. Since the image is a finitely-generated k-algebra and a field, it follows from Zariski's lemma that the image is a finite algebraic extension of k. Since k is algebraically closed, this implies that the image is k itself. Consider the images $\alpha_i = \pi(t_i)$ for $1 \leq i \leq n$. Then clearly, $t_i - \alpha_i \in \mathfrak{m}$ for all i. That is, $(t_i - \alpha_i) \subseteq \mathfrak{m}$. Since this ideal is maximal (its quotient is k itself), we necessarily have an equality of the form

$$\mathfrak{m} = (t_1 - \alpha_1, \ldots, t_n - \alpha_n).$$

18. Note that this result is proved in the text. The proof will follow by induction. Let k be a field and let B be a finitely-generated k-algebra that is also a field. For the base case, if B is generated by one element, x, then x is a unit and so there exists coefficients k_i such that

$$x(k_1 + \ldots + k_n x^n) = 1.$$

    That is, x is algebraic over k so that B is a finite algebraic extension. Assume now that the result holds for all such B generated by $n-1$ elements. Let B be generated by $x_1, \ldots, x_n$ and let $A = k[x_1]$, $K = k(x_1)$. Since B is a finitely-generated K-algebra (generated by $x_2, \ldots, x_n$) and a field, then B is a finite algebraic extension of K. Therefore, each $x_2, \ldots, x_n$ satisfy a monic polynomial with coefficients in K. If f is the product of denominators of the coefficients from these monomials, then each $x_2, \ldots, x_n$ is integral over A_f. Therefore, B is integral over A_f, but $K \subseteq B$ so that K is integral over A_f. If x_1 is transcendental over k, then A is integrally closed (over its field of fractions) since it is a UFD (by rational root theorem and Gauss' lemma). This should imply that A_f is integrally closed as well since localizations of integrally closed domains are integrally closed. Since K is integral over A_f, this is a contradiction. Therefore, x_1 is algebraic over k. This implies that K is a finite algebraic extension of k and since B is a finite algebraic extension of K, B is a finite algebraic extension of k.

19. This is exactly what was done in problem 17. For the first part, any proper ideal is contained in some maximal ideal which has the form above and so there is some common root in every element of $\mathfrak{m}$ so that the corresponding variety is nonempty.

20. Let $A \subseteq B$ be integral domains where B is a finitely-generated A-algebra. Let $S = A \setminus \{0\}$ and $K = S^{-1}A = \operatorname{frac}(A)$. Then $S^{-1}B$ is a finitely-generated K-algebra (with essentially the same generators). By the normalization lemma, there exists $y_1/s_1, \ldots, y_n/s_n \in S^{-1}B$ such that the y_n/s_n are algebraically independent over K and $S^{-1}B$ is integral over $K[y_n/s_n, \ldots, y_n/s_n]$ (it is easy to see that then the y_i are algebraically independent). If $z_1, \ldots, z_m$ generate B as an A-algebra, then $z_i/1$ generate $S^{-1}B$ as a K-algebra. Therefore, each $z_i/1$ is integral over $K[y_1/s_1, \ldots, y_n/s_n]$ and so satisfies an equation of the form

$$(z_i/1)^n + f_{1i}(y_1/s_1, \ldots, y_n/s_n)(z_i/1)^{n-1} + \ldots + f_{ni}(y_1/s_1, \ldots, y_n/s_n) = 0.$$

Let $s \in S$ be a common denominator for all $f_{ij}(y_1/s_1, \ldots, y_n/s_n)$. Then we have $sf_{ij}(y_1/s_1, \ldots, y_n/z_n) = g_{ij}(y_1, \ldots, y_n) \in A[y_1, \ldots, y_n]$. Let $B' = A[y_1, \ldots, y_n]$. With this common denominator, the above is a monic polynomial in B'_s. That is, the $z_i/1$ are integral over B'_s. Since the $z_i/1$ generate $S^{-1}B$, they generate B_s as well. Therefore, B_s is integral over B'_s.

21. Let $A \subseteq B$ be integral domains where B is a finitely-generated A-algebra and let $f : A \mapsto \Omega$ be a ring homomorphism where Ω is algebraically closed. From the above, there exists $s \in A \setminus \{0\}$ such that B_s is integral over B'_s, where $B' = A[y_1, \ldots, y_n]$ for some y_i algebraically independent over A (since they are algebraically independent over $K = \operatorname{frac}(A)$). This $s \in A \setminus \{0\}$ is the specified s in the problem. To see this, assume $f(s) \neq 0$. Then f extends to a map $B' \mapsto \Omega$ defined by sending each $y_i \mapsto 0$. Since $f(s) \neq 0$, this then extends further to a map $B'_s \mapsto \Omega$ (since Ω is a field, $f(s)$ is invertible). Since B_s is integral over B'_s, from problem 2, this extends even further to a map $B_s \mapsto \Omega$. Mapping $B \mapsto B_s \mapsto \Omega$ is then the desired extension.

22. Let $A \subseteq B$ be integral domains such that B is a finitely-generated A-algebra and the Jacobson radical of A $\mathfrak{J}_A = 0$. Let $v \in B$, $v \neq 0$. From the previous problem, since B_v is a finitely-generated A-algebra as well, there exists some $s \in A \setminus \{0\}$ that satisfies the previous problem. Let $\mathfrak{m}$ be a maximal ideal of A not containing s (if they all do, $s = 0$) and let $k = A/\mathfrak{m}$. The projection map $A \mapsto k$ extends to a map $\phi : B_v \mapsto \Omega$, where Ω is an algebraic closure of k. It is clear that $\phi(v) \neq 0$ (if $\phi(v) = 0$, then $\phi(b) = \phi(v)\phi(b/v) = 0$ for all $b \in B$, but this map extends a nontrivial map and so is nontrivial). Therefore, the composition $\varphi : B \mapsto B_v \mapsto \Omega$ is such that $\varphi(v) \neq 0$. The kernel of this map satisfies $\ker \varphi \cap A = \mathfrak{m}$ and so $\ker \varphi$ is maximal (If $x \in \ker \varphi \cap A$, then the projection of x is zero so $x \in \mathfrak{m}$. Conversely, if $x \in \mathfrak{m}$, $x \in \ker \varphi$ and $x \in A$ since $\mathfrak{m} \subseteq A$). Since $v \notin \ker \varphi$, this implies $v \notin \mathfrak{J}_B$. Since $v \in B$, $v \neq 0$ was arbitrary, this implies that $\mathfrak{J}_B = 0$ as well.

23. $(i) \implies (iii)$ Assume every prime ideal of A is the intersection of maximal ideals. If $\mathfrak{p}$ is a prime ideal that is not maximal, then it can be written

$$\mathfrak{p} = \bigcap_{i \in I} \mathfrak{m}_i,$$

where the $\mathfrak{m}_i$ are maximal. From this, we can include any prime ideal that strictly contains $\mathfrak{p}$ (note the maximal ideals already strictly contain $\mathfrak{p}$) in this intersection (just intersect the above line with $\mathfrak{q}$ where $\mathfrak{p} \subseteq \mathfrak{q}$). That is, we may write

$$\mathfrak{p} = \bigcap_{\substack{\mathfrak{q} \text{ prime} \\ \mathfrak{p} \subset \mathfrak{q}}} \mathfrak{q}.$$

$(iii) \implies (ii)$ The contrapositive will be proved. Assuming (ii) is false, there exists a surjective homomorphism $\phi : A \mapsto B$ where $\mathfrak{N}_B \neq \mathfrak{J}_B$. That is, there exists some prime ideal $\mathfrak{q}$ of B such that $\mathfrak{q}$ cannot be written as the intersection of maximal ideals (where the containment is not necessarily strict. That is, $\mathfrak{q}$ may be maximal itself). This is because if every prime ideal can be written as the intersection of maximal ideals, then the intersection of all prime ideals is equal to the intersection of all maximal ideals (remember every maximal ideal is prime) and therefore, we should have $\mathfrak{N}_B = \mathfrak{J}_B$. Since the map $\phi : A \mapsto B$ is surjective, there is the correspondence of prime ideals in B with prime ideals of A containing $\ker \phi$. It is clear from this that $\mathfrak{p} = \mathfrak{q}^c$ cannot be written as an intersection of maximal ideals (since the correspondence preserves inclusions and so maximal ideals correspond). From this, passing to the quotient $A/\mathfrak{p}$, the Jacobson radical $\mathfrak{J}_{A/\mathfrak{p}} \neq 0$ (consider the set of maximal ideals that contain $\mathfrak{p}$, $\mathfrak{p}$ is not equal to their intersection so the containment is proper). Let $f \in \mathfrak{J}_{A/\mathfrak{p}}$ be nonzero. Then $(A/\mathfrak{p})_f \neq 0$ (f is not nilpotent since $A/\mathfrak{p}$ is an integral domain). Therefore, there is some maximal ideal $\mathfrak{m}$ of $(A/\mathfrak{p})_f$. The contraction $\mathfrak{p}' = \mathfrak{m}^c$ is such that $f \notin \mathfrak{p}'$ and is maximal in this set of ideals that do not contain f (by the correspondence of prime ideals in A with $S^{-1}A$). Then $\mathfrak{p}'$ is not maximal since $f \in \mathfrak{J}_{A/\mathfrak{p}}$ and not equal to the intersection of all prime ideals that strictly contain it because any prime ideal that strictly contains $\mathfrak{p}'$ necessarily contains f. By the prime ideal correspondence of A with $A/\mathfrak{p}$, $\mathfrak{p}'^c$ has these same properties.

$(ii) \implies (i)$ The contrapositive will be proved. If there is a prime ideal $\mathfrak{p}$ of A that is not an intersection of maximal ideals, then consider the map $A \mapsto A/\mathfrak{p}$. This is an integral domain so $\mathfrak{N}_{A/\mathfrak{p}} = 0$, but the intersection of all maximal ideals that contains $\mathfrak{p}$ is not equal to $\mathfrak{p}$ and so the containment

$$\mathfrak{p} \subset \bigcap_{\substack{\mathfrak{m} \text{ maximal} \\ \mathfrak{p} \subset \mathfrak{m}}} \mathfrak{m}$$

is proper. That is, $\mathfrak{J}_{A/\mathfrak{p}} \neq 0$.

Rings satisfying these properties is called a Jacobson ring.

24. The contrapositive of (i) will be proved. Let $f : A \mapsto B$ be an A-algebra, and B be integral over $f(A)$. Assume that B is not a Jacobson radical. Then there exists $\mathfrak{q}$, a prime, non-maximal ideal of B and assume that

$$\mathfrak{q} \subset \bigcap_{\substack{\mathfrak{q}' \text{ prime} \\ \mathfrak{q} \subset \mathfrak{q}'}} \mathfrak{q}'$$

is a proper containment. Then there exists g in the right hand side of the above and $g \notin \mathfrak{q}$. That is, the extension $\overline{\mathfrak{q}}$ of $\mathfrak{q}$ in B_g is a maximal ideal (any prime ideal properly containing $\mathfrak{q}$ contains g). Since B is integral over $f(A)$, B_g is integral over $(f(A))_g$. Then $\overline{q} \cap (f(A))_g$ is a maximal ideal of $(f(A))_g$. The contraction of this ideal, $\mathfrak{p}$ is an ideal of $f(A)$ such that $g \notin \mathfrak{p}$ and

$$\mathfrak{p} \subset \bigcap_{\substack{\mathfrak{p}' \text{ prime} \\ \mathfrak{p} \subset \mathfrak{p}'}} \mathfrak{p}'$$

is a proper containment (since every ideal properly containing $\mathfrak{p}$ contains g). Note that this statement is exactly that $f(A)$ is not a Jacobson ring. That is, we've reduced to the case that f is surjective.

Since $f(A)$ is not surjective, there exists some prime ideal $\mathfrak{p}$ of $f(A)$ such that $\mathfrak{p}$ is properly contained in the intersection of all maximal ideals that contain $\mathfrak{p}$. Consider the composition $A \mapsto f(A) \mapsto f(A)/\mathfrak{p}$. This map is surjective and we have $\mathfrak{N}_{f(A)/\mathfrak{p}} = 0 \neq \mathfrak{J}_{f(A)/\mathfrak{p}}$. Therefore, A is not a Jacobson ring.

The contrapositive of the above is that if A is a Jacobson ring and $f : A \mapsto B$ is a A-algebra such that B is integral over $f(A)$, then B is a Jacobson ring as well.

If B is finitely-generated over A, let $\mathfrak{q}$ be a prime ideal of B and $\mathfrak{p} = \mathfrak{q}^c$ be a prime ideal in A. Then the map $f : A \mapsto B$ induces a map $f : A/\mathfrak{p} \mapsto B/\mathfrak{q}$. Since $\mathfrak{p}$ can be written as the intersection of the maximal ideals that contain it, $\mathfrak{J}_{A/\mathfrak{p}} = 0$. From problem 22, this implies that $\mathfrak{J}_{B/\mathfrak{q}} = 0$. That is, the intersection of the maximal ideals that contain $\mathfrak{q}$ is exactly $\mathfrak{q}$. Since $\mathfrak{q}$ was arbitrary, this implies that B is a Jacobson ring.

25. $(i) \implies (ii)$ Let A be a Jacobson ring and $f : A \mapsto B$ a finitely-generated A-algebra that is also a field. Clearly, B is a finitely generated $f(A)$-algebra and B is finite over A if and only if it is finite over $f(A)$ (by definition of the action of A on B). From the previous problems, $f(A)$ is also Jacobson and $\mathfrak{N}_{f(A)} = \mathfrak{J}_{f(A)}$. Since $f(A)$ is contained in a field, it is an integral domain so in particular, $\mathfrak{J}_{f(A)} = \mathfrak{N}_{f(A)} = 0$. Therefore, it suffices to consider $A \subseteq B$.

Since B is a field, we may use problem 21. Let $s \in A \setminus \{0\}$ satisfy the conditions of problem 21 and $\mathfrak{m}$ be a maximal ideal of A such that $s \notin \mathfrak{m}$ (possible since $\mathfrak{J}_A = 0$). Then the homomorphism $A \mapsto A/\mathfrak{m} = k$ extends to a homomorphism $\phi : B \mapsto \Omega$ (since $s \notin \mathfrak{m}$) where Ω is an algebraic closure of k. Since B is a field and ϕ extends a nontrivial map, ϕ is nontrivial and hence injective. Since $\phi(B) \subseteq \Omega$, $\phi(B)$ is an algebraic extension of k. Since B is a finitely-generated A-algebra, $\phi(B)$ is a finite algebraic extension of k (each generator satisfies some polynomial equation, which gives a finite basis of $\phi(B)$ over k). Notice however that ϕ extends the projection $A \mapsto A/\mathfrak{m}$. Since $\ker \phi = \{0\}$, this implies that $\mathfrak{m} = 0$ and $A = k$. That is, $\phi(B)$ is a finite extension of A. This clearly implies that B is a finite extension of A as well.

$(ii) \implies (i)$ Let $\mathfrak{p}$ be a prime ideal of A which is not maximal, and let $B = A/\mathfrak{p}$. For $f \in B$, $f \neq 0$, B_f is a finitely-generated A-algebra (generated by $1/f$). If it is a field, it is a finitely-generated A-module by our assumptions. This then implies that B_f is a finitely-generated B-module (since multiplication by elements of

A is really multiplication by their image in B). In particular, this implies that B_f is integral over B (since B_f is finitely generated as a B-module, every element is integral). Then this would imply that B is a field, but this contradicts that $\mathfrak{p}$ is not maximal. Therefore, B_f is not a field. This implies there is some non-zero prime ideal whose contraction in B is a non-zero prime ideal not containing f. That is, the intersection of all non-zero prime ideals in B is 0. Contracting back to A and using the prime ideal correspondence with quotients, we have that $\mathfrak{p}$ is the intersection of all prime ideals that strictly contain $\mathfrak{p}$. Since $\mathfrak{p}$ was arbitrary, this implies that A is a Jacobson ring.

26. (1) $\implies$ (2) Let X be a topological space. Let E be closed in X. Clearly, $\overline{E \cap X_0} \subseteq E$ since E is closed. For any $x \in E$ and open set U containing X, $U \cap E$ is locally-closed so $(E \cap X_0) \cap U \neq \emptyset$ so that $x \in \overline{E \cap X_0}$. Therefore, $\overline{E \cap X_0} = E$.

    (2) $\implies$ (3) The mapping $U \mapsto U \cap X_0$ of open subsets of X to open subsets of X_0 is surjective by definition of the subspace topology on X_0. Therefore, it suffices to show that this map is injective. Notice that if $U \cap X_0 = V \cap X_0$ for U, V open in X, then $(X \setminus U) \cap X_0 = (X \setminus V) \cap X_0$. Taking closures, we get $X \setminus U = X \setminus V$ so that $U = V$. Therefore, this map is also injective and hence, bijective.

    (3) $\implies$ (1) Let $A = U \cap E$ where U is open and $E = X \setminus V$ is closed. If $A \cap X_0 = U \cap E \cap X_0 = \emptyset$, Then $U \cap X_0 \subseteq X \setminus C = V$. Intersecting with $U \cap X_0$ on both sides, we have $U \cap X_0 = (U \cap V) \cap X_0$. This then implies that $U = U \cap V$ so that $U \subseteq V$. That is, $U \cap E = \emptyset$. Therefore, if $U \cap E \neq \emptyset$, then $A \cap X_0 \neq \emptyset$.

    A set satisfying these properties is said to be "very dense".

    $(i) \implies (ii)$ Let A be a Jacobson ring and $\mathrm{Max}(A)$ be the set of maximal ideals of A. For any ideal $\mathfrak{a}$ of A, since any prime ideal is the intersection of all prime ideals that contain it, we may write

$$r(\mathfrak{a}) = \bigcap_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{a} \subseteq \mathfrak{p}}} \mathfrak{p} = \bigcap_{\substack{\mathfrak{m} \text{ maximal} \\ \mathfrak{a} \subseteq \mathfrak{m}}} \mathfrak{m}.$$

    Now let $\mathfrak{a}$ be an ideal such that $r(\mathfrak{a}) = \mathfrak{a}$ and consider $V(\mathfrak{a})$. Notice

$$V(\mathfrak{a}) \cap \mathrm{Max}(A) = \{\mathfrak{m} \in \max(A) : \mathfrak{a} \subseteq \mathfrak{m}\}.$$

    Therefore, $V(\mathfrak{a})$ contains $V(\mathfrak{a} \cap \mathrm{Max}(A))$ if and only if $\mathfrak{b} \subseteq \mathfrak{m}$ for every maximal ideal of A that contains $\mathfrak{a}$. That is, if and only if

$$\mathfrak{b} \subseteq \bigcap_{\substack{\mathfrak{m} \in \mathrm{Max}(A) \\ \mathfrak{a} \subseteq \mathfrak{m}}} \mathfrak{m} = r(\mathfrak{a}) = \mathfrak{a}.$$

    From this, we have,

$$\overline{V(\mathfrak{a}) \cap \mathrm{Max}(A)} = \bigcap_{\mathfrak{b} \subseteq \mathfrak{a}} V(\mathfrak{b}).$$

    However, $\mathfrak{b} \subseteq \mathfrak{a}$ implies $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ so that the above implies

$$\overline{V(\mathfrak{a}) \cap \mathrm{Max}(A)} \supseteq V(\mathfrak{a}).$$

    The other inclusion holds trivially. Therefore, the two are equal and $\mathrm{Max}(A)$ is very dense.

    $(ii) \implies (iii)$ Let $\{\mathfrak{p}\} = V(\mathfrak{a}) \setminus V(\mathfrak{b})$ be locally closed. Since $\mathrm{Max}(A)$ is very dense, $\{\mathfrak{p}\} \cap \mathrm{Max}(A) \neq \emptyset$. Therefore, this intersection is exactly $\{\mathfrak{p}\}$ so that $\mathfrak{p} \in \mathrm{Max}(A)$. Then $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = \{\mathfrak{p}\}$ so this set is closed.

    $(iii) \implies (i)$ The contrapositive will be proved. Assume there exists a non-maximal prime ideal $\mathfrak{p}$ such that

$$\mathfrak{p} \subset \bigcap_{\substack{\mathfrak{p}' \text{ prime} \\ \mathfrak{p} \subset \mathfrak{p}'}} \mathfrak{p}'$$

    is proper. Then let f be an element of the intersection on the right so that $f \notin \mathfrak{p}$. Then the extension $\overline{p}$ in A_f is maximal by the prime ideal correspondence with localizations. That is, $\{\mathfrak{p}\} = V(\mathfrak{p}) \setminus V(f)$. However, $\overline{\{\mathfrak{p}\}} \neq \{\mathfrak{p}\}$ since $\mathfrak{p}$ is not maximal. Therefore, the locally-closed singleton $\{\mathfrak{p}\}$ is not closed.

27. Let Σ be the set of local subrings of k ordered under the relation of domination. To show that Σ has maximal elements, we will use Zorn's lemma. Let

$$(A_1, \mathfrak{m}_1) \subseteq (A_2, \mathfrak{m}_2) \subseteq \ldots$$

be an increasing chain of local subrings of k. Note that $(A_i, \mathfrak{m}_i) \leq (A_j, \mathfrak{m}_j)$ implies the inclusion map $A_i \hookrightarrow A_j$ induces a nontrivial map (consider the image of 1) $A_i/\mathfrak{m}_i \mapsto A_j/\mathfrak{m}_j$. Since this is a nontrivial map of fields, it is injective (which implies $\mathfrak{m}_i = A_i \cap \mathfrak{m}_j$). Therefore, we have an increasing sequence of fields

$$A_1/\mathfrak{m}_1 \subseteq A_2/\mathfrak{m}_2 \subseteq \ldots.$$

Let $K = \cup A_i/\mathfrak{m}_i$. It is clear that K is a field and $A_i/\mathfrak{m}_i$ injects into K for each i. We now have a sequence of maps $\pi_i : A_i \mapsto K$ with kernel $\mathfrak{m}_i$ and $A_i \subseteq A_j$, $\mathfrak{m}_i \subseteq \mathfrak{m}_j$ for $i \leq j$. Let Λ be the set of pairs (B, g) such that $g : B \mapsto K$ extends all of these maps. Λ is nonempty because there is a well-defined map $\pi : \cup A_i \mapsto K$ defined by $\pi(x) = \pi_i(x)$ for $x \in A_i$. A simple use of Zorn's lemma shows that Λ has maximal elements (do the same as for the A_i for any increasing chain in Λ). That is, there exists (B, g) such that $A_i \subseteq B$ and $\mathfrak{m}_i \subseteq \ker g$ for each i (the latter because g extends π_i). It remains now to be seen that B is local and $\ker g$ is its maximal ideal. Since $g : B \mapsto K$ and K is a field, $\ker g$ is prime. Since $g(s) \neq 0$ for $s \in B \setminus \ker g$, we may localize and get an extension $g : B_{\ker g} \mapsto K$. Since (B, g) is maximal, this implies that $B_{\ker g} = B$ and that every element not in $\ker g$ is a unit. That is, B is a local ring with maximal ideal $\ker g$. Therefore, $(B, g) \in \Sigma$ is the desired upper bound of the sequence $(A_i, \mathfrak{m}_i)$. It follows now by Zorn's lemma that Σ has maximal elements.

If $(B, \mathfrak{n}) \in \Sigma$ is maximal, consider the set Θ of pairings (A, f) where A is a subring of k containing B and $f : A \mapsto \Omega$ where Ω is the algebraic closure of $B/\mathfrak{n}$. Order Θ by $(A_1, f_1) \leq (A_2, f_2)$ if $A_1 \subseteq A_2$ and $f_2\big|_{A_1} = f_1$ (as in the text). By Zorn's lemma, this set easily has maximal elements and similar to above, maximal elements are local rings. Following the text, maximal elements of this set are also valuation rings over k. For such a maximal element (C, g), g extends the projection $\pi : B \mapsto B/\mathfrak{n}$ so that $\mathfrak{n} \subseteq \ker g$ so that $(B, \mathfrak{n})$ is dominated by $(C, \ker g)$. Therefore by maximality, $C = B$ and B is a valuation ring over k.

Conversely, assume $(A, \mathfrak{m}) \in \Sigma$ is a valuation ring over k and $(B, \mathfrak{n}) \in \Sigma$ dominates $(A, \mathfrak{m})$. For $b \in B \subseteq k$, we necessarily have either $b \in A$ or $b^{-1}A$. In the latter case, either b^{-1} is a unit in A or $b^{-1} \in \mathfrak{m}$. If b^{-1} is a unit in A, then $b \in A$. If $b^{-1} \in \mathfrak{m}$, then $b^{-1} \in \mathfrak{n}$, but this implies $1 = bb^{-1} \in \mathfrak{n}$. This is a contradiction. Therefore, in any of the cases above, $b \in A$. This implies that $(A, \mathfrak{m}) = (B, \mathfrak{n})$ and so $(A, \mathfrak{m})$ is maximal.

28. $(i) \implies (ii)$ The contrapositive will be proved. Let $\mathfrak{a}, \mathfrak{b}$ be proper ideals of an integral domain A with $k = \mathrm{frac}(A)$ and assume that $\mathfrak{b} \not\subseteq \mathfrak{a}$ and $\mathfrak{a} \not\subseteq \mathfrak{b}$. Then there exists $a \in \mathfrak{a}$ such that $a \notin \mathfrak{b}$ and there exists $b \in \mathfrak{b}$ such that $b \notin \mathfrak{a}$. Consider $a/b \in k$. Clearly, $a/b \notin A$ since if it were, $a = b(a/b) \in \mathfrak{b}$. Similarly, $b/a \notin A$ since this would imply $b = a(b/a) \in \mathfrak{a}$. Therefore, A is not a valuation ring.

    $(ii) \implies (i)$ Let $x/y \in k$ for $x, y \in A$. Then either $(x) \subseteq (y)$ or $(y) \subseteq (x)$. In the first case, there exists a $z \in A$ such that $x = zy$. From this, we have that $x/y = z \in A$. In the other case, there exists $z \in A$ such that $y = zx$. From this, $y/x = z \in A$. Therefore, either $x/y \in A$ or $y/z \in A$. That is, A is a valuation ring over k.

    Let A be an integral domain that is a valuation ring over its field of fractions and $\mathfrak{p}$ a prime ideal of A. Since the ideals of A are in an order preserving correspondence with the ideals of $A/\mathfrak{p}$ (ideals that contain $\mathfrak{p}$ in this case) and with the ideals of $A_\mathfrak{p}$ (ideals that are contained in $\mathfrak{p}$ in this case), both of these rings have their ideals totally ordered. That is, they are valuation rings over their respective fields of fractions.

29. Let A be a valuation ring (integral domain) over a field k and let $A \subseteq B \subseteq k$ for some integral domain B. It is easy to verify that B is a local ring (the argument given that a valuation ring is local carries over almost word for word). Let $\mathfrak{n}$ be the maximal ideal of B (the set of all non-units) and let $\mathfrak{p} = \mathfrak{n} \cap A$. Clearly, $A \setminus (\mathfrak{n} \cap A)$ is the set of elements of A who, when considered as an element of B, are a unit. For $b \in B$, either $b \in A$ or $b^{-1} \in A$. If $b \in A$, then clearly, $b = b/1 \in A_\mathfrak{p}$. Similarly, if $b^{-1} \in A$, then $b^{-1} \in B$ so that b^{-1} is a unit in B. That is, $b^{-1} \in A \setminus (\mathfrak{n} \cap A)$ so that $b = 1/b^{-1} \in A_\mathfrak{p}$. Therefore, $B \subseteq A_\mathfrak{p}$. Conversely, if $a/s \in A_\mathfrak{p}$ with $s \in A \setminus (\mathfrak{n} \cap A)$ (and so is a unit in B), then considered as an element of K, we see $a/s = as^{-1} \in B$. Therefore, the opposite inclusion holds as well. Therefore, we have $B = A_\mathfrak{p}$ is a local ring of A.

30. Let A be a valuation ring of a field k. Since k is commutative, the set of units, U, of A forms a group under multiplication, which is clearly a subgroup of the multiplicative group k^*. Let $\Gamma = k^*/U$ be the quotient. Order

Γ by $\xi \le \eta$ if $\eta\xi^{-1} \in A$ (for any choice of representative, since different representatives differ by a unit of A). This relation is clearly reflexive and transitive. To show that it is antisymmetric, notice that if $\eta\xi^{-1}, \xi\eta^{-1} \in A$, then $\eta\xi^{-1} \in U$ so that $\eta = \xi$ in Γ. Therefore, this defines a partial order. Since A is a valuation ring, for every $\xi, \eta \in \Gamma$ and representatives, $x, y \in k^*$, either $xy^{-1} \in A$ or $yx^{-1} \in A$. Equivalently, either $\eta \le \xi$ or $\xi \le \eta$ respectively. Therefore, this defines a total order on Γ. If $\theta \in \Gamma$ and $\xi \le \eta$, it is clear by writing out representatives that $\xi\theta \le \eta\theta$ so this order respects the group operation. From this, Γ is a totally ordered abelian group (called the "value group" of A).

Let $v : k^* \mapsto \Gamma$ be the projection map and $x, y \in k^*$. Without loss of generality, assume that $v(x) \le v(y)$ so that $yx^{-1} \in A$ (since x and y are representatives of $v(x)$ and $v(y)$ respectively). Then we have $x^{-1}(x + y) = 1 + yx^{-1} \in A$ so that

$$v(x + y) \ge v(x) = \min\{v(x), v(y)\}.$$

Since $v(xy) = v(x) + v(y)$ (where Γ is written additively), the above implies that v is a valuation on k with values in Γ. Notice that $v(x) \ge 0$ if and only if $x1^{-1} = x \in A$. That is, $A = \{x \in k^* : v(x) \ge 0\} \cup \{0\}$.

31. Let Γ be a totally ordered abelian group (written additively) and let k be a field. A group homomorphism $v : k^* \mapsto \Gamma$ satisfying the inequality from the previous problem is said to be a valuation with values in Γ.

    Let k be a field, Γ a totally ordered abelian group and $v : k^* \mapsto \Gamma$ be a valuation. Let $A = \{x \in k^* : v(x) \ge 0\} \cup \{0\}$. First, this is an additive group since for every $x, y \in A$, the inequality $v(x + y) \ge \min\{v(x), v(y)\}$ ensures that $x + y \in A$ and $0 = v(1) \le v(-1) \le v(1)$ implies that $v(-x) = v(-1) + v(x) = v(x)$ implies that $-x \in A$ for all $x \in A$. The fact that v is a homomorphism ensures that $xy \in A$ for all $x, y \in A$. Therefore, A is a ring. For $x \in k^*$, we have $0 = v(1) = v(xx^{-1}) = v(x) + v(x^{-1})$. If $v(x) \ge 0$, then $x \in A$. If $v(x) \le 0$, then $v(x^{-1}) \ge 0$ (remember the order preserves the group structure and add $v(x^{-1})$ to both sides) implies that $x^{-1} \in A$. That is, A is a valuation ring.

    A is the valuation ring of the valuation v and $v(k^*)$ is the value group of v.

32. Let A be a valuation ring of a field k, $\Gamma = k^*/U$ (where U is the group of units of A), and $v : k^* \mapsto \Gamma$ be the corresponding valuation with values in Γ. For a prime $\mathfrak{p}$ of A, consider the set

    $$\Delta_{\mathfrak{p}} = \{\xi \in \Gamma : \pm\xi \in v(A \setminus \mathfrak{p})\}.$$

    Clearly, this set is closed under addition since $A \setminus \mathfrak{p}$ is multiplicative. Almost by definition, this set has additive inverses (and identity). Therefore, it is a subgroup of Γ. Notice that since $A \setminus \mathfrak{p} \subseteq A$, $v(A \setminus \mathfrak{p}) \subseteq \{\xi \in \Delta_{\mathfrak{p}} : \xi \ge 0\}$. Conversely, for $v(x) \in \{\xi \in \Delta_{\mathfrak{p}} : \xi \ge 0\}$, either $v(x) \in v(A \setminus \mathfrak{p})$ or $-v(x) \in v(A \setminus \mathfrak{p})$. In the latter case, $-v(x) = v(s)$ for some $s \in A \setminus \mathfrak{p}$ so that $v(xs) = 0$ and xs is a unit in A. This implies that both x and s are units in A and hence, not in $\mathfrak{p}$ (the unit group is saturated). That is, $x \in A \setminus \mathfrak{p}$ and $v(x) \in v(A \setminus \mathfrak{p})$. Therefore,

    $$\{\xi \in \Delta_{\mathfrak{p}} : \xi \ge 0\} = v(A \setminus \mathfrak{p}).$$

    If $0 \le v(x) \le v(s)$ for some $x \in A$ and $s \in A \setminus \mathfrak{p}$, then $sx^{-1} \in A$. That is, there exists $a \in A$ such that $s = ax$. From this, it is clear that $x \notin \mathfrak{p}$ since $x \in \mathfrak{p}$ would imply that $s \in \mathfrak{p}$. Therefore, $v(x) \in v(A \setminus \mathfrak{p})$ as well. Therefore, for each prime $\mathfrak{p}$ of A, the subgroup $\Delta_{\mathfrak{p}}$ of Γ is isolated.

    Define a map from $\mathrm{Spec}(A)$ into the set of isolated subgroups of Γ defined by $\mathfrak{p} \mapsto \Delta_{\mathfrak{p}}$. Define a mapping in the other directions as follows. For an isolated subgroup Δ of Γ, define

    $$\mathfrak{p}_\Delta = \{x \in A : v(x) \notin \Delta\} = \{x \in A : \forall \xi \in \Delta,\ v(x) \ge \xi\}.$$

    It is immediate from the second definition (that follows since Δ is isolated) that $\mathfrak{p}_\Delta$ is an ideal. To see that it is prime, assume for that $xy \in \mathfrak{p}_\Delta$ and $y \notin \mathfrak{p}_\Delta$ (that is, $v(y) \in \Delta$). Then for all $\xi \in \Delta$,

    $$v(x) + v(y) = v(xy) \ge \xi.$$

    Since $v(y) \in \Delta \implies -v(y) \in \Delta$ and the ordering preserves the group structure,

    $$v(x) \ge \xi - v(y).$$

    It is clear that this implies $v(x) \ge \xi$ for all $\xi \in \Delta$. That is, $x \in \mathfrak{p}_\Delta$. Therefore, $\mathfrak{p}_\Delta$ is a prime ideal and the assignment $\Delta \mapsto \mathfrak{p}_\Delta$ is well-defined.

For a prime $\mathfrak{p}$ of A, it is clear from the above that

$$\mathfrak{p}_{\Delta_\mathfrak{p}} = \{x \in A : v(x) \notin \Delta_\mathfrak{p}\} = A \setminus v^{-1}(\Delta_\mathfrak{p}) = A \setminus (A \setminus \mathfrak{p}) = \mathfrak{p}.$$

Conversely, for an isolated subgroup Δ of Γ, (this takes a second to verify)

$$\Delta_{\mathfrak{p}_\Delta} = \{v(x) \in \Gamma : \pm v(x) \in v(A \setminus \mathfrak{p}_\Delta)\} = \{v(x) \in \Gamma : \pm v(x) \in \Delta\} = \Delta.$$

Therefore, we have that the correspondence between prime ideals of A and isolated subgroups of Γ is bijective.

33. Let Γ be a totally ordered abelian group. Let k be a fiel and let $A = k[\Gamma]$ be the group-algebra of Γ over k. By definition, A is freely-generated as a k-vector space by elements x_α with $\alpha \in \Gamma$ such that $x_\alpha x_\beta = x_{\alpha+\beta}$. To see that A is an integral domain, simply notice that any nonzero element can be written

$$\lambda_1 x_{a_1} + \ldots + \lambda_n x_{a_n},$$

where $\lambda_i \neq 0$ and $a_1 < \ldots < a_n$ (where strict inequality $a_i < a_j$ is $a_i \leq a_j$ and $a_i \neq a_j$). Then multiplication of two such elements has maximal term given by the sum of maximal terms (with coefficient given as the product of coefficients). From this, any two nonzero element necessarily gives a nonzero element and so A is an integral domain.

If $u \in A$ is an (nonzero) element of the above form, let $v_0(u) = a_1$. This is easily a well-defined mapping $v_0 : A \mapsto \Gamma$. It is clear that this map is a homomorphism since the product of two such elements has minimal term given by the product of the individual minimal terms. Similarly, if $v_0(u) \leq v_0(w)$, then the minimal term of $u + w$ is clearly greater than or equal to $v_0(u)$ (they may cancel). That is, $v_0(u + w) \geq \min\{v(x), v(y)\}$.

Let $K = \text{frac}(A)$ and define $v : K^* \mapsto \Gamma$ by $v(a/s) = v_0(a) - v_0(s)$. This clearly defines a group homomorphism. To see the other property, assume

$$v(a) - v(s) = v(a/s) \leq v(b/t) = v(b) - v(t).$$

Then $v(at) \leq v(bs)$ so that $v(at + bs) \geq v(at) = v(a) + v(t)$. This implies

$$\begin{aligned} v(a/s + b/t) &= v((at + bs)/st) \\ &= v(at + bs) - v(st) \\ &\geq v(a) + v(t) - v(s) - v(t) \\ &= v(a) - v(s) = v(a/s) = \min\{v(a/s), v(b/t)\}. \end{aligned}$$

Therefore, v defines a valuation on K with values in Γ. It is clear that v is surjective since $v(x_\alpha/1) = \alpha$ for all $\alpha \in \Gamma$. Therefore, Γ is exactly the value group of the valuation v.

34. Let A be a valuation ring and k its field of fractions. Let $f : A \mapsto B$ be a ring homomorphism such that $f^* : \text{Spec}(B) \mapsto \text{Spec}(A)$ is closed and $g : B \mapsto k$ be any A-algebra homomorphism. If $C = g(B)$, then clearly, $A \subseteq C$ since $g(a1) = ag(1) = a \in C$. Let $\mathfrak{n}$ be a maximal ideal of C. Since f^* is closed, $\mathfrak{m} = \mathfrak{n} \cap A$ is the maximal ideal of A ($f^*(V(\mathfrak{n}))$ is closed and a singleton. Therefore, the ideal of the image, $\mathfrak{m} = \mathfrak{n} \cap A$ is maximal) (note, we are considering $A \subseteq B$ since an A-algebra is really a $f(A)$-algebra). Since A is a valuation ring, it is a local ring and therefore, $\mathfrak{m}$ is its maximal ideal and $A_\mathfrak{m} = A$. Notice that $A = A_\mathfrak{m}$ is a local ring and a valuation ring over k so that it is maximal in the local subrings of k. Therefore, $C \subseteq C_\mathfrak{n} \subseteq A_\mathfrak{m} = A$. Therefore, $g(B) = C = A$.

35. If $f : A \mapsto B$ is integral and C is any A-algebra, from problem 2, the map $f \otimes \text{Id} : C = A \otimes_A C \mapsto B \otimes_A C$ is integral. Therefore, the induced map $(f \otimes \text{Id})^* : \text{Spec}(B \otimes_A C) \mapsto \text{Spec}(C)$ is closed. Conversely, let $f : A \mapsto B$ be such that for any A-algebra C, the induced map $(f \otimes \text{Id})^* : \text{Spec}(B \otimes_A C) \mapsto \text{Spec}(C)$ is closed and assume B is an integral domain.

As usual, assume that $A \subseteq B$ (it is easy to see that this also does not change the tensor product) so that the map f is injective. Let $k = \text{frac}(B)$ and let A' be a valuation ring of k containing A (which exists since the integral closure of A in k is the intersection of all valuation rings of k that contain A). From this note, it then suffices to show that $B \subseteq A'$ since then B is contained in all valuation rings that contain A and hence, the integral closure of A in k.

Since A' is an A-algebra, the map $(f \otimes \text{Id})^* : \text{Spec}(B \otimes_A A') \mapsto \text{Spec}(A')$ is closed. From the previous problem, we have that the image of the map $\phi : B \otimes_A A' \mapsto k$ is A' (ϕ is the multiplication map) (considering both of these as A'-algebras). Therefore, for all $b \in B$ and $a' \in A'$, $ba' \in A'$. That is, $b = b1 \in A'$ for all $b \in B$ so that $B \subseteq A'$ and so B is contained in the integral closure of A in k. This implies that the map $f : A \mapsto B$ is integral.

Let B be a ring with finitely many minimal primes $\mathfrak{p}_i$ and assume $f : A \mapsto B$ has the property above for any A-module C. Then for any i, the surjective map $B \mapsto B/\mathfrak{p}_i$ induces a surjective map $B \otimes_A C \mapsto B/\mathfrak{p}_i \otimes_A C$. That is, $B/\mathfrak{p}_i \otimes_A C$ is isomorphic to a quotient of $B \otimes_A C$ (all prime ideals that contain the kernel of this map). From this, $\operatorname{Spec}(B/\mathfrak{p}_i \otimes C)$ is a closed subspace of $\operatorname{Spec}(B \otimes_A C)$ and so $(f \otimes \mathrm{Id})^*$ restricts to the pull back of the induced map $\operatorname{Spec}(B/\mathfrak{p}_i \otimes_A C) \mapsto \operatorname{Spec}(C)$. Therefore, this map is closed as well and the composite map $A \mapsto B \mapsto B/\mathfrak{p}_i$ satisfies the above property as well. Since $B/\mathfrak{p}_i$ is an integral domain, this implies this map is integral. Since there are finitely many, this implies the map $A \mapsto \prod_i B/\mathfrak{p}_i$ is integral. Considering the product of projections, $B \mapsto \prod_i B/\mathfrak{p}_i$, the kernel is obviously $\mathfrak{N}_B = \cap_i \mathfrak{p}_i$ so that $\prod_i B/\mathfrak{p}_i \simeq B/\mathfrak{N}_B$. Therefore, the map $A \mapsto B/\mathfrak{N}_B$ is integral. That is, for every $b \in B$, there are coefficients $a_i \in A$ such that

$$b^n + a_1 b^{n-1} + \ldots + a_n \in \mathfrak{N}_B.$$

Raising this polynomial to a high enough power gives a monic polynomial with coefficients in A that b satisfies. That is, B is integral over A so the map $f : A \mapsto B$ is integral.

Chapter 6

1a. Let M be an A-module and $u : M \mapsto M$ be a surjective module homomorphism that is not injective. Consider the sequence of submodules

$$0 \subset \ker u \subset \ker u^2 \subset \ldots.$$

It will be shown by induction that these inclusions are proper. The first inclusion is proper simply because $\ker u \neq \{0\}$. Assuming $\ker u^{k-1} \subset \ker u^k$, let $a \in \ker u^n$, $a \notin \ker u^{n-1}$. Since u is surjective, there exists $b \in M$ such that $u(b) = a$. Then $u^k(b) = u^{k-1}(a) \neq 0$, but $u^{k+1}(b) = u^k(a) = 0$. Therefore, $b \in \ker u^{k+1}$, but $b \notin \ker u^k$. Therefore, M is not Noetherian.

The contrapositive of this statement is that if M is Noetherian and $u : M \mapsto M$ is a surjective module homomorphism, then u is injective and hence, an isomorphism.

1b. Let M be an A-module and $u : M \mapsto M$ be an injective module homomorphism that is not surjective. Consider the sequence of submodules

$$M \supset \operatorname{Im} u \supset \operatorname{Im} u^2 \supset \ldots.$$

It will again be shown by induction that these inclusions are proper. The first inclusion is proper because $\operatorname{Im} u \neq M$. Assuming that $\operatorname{Im} u^{k-1} \supset \operatorname{Im} u^k$, let $a \in \operatorname{Im} u^{k-1}$, but $a \notin \operatorname{Im} u^k$. Then there exists $b \in M$ such that $u^{k-1}(b) = a$ so that $u(a) = u^k(b) \in \operatorname{Im} u^k$. However, if $u(a) = u^{k+1}(c)$, then by injectivity, $a = u^k(c)$ for some c, which is not possible. Therefore, $u(a) \in \operatorname{Im} u^k$, but $u(a) \notin \operatorname{Im} u^{k+1}$. Therefore, M is not Artinian.

The converse of this is that if M is an Artinian ring and $u : M \mapsto M$ is an injective module homomorphism, then u is surjective and hence, an isomorphism.

2. Let M be an A-module such that every non-empty set of finitely generated submodules has a maximal element. Let N be a submodule of M and consider the set of finitely-generated submodules of N. This set has a maximal element, N_0. For $n \in N$, we may consider $N_0 + An$. Since this module is finitely-generated by the generators of N_0 and n, maximality of N_0 implies that $n \in N_0$. That is, we necessarily have $N = N_0$ so that N is finitely-generated. Therefore, M is Noetherian.

   Clearly, the converse holds as well. If M is Noetherian, then every non-empty subset of submodules has a maximal element so that in particular, every non-empty set of finitely-generated submodules has a maximal element.

3. Let M be an A-module and let N_1, N_2 be submodules of M such that M/N_1 and M/N_2 are Noetherian (or Artinian). Consider the ring homomorphism $\phi : M \mapsto (M/N_1) \oplus (M/N_2)$. Clearly, $\ker \phi = N_1 \cap N_2$ so that there is an isomorphism $M/(N_1 \cap N_2) \simeq (M/N_1) \oplus (M/N_2)$ Since M/N_i are Noetherian (resp. Artinian), so is their direct sum. Since these are isomorphism invariants (there is a bijective correspondence between submodules), this implies that $M/(N_1 \cap N_2)$ is Noetherian (resp. Artinian).

4. First, a result which will be useful later. Let M be a finitely-generated faithful A module and let $\mathfrak{p}, \mathfrak{q}$ be ideals such that $\mathfrak{p}M \subseteq \mathfrak{q}M$. Define a module homomorphism $\phi : M \mapsto M$ by $\phi(m) = pm$ for some $p \in \mathfrak{p}$. Then $\phi(M) = \mathfrak{p}M \subseteq \mathfrak{q}M$. By the Cayley-Hamilton theorem, there is some equation of the form

$$\phi^n + q_1\phi^{n-1} + \ldots + q_n = 0$$

for $q_i \in \mathfrak{q}$. That is, the map on the left is the zero map. Since M is a faithful A-module, this then implies

$$p^n + q_1 p^{n-1} + \ldots + q_n = 0.$$

This may be rewritten as follows.

$$p(p^{n-1} + q_1 p^{n-2} + \ldots + q_{n-1}) = -q_n \in \mathfrak{q}$$

Therefore, either $p \in \mathfrak{q}$ or $p^{n-1} + q_1 p^{n-2} + \ldots + q_{n-1} \in \mathfrak{q}$. It is clear that we may continue to decrease the degree so this process eventually terminates with $p \in \mathfrak{q}$. Therefore, $\mathfrak{p} \subseteq \mathfrak{q}$.

Let M be a Noetherian A-module and let $\mathfrak{a} = \operatorname{ann}(M)$. We know M is a faithful $A/\mathfrak{a}$-module. Let

$$\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \ldots$$

be an increasing sequence of prime ideals of $A/\mathfrak{a}$. Then

$$\mathfrak{p}_1 M \subseteq \mathfrak{p}_2 M \subseteq \ldots$$

is an increasing sequence of submodules of M. Therefore, there exists n such that $\mathfrak{p}_m M = \mathfrak{p}_n M$ for $m \geq n$. Since M is Noetherian, M is finitely-generated (as an $A/\mathfrak{a}$-module) and the above then implies that $\mathfrak{p}_m = \mathfrak{p}_n$ for $m \geq n$. That is, the sequence of prime ideals above is stationary and so $A/\mathfrak{a}$ is Noetherian.

It is clear that the same process works with Artinian in place of Noetherian and inclusions reversed in the chains.

5. Let X be a Noetherian topological space and $Y \subseteq X$ be a subspace. First, notice that for open sets $U_1 \subseteq U_2$ of Y, there exists open sets W_1, W_2 of X such that $W_i \cap Y = U_i$. We may choose W_2 to contain W_1 by taking the union if necessary. That is, for any open sets $U_1 \subseteq U_2$ of Y, there exists open sets $W_1 \subseteq W_2$ of X such that $W_i \cap Y = U_i$.

Now let $U_1 \subseteq U_2 \subseteq \ldots$ be an ascending chain of open subsets of Y. From the above, we can inductively find open subsets $W_1 \subseteq W_2 \subseteq \ldots$ of X such that $W_i \cap Y = U_i$. Since X is Noetherian, there exists some n such that for $m \geq n$, $W_m = W_n$. Then it is clear that $U_m = W_m \cap Y = W_n \cap Y = U_n$. Therefore, the sequence of open sets $U_1 \subseteq U_2 \subseteq \ldots$ is stationary and so Y is a Noetherian topological space as well.

If $\{U_i\}_{i \in I}$ is an open cover of X, consider the set of finite unions of the U_i, $\{V_J\}_{J \subseteq I}$. Since these sets are open and X is Noetherian, there is necessarily a maximal element $V_0 \in \{V_J\}_{J \subseteq I}$. For $x \in X$, we may find an open subset U_0 containing x. Then $V_0 \cup U_0$ is a finite union of the U_i and by maximality is equal to V_0. That is, $x \in V_0$. Therefore, $V_0 = X$ and X is a finite union of the U_i. Therefore, X is a Noetherian topological space.

6. $(i) \implies (iii)$ This follows from the above since every subspace of a Noetherian topological space is Noetherian and a Noetherian topological space is quasi-compact.

$(iii) \implies (ii)$ This is immediate.

$(ii) \implies (i)$ Let X be a topological space such that every open subspace of X is quasi-compact and consider an ascending chain of open sets $U_1 \subseteq U_2 \subseteq \ldots$ of X. Then the set $U = \cup U_i$ is open and so is quasi-compact. If we write each U_i as a finite union $U_i = \cup_j V_{ij}$, then each V_{ij} is open in U and these sets cover U. Therefore, there are finitely many $V_{i_1 j_1}, \ldots V_{i_n j_n}$ that cover U. If $n \geq \max\{i_k\}_{k=1}^n$, then we have that $U_m = U_n$ for $m \geq n$, since $U_n = U$. That is, the sequence of open sets of X is stationary and so X is Noetherian.

7. Let X be a Noetherian topological space. Let Σ be the set of closed subsets of X that are not finite unions of closed, irreducible spaces. If Σ is nonempty, since X is Noetherian, there is some minimal element $Y \in \Sigma$. Since Y itself cannot be irreducible (then it would be a finite union of closed, irreducible spaces, itself) there exists two nonempty open sets U, V of Y such that $U \cap V = \emptyset$. The complements of these sets A, B are then

proper subsets of Y and such that $A \cup B = Y$. By minimality of Y, we necessarily have that A and B are finite unions of closed, irreducible sets, but then $Y = A \cup B$ is a finite union of closed, irreducible sets. This is a contradiction. Therefore, $\Sigma = \emptyset$. Since X itself is a union of closed, irreducible sets (from chapter 1), this implies that X can be written as a finite union of closed, irreducible sets.

8. Let A be a Noetherian ring and $V(\mathfrak{a}_1) \supseteq V(\mathfrak{a}_2) \supseteq \ldots$ be a decreasing sequence of closed subsets of $\mathrm{Spec}(A)$. We see for $i \leq j$,

$$r(\mathfrak{a}_i) = \bigcap_{\mathfrak{p} \in V(\mathfrak{a}_i)} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in V(\mathfrak{a}_j)} \mathfrak{p} = r(\mathfrak{a}_j).$$

Therefore, we have an increasing sequence of ideals

$$r(\mathfrak{a}_1) \subseteq r(\mathfrak{a}_2) \subseteq \ldots.$$

Therefore, for some n, $r(\mathfrak{a}_m) = r(\mathfrak{a}_n)$ for $m \geq n$. Then $V(\mathfrak{a}_m) = V(r(\mathfrak{a}_m)) = V(r(\mathfrak{a}_n)) = V(\mathfrak{a}_n)$ so that the sequence of closed subsets is stationary. That is, $\mathrm{Spec}(A)$ is a Noetherian topological space.

To show the converse is not true, let k be a field, consider the polynomial ring $k[x_1, x_2, \ldots]$ in countably many indeterminants, the ideal $\mathfrak{a}$ generated by the indeterminants $x_1, \ldots$, and the quotient $A = k[x_1, \ldots]/\mathfrak{a}^2$. Clearly, the sequence of ideals

$$(x_1) \subset (x_1, x_2) \subset \ldots$$

is strictly increasing so that A is not Noetherian. However, $\mathrm{Spec}(A)$ is finite and so necessarily is Noetherian (in fact, $\mathrm{Spec}(A)$ is a singleton. Every prime ideal contains every x_i since they are nilpotent. Conversely, the ideal generated by the x_i is maximal since the quotient is the field k).

9. Let A be a Noetherian ring. From the previous exercise, $\mathrm{Spec}(A)$ is Noetherian. Therefore, $\mathrm{Spec}(A)$ has finitely many irreducible components. Since the irreducible components of $\mathrm{Spec}(A)$ are in bijection with the minimal primes of A (from chapter 1), there are finitely many minimal primes of A.

10. Let M be a Noetherian A-module and $\mathfrak{a} = \mathrm{ann}(M)$. From a previous problem, since M is finitely-generated, we have

$$\mathrm{Supp}(M) = V(\mathfrak{a}) = \mathrm{Spec}(A/\mathfrak{a}).$$

From an earlier problem in this chapter, we know that $A/\mathfrak{a}$ is a Noetherian ring since M is Noetherian. Therefore, $\mathrm{Spec}(A/\mathfrak{a})$ is a closed, Noetherian subspace of $\mathrm{Spec}(A)$.

11. First, it will be shown that every ideal in a Noetherian ring has a primary decomposition. From the primary decomposition chapter, we need only show that a Noetherian ring satisfies $(L1)$ and $(L2)$ (defined in a previous problem).

Let A be a Noetherian ring, $\mathfrak{a}$ be an ideal and $\mathfrak{p}$ be any prime ideal. We know that

$$S_{\mathfrak{p}}(\mathfrak{a}) = \bigcup_{s \in A \setminus \mathfrak{p}} (\mathfrak{a} : s).$$

Consider $\Sigma = \{(\mathfrak{a} : s) : s \in A \setminus \mathfrak{p}\}$. Since A is Noetherian, there is some maximal element of Σ, say $(\mathfrak{a} : x)$ for $x \in A \setminus \mathfrak{p}$. For $y \in S_{\mathfrak{p}}(\mathfrak{a})$, there exists some $s \in A \setminus \mathfrak{p}$ such that $sy \in \mathfrak{a}$. Consider $(\mathfrak{a} : sx)$ (where $sx \in A \setminus \mathfrak{p}$ since they both are). It is clear that $(\mathfrak{a} : x) \subseteq (\mathfrak{a} : sx)$ so that

$$y \in (\mathfrak{a} : sx) = (\mathfrak{a} : x).$$

Therefore, $S_{\mathfrak{p}}(\mathfrak{a}) = (\mathfrak{a} : x)$ (since the other inclusion holds trivially). Therefore, $(L1)$ is satisfied.

Let $\mathfrak{a}$ be an ideal and $S_1 \supseteq S_2 \supseteq \ldots$ be a decreasing sequence of multiplicatively closed subsets. Clearly, $S_1(\mathfrak{a}) \subseteq S_2(\mathfrak{a}) \subseteq \ldots$ is an increasing sequence of ideals and so is stationary. Therefore, $(L2)$ is satisfies almost trivially.

Since these are both satisfied, every ideal of a Noetherian ring has a primary decomposition.

From a problem last chapter, it was shown that if $f : A \mapsto B$ is such that $f^* : \mathrm{Spec}(B) \mapsto \mathrm{Spec}(A)$ is a closed map, then f has the going-up property.

Assume $\mathrm{Spec}(B)$ is Noetherian, $f : A \mapsto B$ has the going-up property, and $f^* : \mathrm{Spec}(B) \mapsto \mathrm{Spec}(A)$ is the induced map on spectra. As usual, we can reduce to the case that $A \subseteq B$ and $f : A \mapsto B$ is the inclusion map since the map $f^* : \mathrm{Spec}(f(A)) \mapsto \mathrm{Spec}(A)$ is a homeomorphism onto the closed set, $V(\ker f)$.

Let $V(\mathfrak{b}) \subseteq \mathrm{Spec}(B)$ be an arbitrary closed subset of $\mathrm{Spec}(B)$. From the above, there exists a primary decomposition

$$\mathfrak{b} = \bigcap_{i=1}^{n} \mathfrak{q}_i,$$

where $r(\mathfrak{q}_i) = \mathfrak{p}_i$. This implies

$$V(\mathfrak{b}) = \bigcup_{i=1}^{n} V(\mathfrak{q}_i) = \bigcup_{i=1}^{n} V(\mathfrak{p}_i).$$

Since then

$$f^*(V(\mathfrak{b})) = \bigcup_{i=1}^{n} f^*(V(\mathfrak{p}_i)),$$

it suffices to show that each $f^*(V(\mathfrak{p}_i))$ is closed (where f is the inclusion map). However, the going-up property immediately implies that $f^*(V(\mathfrak{p})) = V(\mathfrak{p}^c) = V(\mathfrak{p} \cap A)$ for any prime $\mathfrak{p}$ of B. The inclusion $\subseteq$ holds trivially. For the other inclusion, consider any containment of prime ideals $\mathfrak{p} \cap A \subseteq \mathfrak{p}'$. By the going-up property, there exists a prime ideal $\mathfrak{q}$ of B such that $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{q}^c = \mathfrak{q} \cap A = \mathfrak{p}'$. Therefore, $f^*(V(\mathfrak{p}_i)) = V(\mathfrak{p}_i \cap A)$ and so f^* is a closed map.

12. Let A be a ring such that $\mathrm{Spec}(A)$ is Noetherian. Consider a sequence of increasing prime ideals $\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \ldots$. This gives rise to the decreasing sequence of closed sets

$$V(\mathfrak{p}_1) \supseteq V(\mathfrak{p}_2) \supseteq \ldots.$$

Since $\mathrm{Spec}(A)$ is Noetherian, we eventually have $V(\mathfrak{p}_m) = V(\mathfrak{p}_n)$ for $m \geq n$. We then have

$$\mathfrak{p}_m = r(\mathfrak{p}_m) = \bigcap_{\mathfrak{q} \in V(\mathfrak{p}_m)} \mathfrak{q} = \bigcap_{\mathfrak{q} \in V(\mathfrak{p}_n)} \mathfrak{q} = r(\mathfrak{p}_n) = \mathfrak{p}_n.$$

Therefore, the sequence of prime ideals is stationary. That is, the set of prime ideals satisfies the ascending chain condition.

To show that the converse does not hold, consider the ring $A = \prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$. It is clear that every prime ideal is maximal (the quotient by a prime ideal has two elements and so is a field). Therefore, A satisfies the ascending chain condition for prime ideals. However, there is a strictly decreasing sequence of closed subsets of $\mathrm{Spec}(A)$.

$$V(e_1) \supset V(e_1, e_2) \supset \ldots$$

To see that each containment is proper, notice that $(1 - e_k) \in V(e_1, \ldots, e_{k-1})$, but $(1 - e_k) \notin V(e_1, \ldots, e_k)$ (and $(1 - e_k) \in \mathrm{Spec}(A)$ clearly).

Chapter 7

1. Let A be a non-Noetherian ring, Σ be the set of ideals in A that are not finitely-generated. By Zorn's lemma, it is easy to see that there are maximal elements of Σ (for an increasing chain of not finitely-generated ideals, their union is an ideal that is not finitely-generated). Let $\mathfrak{a} \in \Sigma$ be maximal and let $xy \in \mathfrak{a}$ with $x \notin \mathfrak{a}$. Since $x \notin \mathfrak{a}$, $\mathfrak{a} \subset \mathfrak{a} + (x)$ so that $\mathfrak{a} + (x)$ is necessarily finitely-generated by say $\{a_i + b_i x\}_{i=1}^{n}$. Let $\mathfrak{a}_0 = (a_1, \ldots, a_n)$. Clearly, $\mathfrak{a}_0$ is finitely-generated and $\mathfrak{a}_0 + (x) \subseteq \mathfrak{a} + (x)$. The other inclusion follows immediately since $\mathfrak{a} + (x) = (a_1 + b_1 x, \ldots, a_n + b_n x)$. Therefore,

$$\mathfrak{a}_0 + (x) = \mathfrak{a} + (x).$$

We clearly have $\mathfrak{a} \supseteq \mathfrak{a}_0 + x \cdot (\mathfrak{a} : x)$. For $a \in \mathfrak{a} \subseteq \mathfrak{a} + (x)$, we may write $a = a_0 + bx$ for $a_0 \in \mathfrak{a}_0$. Subtracting a_0, we get that $bx \in \mathfrak{a}$ so $b \in (\mathfrak{a} : x)$. That is,

$$a = a_0 + bx \in \mathfrak{a}_0 + x \cdot (\mathfrak{a} : x).$$

Therefore, we have $\mathfrak{a} = \mathfrak{a}_0 + x \cdot (\mathfrak{a} : x)$. If $(\mathfrak{a} : x)$ is finitely-generated, we get that $\mathfrak{a}$ is finitely-generated, which is a contradiction. Therefore, $(\mathfrak{a} : x) \in \Sigma$, but $\mathfrak{a} \subsetneq (\mathfrak{a} : x)$ so that they are necessarily equal $\mathfrak{a} = (\mathfrak{a} : x)$. Therefore,

$$y \in (\mathfrak{a} : x) = \mathfrak{a}.$$

That is, $\mathfrak{a}$ is necessarily prime. Therefore, if A is non-Noetherian, then there exists prime ideals that are not finitely-generated. The contrapositive of this statement is that if all prime ideals of a ring are finitely-generated, then the ring itself is Noetherian.

2. The implication that if $f \in A[[x]]$ is nilpotent than the coefficients are nilpotent was an exercise from the first chapter. It is very easy to see that if $f \in A[[x]]$ is nilpotent, then the term of lowest power is nilpotent (and so the coefficient as well). Subtracting, we get $g \in A[[x]]$ of higher lowest term that is still nilpotent. Continuing in this way by induction, it follows that every coefficient is nilpotent.

   If $f \in A[[x]]$ has coefficients $a_i \in \mathfrak{N}_A$, then since A is Noetherian, $\mathfrak{N}_A$ is nilpotent and there exists $m > 0$ such that $\mathfrak{N}_A^m = 0$ (that is, all products of m nilpotents is zero). Then $f^m = 0$ since every coefficient is in $\mathfrak{N}_A^m = 0$.

3. $(i) \implies (ii)$ Let $\mathfrak{a}$ be a proper irreducible ideal of a ring A. Assume $\mathfrak{a}$ is primary with $r(\mathfrak{a}) = \mathfrak{p}$ and S is a multiplicative subset of A. If $S \cap \mathfrak{p} \neq \emptyset$, then $\mathfrak{a}^{ec} = (1) = (\mathfrak{a} : s^n)$ for $s \in S \cap \mathfrak{p}$ and n sufficiently large (so that $s \in \mathfrak{a}$). If $S \cap \mathfrak{p} = \emptyset$, then for every $s \in S$, if $x \in (\mathfrak{a} : s)$, since $s \notin \mathfrak{p}$, $x \in \mathfrak{a}$. That is,

   $$\mathfrak{a} \subseteq (\mathfrak{a} : s) \subseteq \mathfrak{a}.$$

   Therefore, we have equality.

   $(ii) \implies (i)$ From a previous problem (chapter 4, problem 17) if this property holds, the ideal $\mathfrak{a}$ can be decomposed into the intersection of a primary ideal and an ideal that properly contains $\mathfrak{a}$. Since $\mathfrak{a}$ is irreducible, this implies that $\mathfrak{a}$ is then primary.

   $(i) \implies (iii)$ Let $\mathfrak{a}$ be primary and $r(\mathfrak{a}) = \mathfrak{p}$. If $x \in \mathfrak{p}$, then $x^n \in \mathfrak{a}$ for some $n > 0$ so that $(\mathfrak{a} : x^n) = (1)$ and so this sequence is stationary. If $x \notin \mathfrak{p}$, then it is easy to see that $(\mathfrak{a} : x) \subseteq \mathfrak{a}$. Since the other inclusion is immediate, we have equality and so this sequence is stationary as well.

   $(iii) \implies (i)$ Passing to the quotient $A/\mathfrak{a}$, it suffices to show that every nonzero zero-divisor is nilpotent. Our hypotheses are then (by the correspondence of ideals under quotients) that for every $x \in A$, the sequence $\mathrm{ann}(x^n)$ is stationary. For $xy = 0$ and $y \neq 0$, our hypotheses imply that $(x^n) \cap (y) = 0$ for n such that the sequence of annihilators is stationary at n. Indeed, if $a \in (y) \cap (x^n)$, $ax = 0$ (since $a \in (y)$), and $a = bx^n$ for some $b \in A$. But then, $bx^{n+1} = ax = 0$ implies that $b \in \mathrm{ann}(x^{n+1}) = \mathrm{ann}(x^n)$. That is, $a = bx^n = 0$. Since $\mathfrak{a}$ is irreducible, 0 is irreducible in $A/\mathfrak{a}$. Since $y \notin 0$, this then implies that $x^n = 0$ so x is nilpotent. Therefore, $\mathfrak{a}$ is primary.

4a. Let A_1 be the ring of rational functions $\mathbb{C}(x)$ with no pole on the unit circle, $\mathbb{S}^1$. Consider the inclusion $\mathbb{C}[x] \hookrightarrow A_1$. For any ideal $\mathfrak{a}$ of A_1, $\mathfrak{a}^c$ is clearly the set of numerators of elements of A_1. Then an increasing sequence of ideals $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \ldots$ induces an increasing sequence of ideals $\mathfrak{a}_1^c \subseteq \mathfrak{a}_2^c \subseteq \ldots$. Since $\mathbb{C}[x]$ is a finitely-generated algebra over a field, it is Noetherian and this sequence is stationary. It is clear that the extension of a contracted ideal is the original ideal itself (just put denominators back). Therefore, the original sequence of ideals is stationary as well.

4b. Let A_2 be the ring of analytic functions with convergent power series at $z = 0$ (that is, positive radius of convergence). For any ideal $\mathfrak{a}$, let

$$f(z) = \sum_{n=k}^{\infty} a_n z^n \in \mathfrak{a}$$

be such that k is minimal with $a_k \neq 0$ (Note that $k = 0$ implies f is a unit). Then for any

$$g(z) = \sum_{n=l}^{\infty} b_n z^n \in \mathfrak{a},$$

g/f has a removable singularity at $z = 0$ and therefore is analytic in some neighborhood of $z = 0$. That is, $g/f \in A_2$. Then $g = f(g/f) \in (f)$. This then implies that $\mathfrak{a} = (f)$ so that A_2 is a PID and hence, Noetherian.

4c. Let A_3 be the ring of entire functions and consider the increasing chain of ideals

$$(\sin z) \subset (\sin z/2) \subset (\sin z/4) \subset \dots.$$

These inclusions are clearly proper since if $g(w)\sin w = \sin w/2$, then $g(w)$ necessarily has poles (for example at $w = \pi$) and so g is not entire. Therefore, A_3 is not Noetherian.

4d. Let A_4 be the set of polynomials whose first k derivatives vanish (note $k = 0$ is not included so the ring has an identity). It is clear that every element of $f \in A_4$ may be written

$$f(z) = a_0 + \sum_{n=k+1}^{m} a_n z^n.$$

From this, A_4 is a finitely-generated $\mathbb{C}$-algebra and hence, Noetherian (generated by $\{1, z^{k+1}, z^{k+2}, \dots, z^{2k+1}\}$).

4e. Let A_5 be the ring of polynomials $\mathbb{C}[z, w]$ such that all partial derivatives (except the 0-th) with respect to w vanish for $z = 0$. Consider the increasing chain of ideals

$$(zw) \subset (zw, zw^2) \subset (zw, zw^2, zw^3) \subset \dots.$$

To show this chain is strictly increasing, we need only show $zw^{n+1} \notin (zw, zw^2, \dots, zw^n)$. As an equality of polynomials, it is clear that if $zw^{n+1} = g(z, w)zw^k$ for $k \leq n$, then g is independent of z. However, this should imply that g is a constant since the partial derivatives then need vanish. This is a contradiction. Therefore, the inclusions are proper and so A_5 is not Noetherian.

5. Let A be a Noetherian ring, B a finitely-generated A-algebra, G a finite group of A-automorphisms of B, and B^G the set of elements fixed by all elements of G. As usual, we may consider $A \subseteq B$. Then this follows immediately from proposition 7.8. Indeed, A is Noetherian, B is a finitely-generated A-algebra, and B is integral over B^G. Therefore, B^G is a finitely-generated A-algebra.

6. Let K be a finitely-generated ring that is also a field. If $\operatorname{char} K = 0$, then $\mathbb{Z} \subseteq \mathbb{Q} \subseteq K$ and so K is finitely-generated over $\mathbb{Q}$. By Zariski's lemma, K is a finite algebraic extension of $\mathbb{Q}$ and therefore a finitely-generated $\mathbb{Q}$-module. Since $\mathbb{Z}$ is Noetherian, K is a finitely-generated $\mathbb{Z}$-algebra (it is a finitely-generated ring), and K is finitely-generated as a $\mathbb{Q}$-module, by proposition 7.8, we get a contradiction since this should imply that $\mathbb{Q}$ is a finitely-generated $\mathbb{Z}$-algebra. Therefore, $\operatorname{char} K > 0$.

   Therefore, K is finitely-generated as a $\mathbb{Z}/p\mathbb{Z}$-algebra for some prime $p > 0$. By Zariski's lemma, this implies that K is a finite algebraic extension of $\mathbb{Z}/p\mathbb{Z}$ and is therefore a finite field.

7. Let X be an affine algebraic variety over a field k given by the family of equations $f_\alpha(t_1, \dots, t_n) = 0$ for $\alpha \in I$ for some index set I. Consider the corresponding ideal $\mathfrak{a}$ generated by the f_α. By the Hilbert basis theorem, this ideal is finitely-generated and so we may write

$$\mathfrak{a} = (f_1, \dots, f_n).$$

   That is, each of the above f_α can be written as linear combinations of these f_i and vice versa (so if one set is zero, so is the either set). Therefore, X is determined by finitely many polynomials, $f_1, \dots, f_n$.

8. Let A be a ring such that $A[x]$ is Noetherian. For any increasing chain of ideals $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \dots$ of A, consider the chain of ideals

$$\mathfrak{a}_1 + (x) \subseteq \mathfrak{a}_2 + (x) \subseteq \dots$$

   of $A[x]$. Since $A[x]$ is Noetherian, this sequence is stationary. Taking contractions and using that $(\mathfrak{a}_i + (x))^c = (\mathfrak{a}_i + (x)) \cap A = \mathfrak{a}_i$, we get that the original sequence of ideals is stationary as well. That is, A is Noetherian as well.

9. Let A be a ring such that for every maximal ideal $\mathfrak{m}$ of A, $A_\mathfrak{m}$ is Noetherian and for every $x \in A$, $x \neq 0$, there are finitely many maximal ideals that contain x. Let $\mathfrak{a}$ be a nonzero ideal of A and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ be the set of maximal ideals that contain $\mathfrak{a}$ (there are finitely many because if there were infinitely many, every point of $\mathfrak{a}$ is contained in infinitely many maximal ideals). Let $x_0 \in \mathfrak{a}$, $x_0 \neq 0$ and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_{r+s}$ be the set of maximal ideals that contain x_0. Since $\mathfrak{m}_{r+1}, \ldots, \mathfrak{m}_{r+s}$ don't contain $\mathfrak{a}$, there exists $x_j \in \mathfrak{a}$ such that $x_j \notin \mathfrak{m}_{r+j}$ for $1 \leq j \leq s$. Since $A_{\mathfrak{m}_i}$ is Noetherian for each $1 \leq i \leq r$, the extension of $\mathfrak{a}$ to $A_{\mathfrak{m}_i}$ is finitely-generated. Then there are $x_{r+s+1}, \ldots x_t \in \mathfrak{a}$ whose images generate $\mathfrak{a}A_{\mathfrak{m}_i}$ for $1 \leq i \leq r$. Let $\mathfrak{a}_0 = (x_0, \ldots, x_t)$.

   For any $\mathfrak{m}_i$, $1 \leq i \leq r$, we know that the images of the $x_{r+s+1}, \ldots, x_t$ generate $\mathfrak{a}A_{\mathfrak{m}_i}$ so that the extensions of $\mathfrak{a}_0$ and $\mathfrak{m}$ agree in these localizations. For the remaining maximal ideals that contain x_0, $\mathfrak{m}_{r+1}, \ldots, \mathfrak{m}_{r+s}$, $x_j \in A \setminus \mathfrak{m}_{r+j}$ with $x_j \in \mathfrak{a} \cap \mathfrak{a}_0$ so that the extensions of these ideals to these localizations are the unit ideal and so agree. For the remaining maximal ideals that don't contain x_0, $x_0 \in \mathfrak{a} \cap \mathfrak{a}_0$ so that once again, the extensions of these ideals are the unit ideal and so agree. That is, the extensions of $\mathfrak{a}$ and $\mathfrak{a}_0$ agree in all localizations of maximal ideals.

   Consider the A-module homomorphism $\mathfrak{a}_0 \hookrightarrow \mathfrak{a}$. Since localizations of this map at all maximal ideals is surjective, the map itself is surjective and so we necessarily have $\mathfrak{a} = \mathfrak{a}_0$ is finitely generated. Since $\mathfrak{a}$ was an arbitrary ideal, this implies that A is Noetherian.

10. Let M be a Noetherian A-module and let $M[x]$ be the polynomial ring with coefficients in M be an $A[x]$-module. By imitating the proof of the Hilbert basis theorem, it will follow that $M[x]$ is a Notherian A-module. Let N be a submodule of $M[x]$. Consider the submodule P of M generated by the leading coefficients of elements of N (it is clear that this is an A-module). Since M is a Noetherian A-module, P is finitely generated, say by $a_1, \ldots, a_n$. For each $1 \leq i \leq n$, there then exists $f_i \in N$ such that

$$f_i(x) = a_i x^{r_i} + \sum_{j=0}^{r_i - 1} f_{ij} x^j.$$

   Let $r = \max\{r_i\}_{i=1}^n$ and N' be the submodule of $M[x]$ generated by the f_i. For any $f \in N$, if $\deg f = m \geq r$, the leading term of f is necessarily given by ax^m with $a \in P$. We may write

$$a = \sum_{j=1}^{n} u_j a_j$$

   for some $u_j \in A$. Then

$$f(x) - \sum_{j=1}^{n} u_j x^{m-r_j} f_j(x)$$

   has degree strictly less than m. That is, we may reduce the degree of f until $\deg f < r$. If N'' is the A-module consisting of polynomials in $M[x]$ of degree strictly less than r, then from the above degree reduction, we have that $N = N' + N''$. Notice that as $N'' \simeq M^r$ and so is Noetherian (since M is Noetherian). If $g_1, \ldots, g_m$ generate N'' as an A-module, then the finite collection $f_1, \ldots, f_n, g_1, \ldots, g_m$ generates N (as an $A[x]$-module, but you really only need coefficients from A for the g_j). Therefore, $M[x]$ is Noetherian.

11. Being Noetherian is not a local property. Let $A = \oplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$ (direct sum). It is easy to see that A is not Noetherian since the sequence of ideals $(e_1) \subset (e_1, e_2) \subset \ldots$ is not stationary. It isn't hard to see that the only prime ideals of A are $(e_j)_{j \neq i}$ (since if $\mathfrak{p} \neq (1)$, there is necessarily some $e_i \notin \mathfrak{p}$. Then $e_i e_j = 0 \in \mathfrak{p}$ implies $e_j \in \mathfrak{p}$. Conversely, for any element of $\mathfrak{p}$, if the i-th component is nonzero, multiplying by e_i gives that $e_i \in \mathfrak{p}$. From this, every element of $\mathfrak{p}$ is a sum of e_j for $j \neq i$. Therefore, $\mathfrak{p} = (e_j)_{j \neq i}$). It is not too difficult to see that for $\mathfrak{p} = (e_j)_{j \neq i}$, $A \setminus \mathfrak{p} = \{a \in A : ae_i = e_i\}$. Then for $f/g \in A_\mathfrak{p}$,

$$\frac{f}{g} = \frac{fe_i}{ge_i} = \frac{fe_i}{e_i}.$$

   Here, either $fe_i = 0$ or $fe_i = e_i$ so that $A_\mathfrak{p}$ has exactly two elements and so is a field. Therefore, $A_\mathfrak{p}$ is Noetherian for each prime $\mathfrak{p}$, but A is not Noetherian.

12. Let A be a ring and $f : A \mapsto B$ be a faithfully-flat Noetherian A-algebra. If $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \ldots$ is an ascending chain of ideals in A, then

$$\mathfrak{a}_1^e \subseteq \mathfrak{a}_2^e \subseteq \ldots$$

is an ascending chain of ideals in B. Since B is a Noetherian, this sequence is stationary. Since B is faithfully-flat, contracting the above sequence gives the original sequence. Since this sequence is stationary however, the original sequence is necessarily stationary as well. Therefore, A is Noetherian.

13. Let $f : A \mapsto B$ be a finitely-generated A-algebra (for which we can assume $A \subseteq B$ and f is the inclusion map) and $f^* : \mathrm{Spec}(B) \mapsto \mathrm{Spec}(A)$ be the induced map on spectra. From a previous problem, we know fibers are given by

$$(f^*)^{-1}(\mathfrak{p}) = \mathrm{Spec}(k(\mathfrak{p}) \otimes_A B),$$

where $k(\mathfrak{p})$ is the residue field of A at $\mathfrak{p}$. From this, it suffices to show that the ring $k(\mathfrak{p}) \otimes_A B$ is Noetherian. From the Hilbert basis theorem, we know that $k(\mathfrak{p})[t_1, \ldots, t_n] \simeq k(\mathfrak{p}) \otimes_A A[t_1, \ldots, t_n]$ is a Noetherian ring for all $n > 0$. Since there is a surjective map $A[t_1, \ldots, t_n] \mapsto B$, tensoring with $k(\mathfrak{p})$ gives a surjective map $k(\mathfrak{p})[t_1, \ldots, t_n] \mapsto k(\mathfrak{p}) \otimes_A B$. Since this is surjective, we have that $k(\mathfrak{p}) \otimes_A B$ is Noetherian.

14. Let k be an algebraically closed field, $A = k[t_1, \ldots, t_n]$, $\mathfrak{a}$ be an ideal of A, X be the variety in k^n determined by $\mathfrak{a}$, and $I = I(X)$ be the ideal of polynomials that vanish identically on X. Clearly, $r(\mathfrak{a}) \subseteq I$ since k is a field. If $f \notin r(\mathfrak{a})$, then there is some prime ideal $\mathfrak{p}$ that contains $\mathfrak{a}$ but not f (localize at $(f^n)_{n \geq 0}$). Let $\overline{f}$ be the image of f in $B = A/\mathfrak{p}$, $C = B_{\overline{f}}$, and $\mathfrak{m}$ be a maximal ideal of C. Since C is generated by the $\overline{t_i}/1$ and $1/\overline{f}$ as a k-algebra, C is finitely-generated as a k-algebra and also a field so that $C/\mathfrak{m} \simeq k$ by Zariski's lemma. Let $(x_1, \ldots, x_n) = (\pi(\overline{t_1}/1), \ldots, \pi(\overline{t_n}/1)) \in (C/\mathfrak{m})^n = k^n$ be the images of the generators of the t_i in $C/\mathfrak{m} = k$.

    For $g \in \mathfrak{a}$, using that the maps above are all ring homomorphisms, we get that

    $$g(x_1, \ldots, x_n) = \pi(\overline{g(t_1, \ldots, t_n)}/1) = 0,$$

    since $g(t_1, \ldots, t_n) \in \mathfrak{a} \subseteq \mathfrak{p}$. Therefore, $(x_1, \ldots, x_n) \in X$. Similarly, we may write

    $$f(x_1, \ldots, x_n) = \pi(\overline{f(t_1, \ldots, t_n)}/1).$$

    Since $f \notin \mathfrak{p}$, $\overline{f(t_1, \ldots, t_n)} \neq 0$. Since B is an integral domain, $\overline{f(t_1, \ldots, t_n)}/1 \neq 0$ ($B \mapsto B_{\overline{f}}$ is injective), and since this is a unit in C, $\pi(\overline{f(t_1, \ldots, t_n)}/1) \neq 0$. That is, $f(x_1, \ldots, x_n) \neq 0$. Therefore, $f \notin I$. The contrapositive of this is that $I \subseteq r(\mathfrak{a})$. Combining inclusions, we get that $I = r(\mathfrak{a})$.

15. $(i) \implies (ii)$ Let $(A, \mathfrak{m})$ be a Noetherian local ring with residue field k and M be a finitely-generated A-module. If M is free, then M is isomorphic to a direct sum of A and so is flat.

    $(ii) \implies (iii)$ If M is flat, then clearly the injective map $\mathfrak{m} \hookrightarrow A$ induces an injective map $\mathfrak{m} \otimes_A M \hookrightarrow A \otimes_A M = M$.

    $(iii) \implies (iv)$ This follows immediately by writing out the induced Tor sequence for the exact sequence

    $$0 \to \mathfrak{m} \to A \to k \to 0.$$

    $(iv) \implies (i)$ Let $x_1, \ldots, x_n$ be elements of M whose images in $M/\mathfrak{m}M$ form a basis of M as a k-vector space. As a consequence of Nakayama's lemma, the x_i generate M. Let F be the free A-module with basis $e_1, \ldots, e_n$ so that $\phi : F \mapsto M$ defined by $\phi(e_i) = x_i$ is surjective. If $E = \ker \phi$, we have an exact sequence

    $$0 \to E \to F \to M \to 0.$$

    Tensoring with k, we get a new exact sequence (using that $\mathrm{Tor}_1^A(k, M) = 0$)

    $$0 \to k \otimes_A E \to k \otimes_A F \to k \otimes_A M \to 0.$$

    Since the latter map is a surjective map between vector spaces of the same dimension, it is necessarily injective so that $k \otimes_A E = 0$. From a previous problem (chapter 2, problem 2), we have that $k \otimes_A E \simeq E/\mathfrak{m}E = 0$. Since A is a local ring, $\mathfrak{J}_A = \mathfrak{m}$ so that by Nakayama's lemma, $E = 0$. Therefore, the map ϕ itself is injective and so is an isomorphism. Therefore, M is free.

16. $(i) \implies (ii)$ Let A be a Noetherian ring and M a finitely-generated A-module. If M is a flat A-module, then $M_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module for each prime $\mathfrak{p}$. From the last problem, since $A_\mathfrak{p}$ is a Noetherian local ring and $M_\mathfrak{p}$ is finitely generated as a $A_\mathfrak{p}$-module, this implies that $M_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module for any prime $\mathfrak{p}$.

    $(ii) \implies (iii)$ This is immediate.

    $(iii) \implies (i)$ If $M_\mathfrak{m}$ is a free $A_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}$, then $M_\mathfrak{m}$ is a flat $A_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}$ from the previous problem. Since flatness is a local property, this implies that M is a flat A-module.

17. Let A be a ring and M a Noetherian A-module. Let a submodule N of M be irreducible if for every submodules N', N'' such that $N = N' \cap N''$, either $N' = N$ or $N'' = N$.

   Suppose there is a submodule that is not a finite intersection of irreducible submodules. Then the set Σ of such submodules is non-empty. Since M is Noetherian, it has some maximal element N. Note that N itself is not irreducible since then it is a finite intersection of irreducible submodules. Therefore, $N = N' \cap N''$ for some submodules N', N'' where neither is equal to N. Since $N \subset N', N''$, N' and N'' are finite intersections of irreducible submodules so $N = N' \cap N''$ is a finite intersection of irreducible submodules. This is a contradiction. Therefore, every submodule of M is a finite intersection of irreducible submodules.

   Let N be an irreducible submodule of M. We wish to show that every zero-divisor of M/N is nilpotent. Since N is irreducible, 0 is irreducible in M/N. Let $x \in A$ be such that $\phi_x : M \mapsto M$ defined by $m \mapsto xm$ is not injective. Consider the ascending sequence of submodules

$$\ker \phi_x \subseteq \ker \phi_{x^2} \subseteq \dots.$$

   Since M is Noetherian, M/N is Noetherian and so this sequence is stationary. Let n be such that $\ker \phi_{x^n} = \ker \phi_{x^{n+1}}$. Then $\operatorname{Im} \phi_{x^n} \cap \ker \phi_x = 0$. Indeed, if $m \in \ker \phi_x$, then $xm = 0$. But $m \in \operatorname{Im} \phi_{x^n}$ implies $m = x^n m'$ so that $xm = x^{n+1} m' = 0$ so $m' \in \ker \phi_{x^{n+1}} = \ker \phi_{x^n}$ so that $m = x^n m' = 0$. Since $\ker \phi_x \neq 0$, irreducibility of 0 implies that $\operatorname{Im} \phi_{x^n} = 0$ so that $\phi_{x^n} = 0$ and x is nilpotent. Therefore, an irreducible module is primary.

   As a corollary of these results, we have that any submodule of a Noetherian module can be written as a finite intersection of primary modules. That is, every submodule of a Noetherian module has a primary decomposition.

18. $(i) \implies (ii)$ Let A be a Noetherian ring, $\mathfrak{p}$ a prime ideal, and M a finitely-generated A-module. If $\mathfrak{p}$ is an associated prime of 0 in M, we may write (from the previous problem) a minimal decomposition

$$0 = \bigcap_{i=1}^{n} N_i,$$

   such that $r_M(N_1) = \mathfrak{p}$. Let $\mathfrak{a} = \cap_{i=2}^{n} N_i$. For $x \in \mathfrak{a}$, $x \neq 0$, since $x \notin N_1$, we have $r(N_1 : x) = \mathfrak{p}$ (from chapter 4, problem 21). That is, $(N_1 : x) \subseteq \mathfrak{p}$.

   It will now be shown that for every submodule N of M, there exists an integer $k > 0$ such that $(r_M(N))^k \subseteq (N : M)$. Notice that $r_M(N)M$ is a submodule of M and so is finitely-generated by say $x_1 m_1, \dots, x_n m_n$. Then for each $1 \leq i \leq n$, there exists k_i such that $x_i^{k_i} \in (N : M)$. Let $k = 1 + \sum_{i=1}^{n}(k_i - 1)$. Then by the pigeon-hole principle, for every product of k elements from the set $x_1, \dots, x_n$, the product lies in $(M : N)$. Consider an single term

$$y_1 \dots y_k m \in (r_M(N))^k M.$$

   Clearly, $y_k m \in r_M(N)M$ so it can be written as a linear combination

$$y_k m = \sum_{i=1}^{n} a_{i1} x_i m_i \implies y_1 \dots y_k m = \sum_{i=1}^{n} a_i x_i y_1 \dots y_{k-1} m_i.$$

   From here, $y_{k-1} m_i \in r_M(N)M$ and so we may continue this process. This process is finite since there are finitely many y_j and each time we add finitely many terms. In the end, we get a linear combination of the m_i with coefficients given by an element of A times a product of k of the x_i and therefore lies in $(M : N)$. It follows from this that $y_1 \dots y_k m \in N$. It follows from this that $(r_M(N))^k \subseteq (N : M)$.

   Letting $N = N_1$ in the above, we get that there is some integer $k > 0$ such that $\mathfrak{p}^k = (r_M(N_1))^k \subseteq (N_1 : M)$. It follows that

$$\mathfrak{p}^k \mathfrak{a} \subseteq \mathfrak{a} \cap N_1 = 0.$$

   Let k be minimal such that the above holds. Then for $x \in \mathfrak{p}^{k-1}\mathfrak{a}$, $x \in \mathfrak{a}$ so that $\operatorname{ann}(x) \subseteq \mathfrak{p}$. Conversely, for every $p \in \mathfrak{p}$, $px \in \mathfrak{p}^k \mathfrak{a} = 0$ so that $\mathfrak{p} \subseteq \operatorname{ann}(x)$. Therefore, $\operatorname{ann}(x) = \mathfrak{p}$ as desired.

   $(ii) \implies (i)$ This is trivial from the fact that the primes associated to 0 are the primes of the form $r(\operatorname{ann}(x))$.

   $(ii) \implies (iii)$ For a fixed prime $\mathfrak{p}$ and $x \in M$ such that $\mathfrak{p} = \operatorname{ann}(x)$, this is evident by the A-module homomorphism $\phi : A \mapsto M$ defined by $\phi(a) = ax$. The kernel is clearly $\operatorname{ann}(x) = \mathfrak{p}$ and so the image (the submodule generated by x) is a submodule isomorphic to $A/\mathfrak{p}$.

$(iii) \implies (ii)$ Let $\phi : A/\mathfrak{p} \mapsto M$ be injective (so an isomorphism with a submodule of M) and let $x = \phi(1+\mathfrak{p})$. By the fact that ϕ is well-defined, we have that $\mathfrak{p} \subseteq \mathrm{ann}(x)$ (for any $p \in \mathfrak{p}$, we necessarily have $px = 0$ for ϕ to be well-defined). Then $ax = 0$ implies that $\phi(a + \mathfrak{p}) = ax = 0$ so that $a \in \mathfrak{p}$ (since ϕ is injective. Therefore, $\mathrm{ann}(x) \subseteq \mathfrak{p}$. Therefore, $\mathrm{ann}(x) = \mathfrak{p}$.

For any prime $\mathfrak{p}_1$ associated to 0 in M, there exists a submodule M_1 such that $0 \subset M_1$ and $M_1 \simeq A/\mathfrak{p}_1$. Considering the quotient M/M_1, the zero module is again decomposable and for any prime $\mathfrak{p}_2$ associated with it, there is a submodule N_2 of M/M_1 such that $N_2 \simeq A/\mathfrak{p}_2$. Then via the correspondence of submodules under quotients, there is a submodule M_2 of M such that $M_1 \subset M_2$ and $M_2 \simeq N_2 \simeq A/\mathfrak{p}_2$. Continue in this way to get an ascending chain of submodules

$$0 \subset M_1 \subset M_2 \subset \dots.$$

Since M is Noetherian, this sequence is stationary. Note that at every step, if $M_i \neq M$, then the above process gives a submodule M_{i+1} that strictly contains M_i. That is, when this chain becomes constant, it is equal to M. Therefore, there is an ascending chain

$$0 \subset M_1 \subset M_2 \subset \ldots \subset M_n \subset M$$

such that each consecutive quotient is isomorphic to $A/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$.

*19.

20a. Let X be a topological space and $\mathfrak{F}$ be the smallest collection of subsets of X which contains all open subsets of X and is closed with under finite intersections and complements ($\mathfrak{F}$ is the intersection of all subsets of X that contain the open sets and are closed under finite intersections and complements). $\mathfrak{F}$ is therefore closed under finite unions as well. Since all open sets belong to $\mathfrak{F}$, all closed sets belong to $\mathfrak{F}$, all intersections $U \cap C$ of an open set and a closed set belong to $\mathfrak{F}$, and all finite unions of these intersections therefore belong to $\mathfrak{F}$. Conversely, the collection of finite unions of sets of the form $U \cap C$ for U open and C closed contains all open sets (take $C = X$) and is closed under finite intersections and complements (show the intersection of two elements is in the set and so by induction, it is closed under finite intersections. Then the complement of a finite union is the intersection of the complements, which is also in the set). Therefore, any element of $\mathfrak{F}$ is in this set and so can be written as a finite union of sets of the form $U \cap C$ where U is open and C is closed.

20b. Let X be irreducible and $E \in \mathfrak{F}$. If E contains a nonempty open set, $U \subseteq E$, since every open set is dense in X, we have $X = \overline{U} \subseteq \overline{E}$ so that E is dense in X. Conversely, if E is dense in X, write

$$E = \bigcup_{i=1}^{n} (U_i \cap C_i),$$

where none of the U_i or C_i are empty. Let $V = \cap_{i=1}^{n} (X \setminus C_i)$. We have

$$E \cap V = \left(\bigcup_{i=1}^{n} (U_i \cap C_i) \right) \cap \left(\bigcap_{i=1}^{n} (X \setminus C_i) \right) = \bigcup_{i=1}^{n} \left(U_i \cap C_i \cap \left(\bigcap_{i=1}^{n} (X \setminus C_i) \right) \right) = \emptyset.$$

Since E is dense and V is open, this implies $V = \emptyset$. Since the sets $X \setminus C_i$ are open, they are dense and since their intersection is empty, there exists i such that $X \setminus C_i = \emptyset$. That is, $C_i = X$ for some i. Then $U_i \subseteq E$ so that E contains a non-empty open set.

21. Let X be a Noetherian topological space and $E \subseteq X$. From the previous problem, if $E \in \mathfrak{F}$, then for every irreducible subset $X_0 \subseteq X$, $E \cap X_0$ is dense (in X_0) if and only if $E \cap X_0$ contains a nonempty open subset of X_0. That is, either $\overline{E \cap X_0} \neq X_0$ or $E \cap X_0$ contains some nonempty open subset of X_0.

Conversely, if $E \notin \mathfrak{F}$, then there exists a closed set $C \subseteq X$ such that $E \cap C \notin \mathfrak{F}$ (one can take $C = X$). Since X is Noetherian, the set of closed sets $C \subseteq X$ such that $E \cap C \notin \mathfrak{F}$ has a minimal element X_0 (so X_0 is closed and $E \cap X_0 \notin \mathfrak{F}$). X_0 is necessarily irreducible since if it can be written as a union of two proper closed sets $X_0 = C_1 \cup C_2$, then either $E \cap C_1 \notin \mathfrak{F}$ or $E \cap C_2 \notin \mathfrak{F}$ (since if they both are, $E \cap X_0 \in \mathfrak{F}$). This contradicts minimality of X_0. Therefore, X_0 is irreducible and closed.

If $\overline{E \cap X_0} \subset X_0$, then $E \cap \overline{E \cap X_0} \in \mathfrak{F}$. However, $E \cap X_0 \subseteq E$ and $E \cap X_0 \subseteq \overline{E \cap X_0}$ so $E \cap X_0 \subseteq E \cap \overline{E \cap X_0}$ and similarly, $E \cap X_0 \subseteq X_0$ implies $\overline{E \cap X_0} \subseteq X_0$ so that $E \cap \overline{E \cap X_0} \subseteq E \cap X_0$. Therefore, the two are equal so that $E \cap X_0 \in \mathfrak{F}$ is a contradiction.

If $U \subseteq E \cap X_0$ for some nonempty open set U of X_0, let $C = X_0 \setminus U$ so that $C \subset X_0$ is a closed subset of X and $E \cap C \in \mathfrak{F}$. Then $U = V \cap X_0$ for some open set V of X and $E \cap U = U = V \cap X_0 \in \mathfrak{F}$ so that $E \cap X_0 = (E \cap U) \cup (E \cap C) \in \mathfrak{F}$. This is a contradiction as well.

22. Let X be a Noetherian topological space and $E \subseteq X$. If E is open in X, then for every irreducible subspace $X_0 \subseteq X$, either $E \cap X_0$ is empty, or $E \cap X_0$ is open in X_0 by definition.

    Conversely, if E is not open, then there is some closed set C such that $E \cap C$ is not open in C (take $C = X$ for example). Since X is Noetherian, there is a minimal element of the set of closed sets C such that $E \cap C$ is not open, X_0.

    If $X_0 = C_1 \cup C_2$ where $X_0 \subset C_1, C_2$, then $E \cap C_1, E \cap C_2$ are necessarily open so that $E \cap X_0 = (E \cap C_1) \cup (E \cap C_2)$ is open as well, contradicting that $E \cap X_0$ is not closed. Therefore, C_0 is irreducible.

    By definition, $E \cap X_0$ is not open in X_0. In particular, this implies that $E \cap X_0 \neq \emptyset$. If $U \subseteq E \cap X_0$ for some nonempty open subset U of X_0, then $D = X_0 \setminus U \subset X_0$ is a closed subset of X so that $E \cap D$ is open in D. Then there exists V_1 open in X_0 such that $E \cap D = V_1 \cap X_0$. Similarly, there exists V_2 such that $E \cap U = V_2 \cap X_0$ (Since $E \cap U = U$ is open in X_0). From this, $E \cap X_0 = (E \cap U) \cup (E \cap D)$ is open in X_0, contrary to the assumption that $E \cap X_0$ is not open in X_0. That is, $E \cap X_0$ contains no nonempty open subset of X_0.

23. Let A be a Noetherian ring, $f : A \mapsto B$ be a finitely-generated A-algebra, $f^* : \mathrm{Spec}(B) \mapsto \mathrm{Spec}(A)$ be the induced map on spectra, and E be a constructible set (that is, belongs to the collection $\mathfrak{F}$ from the previous problems). Since functions preserve unions, from problem 20, it suffices to show that the image of every set of the form $U \cap C$ is constructible where U is open and C is closed in $\mathrm{Spec}(B)$. Writing $C = V(\mathfrak{b}) = \mathrm{Spec}(B/\mathfrak{b})$, the image of E is the image of an open set of $\mathrm{Spec}(B/\mathfrak{b})$ under the induced map on spectra of the composition $A \mapsto B \mapsto B/\mathfrak{b}$. That is, we may reduce to the case that E is open in $\mathrm{Spec}(B)$ (the composition still gives a finitely-generated A-algebra as well). Since A is Noetherian, B is Noetherian and $\mathrm{Spec}(B)$ is Noetherian so that E is quasi-compact (from chapter 6). From chapter 1, an open, quasi-open subset of a spectrum of a ring is necessarily a finite union of sets $Y_g = \mathrm{Spec}(B_g)$. Using that functions preserve unions again, it suffices to show that the image of each $\mathrm{Spec}(B_g)$ is constructible. Considering the composition $A \mapsto B \mapsto B_g$ and using that B_g is then a finitely-generated A-algebra, it suffices now to assume that $E = \mathrm{Spec}(B)$.

    Using problem 21, it suffices now to take an irreducible subset $X_0 \subseteq \mathrm{Spec}(A)$ such that $f^*(\mathrm{Spec}(B)) \cap X_0$ is dense and show that $f^*(\mathrm{Spec}(B)) \cap X_0$ contains some nonempty open set. From chapter 1, every irreducible subspace of $\mathrm{Spec}(A)$ has the form $X_0 = V(\mathfrak{p})$ for some prime $\mathfrak{p} \in \mathrm{Spec}(A)$. We wish to show

    $$f^*(\mathrm{Spec}(B)) \cap X_0 = f^*((f^*)^{-1}(X_0))$$

    is constructible in $\mathrm{Spec}(A)$ where $X_0 = V(\mathfrak{p}) = \mathrm{Spec}(A/\mathfrak{p})$ and

    $$(f^*)^{-1}(X_0) = V(\mathfrak{p}^e) = \mathrm{Spec}(B/\mathfrak{p}B) = \mathrm{Spec}(A/\mathfrak{p} \otimes_A B).$$

    That is, we want to show that the image of the induced map on spectra of the map $A \mapsto A/\mathfrak{p} \otimes_A B$ is constructible. Clearly, $\mathfrak{p}$ is contained in the kernel of this map so that it factors into the map $f : A/\mathfrak{p} \mapsto B/\mathfrak{p}^e = A/\mathfrak{p} \otimes_A B$ (that is, the image of the map on spectra is contained in $V(\mathfrak{p})$, so we may consider this new map instead). That is, we may assume A is an integral domain and $f : A \mapsto B$ is injective (since $A/\mathfrak{p} \otimes_A B \simeq B/\mathfrak{p}^e$) (this map is still such that $B/\mathfrak{p}^e$ is a finitely-generated $A/\mathfrak{p}$-module).

    If $Y_1, \dots, Y_n$ are the irreducible components of $\mathrm{Spec}(B)$ (only finitely many since Y is Noetherian) then it suffices to show that $f^*(Y_i)$ contains an open subset of $\mathrm{Spec}(A)$. That is, we may consider the map $g : A \mapsto B \mapsto B/\mathfrak{q}_i$ where $Y_i = V(\mathfrak{q}_i)$ (this still defines a finitely-generated A-algebra). Then since $A \mapsto B$ is injective and $B/\mathfrak{q}_i$ is an integral domain, the image of A in $B/\mathfrak{q}_i$ is an integral domain. Since the map $g : A \mapsto g(A)$ is surjective, the map of spectra is a homeomorphism onto $V(\ker g)$ and so we may finally consider only the inclusion map $i : f(A) \hookrightarrow B/\mathfrak{q}_i$. That is, we have reduced to the case that A, B are integral domains and f is injective.

    Let A, B be integral domains and $f : A \mapsto B$ be injective (we need only show that $f^*(\mathrm{Spec}(B))$ contains some nonempty open subset of $\mathrm{Spec}(A)$). From chapter 5 problem 21, there exists $s \in A$, $s \neq 0$, such that any map of A into an algebraically closed field and the image of s is nonzero, then the map extends to a map of B into the same field. Consider the set X_s (where $X = \mathrm{Spec}(A)$). For $\mathfrak{p} \in X_s$, consider the map

$g : A \mapsto A/\mathfrak{p} \mapsto \text{frac}(A/\mathfrak{p}) \mapsto \Omega$ where Ω is an algebraic closure of $\text{frac}(A/\mathfrak{p})$. Since $s \notin \mathfrak{p}$, $g(s) \neq 0$ and so there is an extension $\widetilde{g} : B \mapsto \Omega$. Then $\ker \widetilde{g} \in \text{Spec}(B)$ and $f^*(\ker \widetilde{g}) = \ker \widetilde{g} \cap A = \ker g = \mathfrak{p}$. That is, $X_s \subseteq f^*(\text{Spec}(B))$. Therefore, $f^*(\text{Spec}(B))$ contains some open subset of $\text{Spec}(A)$ and so the result follows.

24. Let A be a Noetherian ring, $f : A \mapsto B$ be such that B is a finitely-generated A-algebra, and $f^* : \text{Spec}(B) \mapsto \text{Spec}(A)$ be the induced map on spectra. We've shown previously that if f is injective f^* open implies that f has the going-down property (Note, I'm assuming that f is injective because I'm not convinced that the problem is correct without this assumption. See counterexamples to the results of chapter 5 problem 10 from above). Conversely, assume that $f : A \mapsto B$ has the going-down property and $U \subseteq \text{Spec}(B)$ is open. Since $\text{Spec}(B)$ is Noetherian, U is quasi-compact and so a finite union of sets of the form Y_g (where $Y = \text{Spec}(B)$. Using that functions preserve unions, we need show only that $f^*(Y_g) = f^*(\text{Spec}(B_g))$ is open in $\text{Spec}(A)$. That is, we need only show the map $f : A \mapsto B_g$ is such that the image of the induced map on spectra is open (this still gives B_g as a finitely-generated A-algebra and still has the going-up property from the prime ideal correspondence with localizations). From this, we may assume $E = Y = \text{Spec}(B)$. By the usual reduction, since the map $f : A \mapsto f(A)$ is surjective From problem 22, it suffices now to show that for any irreducible subset $X_0 = V(\mathfrak{p})$ of $\text{Spec}(A)$, if $f^*(\text{Spec}(B)) \cap X_0 \neq \emptyset$, then it contains a non-empty open subset of X_0.

    The going-down property as given (in chapter 5, problem 10) is not sufficient. Using the modified going-down property (see chapter 5, problem 10) is enough. Indeed, if $f : A \mapsto B$ has the going-down property, then for every prime ideal $\mathfrak{p}' \in f^*(\text{Spec}(B))$, if $\mathfrak{p} \subseteq \mathfrak{p}'$, then $\mathfrak{p} \in f^*(\text{Spec}(B))$. That is, if $X_0 = V(\mathfrak{p})$ is an irreducible subset of $\text{Spec}(A)$ such that $f^*(\text{Spec}(B)) \cap X_0 \neq \emptyset$, then there exists $\mathfrak{p}' \in f^*(\text{Spec}(B)) \cap X_0$ (so $\mathfrak{p} \subseteq \mathfrak{p}'$) implies $\mathfrak{p} \in f^*(\text{Spec}(B))$ and $\mathfrak{p} \in V(\mathfrak{p}) = X_0$. Therefore, $\mathfrak{p} \in f^*(\text{Spec}(B)) \cap X_0$ and so

    $$X_0 = V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}} \subseteq \overline{f^*(\text{Spec}(B)) \cap X_0}.$$

    That is, $f^*(\text{Spec}(B)) \cap X_0$ is dense in X_0. Since $f^*(\text{Spec}(B))$ is constructible, it is dense in an irreducible subspace Z if and only if $f^*(\text{Spec}(B)) \cap Z$ contains a nonempty open subset of Z. That is, $f^*(\text{Spec}(B)) \cap X_0$ necessarily contains a nonempty open subset of X_0. From problem 22, this implies that $f^*(\text{Spec}(B))$ is open in $\text{Spec}(A)$. The result follows.

25. Let A be a Noetherian ring and $f : A \mapsto B$ be a finitely-generated A-algebra. From chapter 5, problem 10, f has the going-down property and from the previous problem, this implies $f^* : \text{Spec}(B) \mapsto \text{Spec}(A)$ is an open map.

26a. Let A be a Noetherian ring and $F(A)$ be the set of isomorphism classes of finitely-generated A-modules and C be the free abelian group generated by $F(A)$. Let D be the subgroup of C generated by elements of the form $(M') - (M) + (M'')$ where there is some short exact sequence

    $$0 \to M' \to M \to M'' \to 0.$$

    The let $K(A) = C/D$ be the Grothendieck group of A. For a finitely-generated A-module M, let $\gamma(M) = \gamma_A(M)$ be the image of (M) under the projection $C \mapsto C/D = K(A)$.

    Let λ be an additive function on the class of finitely-generated A-modules with values in some abelian group G. It is clear that λ takes the same values on isomorphic A-modules by writing an exact sequence and so defines a function $\lambda : F(A) \mapsto G$. Then λ can be extended linearly to a function $\lambda : C \mapsto G$. Since λ is additive, it is clear that $D \subseteq \ker \lambda$ (D is defined above). Therefore, there is an induced map $\lambda_0 : K(A) = C/D \mapsto G$. Clearly, this map satisfies $\lambda(M) = \lambda_0(\gamma(M))$ for all finitely-generated A-modules M. If $\theta : K(A) \mapsto G$ is any other map such that $\lambda(M) = \theta(\gamma(M))$, it is easy to see that λ_0 and θ agree on individual elements of $K(A)$ and so to all of $K(A)$ since they are group homomorphisms.

26b. From problem 18, for any A-module M, there exists a chain of submodules

    $$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_{n-1} \subseteq M_n = M$$

    where each quotient $M_i/M_{i-1} \simeq A/\mathfrak{p}_i$ for some prime $\mathfrak{p}$. Then for $1 \leq i \leq n$, we have exact sequences

    $$0 \to M_{i-1} \to M_i \to A/\mathfrak{p}_i \to 0,$$

    so that

    $$\gamma(M_i) = \gamma(M_{i-1}) + \gamma(A/\mathfrak{p}_i).$$

It is clear from this that

$$\gamma(M) = \gamma(M_n) = \gamma(M_0) + \sum_{i=1}^{n} \gamma(A/\mathfrak{p}_i) = \sum_{i=1}^{n} \gamma(A/\mathfrak{p}_i).$$

Therefore, $K(A)$ is generated by the set of elements of the form $\gamma(A/\mathfrak{p})$ where $\mathfrak{p} \in \operatorname{Spec}(A)$.

26c. If A is a PID, then for any nonzero prime ideal $\mathfrak{p} = (p)$, there is an exact sequence

$$0 \to A \to A \to A/\mathfrak{p} \to 0$$

where the first map is multiplication by p (which is injective since A is an integral domain). Therefore, $\gamma(A/\mathfrak{p}) = 0$. Since $K(A)$ is generated by sets of this form, $K(A)$ is then necessarily generated solely by $\gamma(A/(0)) = \gamma(A)$. That is, there is a surjective group homomorphism $\mathbb{Z} \mapsto K(A)$ defined by $1 \mapsto \gamma(A)$.

A finitely-generated module over a PID has a well-defined rank which is an isomorphism invariant. This rank is an additive function with values in $\mathbb{Z}$ and so there is a group homomorphism $K(A) \mapsto \mathbb{Z}$ such that $\operatorname{rank}(\gamma(A)) = 1$. From this, it is clear that the maps described are isomorphisms and so $K(A) \simeq \mathbb{Z}$.

26d. Let $f : A \mapsto B$ be a finite ring homomorphism (B is a finitely-generated A-module). For a finitely-generated B-module M, Let M_A denote the A-module given by restriction of scalars to M (which is finitely-generated since B is a finitely-generated A-module). It is simple to check that for B-modules $M \simeq N$, then $M_A \simeq N_A$ so that restriction of scalars defines a map $f_! : F(B) \mapsto F(A)$. Extending linearly, we get a group homomorphism $f_! : C(B) \mapsto C(A)$. Composing with the projection, we get $f_! : C(B) \mapsto C(A)/D(A) = K(A)$. Finally, it is clear that $D(B) \subseteq \ker f_!$ since short exact sequences of B-modules are short exact sequences of A-modules after restrictions of scalars. Therefore, there is an induced map $f_! : K(B) = C(B)/D(B) \mapsto C(A)/D(A) = K(A)$. By definition of $f_!$, we have $f_!(\gamma_B(N)) = \gamma_A(N_A)$ for any finitely-generated B-module N.

If $g : B \mapsto C$ is another finite ring homomorphism, then for all finitely-generated C-modules O, we have

$$(f_! \circ g_!)(\gamma_C(O)) = f_!(\gamma_B(O_B)) = \gamma_A(O_A) = (g \circ f)_!(\gamma_C(O)).$$

Since sets of these form span $K(C)$ by linearity, we have that $(g \circ f)_! = f_! \circ g_!$.

27a. Let A be a Noetherian ring, $F_1(A)$ be the set of isomorphism classes of finitely-generated flat A-modules and repeat the construction from above to get a group $K_1(A) = C_1(A)/D_1(A)$ where $\gamma_1(M)$ is the image of the finitely-generated flat A-module M in $K_1(A)$.

Clearly, the tensor product of two finitely-generated flat A-modules M, N is finitely-generated and flat as well. Define

$$\gamma_1(M) \cdot \gamma_1(N) = \gamma_1(M \otimes_A N),$$

Clearly, this product is well-defined since if $M \simeq M'$ and $N \simeq N'$ then $M \otimes_A N \simeq M' \otimes_A N'$. Since the modules are flat, they preserve exact sequences and so products with zero are zero (that is, it remains well-defined if we extend using the distributive law). This product is clearly commutative and $\gamma_1(A)$ is an identity element since $M \otimes_A A \simeq M$ for all A-modules M. Therefore, $K_1(A)$ has a ring structure.

27b. Let $\gamma_1(M) \in K_1(A)$ and $\gamma(N) \in K(A)$. Define

$$\gamma_1(M) \cdot \gamma(N) = \gamma(M \otimes_A N).$$

As above, this is well-defined. Since M is flat, this takes zero sums to zero sums and so we may extend via the distributive law. It is then clear that $\gamma_1(A)$ acts as the identity map. Finally, products in $K_1(A)$ are preserved by definition. Therefore, this defines a $K_1(A)$-module structure on $K(A)$.

27c. If $(A, \mathfrak{m})$ is a Noetherian local ring, then from problem 15, every flat A-module is free. It is clear that the same universal property holds so that using the additive rank function, there is a map $\mu : K_1(A) \mapsto \mathbb{Z}$ defined exactly as the rank function extended linearly. This map is clearly surjective since $\mu(A) = 1$ and injective because any two free modules of the same rank are isomorphic. Therefore, μ is an isomorphism.

27d. Let $f : A \mapsto B$ be a ring homomorphism with B Noetherian. For an A-module M, let $M_B = B \otimes_A M$ denote the extension of scalars of M to B. This is clearly defined on isomorphism classes and so defines a map $f^! : F_1(A) \mapsto F_1(B)$. Extending linearly gives $f^! : C_1(A) \mapsto C_1(B)$. Projective gives $f^! : C_1(A) \mapsto C_1(B)/D_1(B) = K_1(B)$. Finally, $D_1(A) \subseteq \ker f^!$ so there is a group homomorphism $f^! : K_1(A) = C_1(A)/D_1(A) \mapsto C_1(B)/D_1(B) = K_1(B)$. It is easy to see that this map satisfies $f^!(\gamma_1(M)) = \gamma_1(M_B)$ and so is also a ring homomorphism (just write it out). Again from the equation $f^!(\gamma_1(M)) = \gamma_1(M_B)$, it is easy to see that $(f \circ g)^! = f^! \circ g^!$ (similar to the restriction of scalars case).

27e. Let $f : A \mapsto B$ be a finite ring homomorphism. To show that $f_!(f^!(x)y) = x f_!(y)$, by linearity and distribution, it suffices to consider the case that $x = \gamma_1(M) \in K_1(A)$ and $y = \gamma_B(N) \in K(B)$. We see

$$f_!(f^!(\gamma_1(M))\gamma_B(N)) = f_!(\gamma_1(M_B)\gamma_B(N)) = f_!(\gamma_B((B \otimes_A M) \otimes_B N)) = f_!(\gamma_B(M \otimes_A N)) = \gamma_A(M \otimes_A N).$$

Similarly, we have

$$\gamma_1(M) f_!(\gamma_B(N)) = \gamma_1(M)\gamma_A(N) = \gamma_A(M \otimes_A N).$$

Therefore, the two are equal on elements of this form and so are equal on all elements of $K_1(A)$ and $K(B)$. That is, considering $K(B)$ as a $K_1(A)$-module via restriction of scalars (from the map $f^! : K_1(A) \mapsto K_1(B)$), the map $f_!$ is a $K_1(A)$-module homomorphism.

Chapter 8

1. Let A be a Noetherian ring and $0 = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ with $\mathfrak{p}_i = r(\mathfrak{q}_i)$ be a minimal primary decomposition of the zero ideal. Since A is Noetherian, for each i there exists r_i such that $\mathfrak{p}_i^{r_i} \subseteq \mathfrak{q}_i$. Then for all $s \in A \setminus \mathfrak{p}_i$, if $sx \in \mathfrak{p}_i^{r_i} \subseteq \mathfrak{q}_i$, then $s \notin \mathfrak{p}_i$ implies $x \in \mathfrak{q}_i$. That is,

$$\mathfrak{p}_i^{(r_i)} = S_{\mathfrak{p}_i}(\mathfrak{p}_i^{r_i}) = \bigcup_{s \in A \setminus \mathfrak{p}_i} (\mathfrak{p}_i^{r_i} : s) \subseteq \mathfrak{q}_i.$$

From this,

$$0 = 0 \cap \mathfrak{p}_i^{(r_i)} = (\mathfrak{q}_i \cap \mathfrak{p}_i^{(r_i)}) \cap \left(\bigcap_{j \neq i} \mathfrak{q}_j \right) = \mathfrak{p}_i^{(r_i)} \cap \left(\bigcap_{j \neq i} \mathfrak{q}_j \right)$$

is another (minimal) primary decomposition. By invariance of the isolated components, we have that $\mathfrak{q}_i = \mathfrak{p}_i^{(r_i)}$. If $\mathfrak{q}_i$ is an embedded component, then $A_{\mathfrak{p}_i}$ is not Artinian (since the dimension is greater than or equal to 1). Therefore, the powers of the maximal ideal $\mathfrak{m}_i = \mathfrak{p}_i^e$ are all distinct and so are the $\mathfrak{p}_i^{(r)}$ (there is a correspondence of ideals of a localization with the contracted ideals of the original ring). Intersecting the primary decomposition as above, we get infinitely-many distinct primary decompositions of 0 that differ in the $\mathfrak{p}_i$-th component.

2. $(i) \implies (ii) \implies (iii)$ This is fairly obvious. The first implication takes a second using that $\mathrm{Spec}(A)$ is finite and that every singleton is closed (then take unions to get singletons are open).

   $(iii) \implies (i)$ If $\mathrm{Spec}(A)$ is discrete, it is Hausdorff in particular so that every prime ideal is maximal (chapter 3, problem 11). Therefore, A is Noetherian and has dimension 0. That is, A is Artinian.

3. $(i) \implies (ii)$ Writing $A = \prod_{i=1}^n A_i$, where the A_i are local Artinian rings and finitely-generated over k, it suffices to show that each A_i is finite as a k-module. That is, we may reduce to the case that $(A, \mathfrak{m})$ is a local Artinian ring. By Zariski's lemma, since A is a finitely-generated k-algebra, $A/\mathfrak{m}$ is a finitely-generated k-algebra that is also a field. Therefore, $A/\mathfrak{m}$ is a finite algebraic extension of k. That is, $A/\mathfrak{m}$ is finite as a k-module. Since A is Artinian, it is also Noetherian and has a finite composition series

$$0 = M_0 \subset \ldots \subset M_n = A,$$

   where we may choose each quotient to be isomorphic to $A/\mathfrak{m}$ (since this is the only prime ideal of A. Then we have $\dim M_i = \dim M_{i-1} + \dim(A/\mathfrak{m})$, which implies $\dim A = n \dim(A/\mathfrak{m}) < \infty$. That is, A is finite as a k-module.

   $(ii) \implies (i)$ If A is a finite k-algebra, then from chapter 6, we know that A satisfies the descending chain condition and so is an Artinian k-module. Notice that the ideals of A are k-submodules and so satisfy the descending chain property as well. That is, A is Artinian.

4. $(i) \implies (iii)$ Let $f : A \mapsto B$ be a finitely-generated A-algebra. If f is finite (that is, B is a finite A-module) then for every prime $\mathfrak{p}$ of A, $k(\mathfrak{p}) \otimes_A B$ is a finite $k(\mathfrak{p})$-algebra generated by the generators of B tensored with 1.

   $(iii) \implies (ii)$ From the previous problem, if $k(\mathfrak{p}) \otimes_A B$ is a finite $k(\mathfrak{p})$-algebra, then $k(\mathfrak{p}) \otimes_A B$ is Artinian. From the problem before that, this implies that

   $$(f^*)^{-1}(\mathfrak{p}) = \mathrm{Spec}(k(\mathfrak{p}) \otimes_A B)$$

   is discrete.

   $(ii) \implies (iii)$ If the fibers are discrete, since $k(\mathfrak{p}) \otimes_A B$ is Noetherian (finite dimensional since $f : A \mapsto B$ is finite), this implies $k(\mathfrak{p}) \otimes_A B$ is Artinian for every prime ideal $\mathfrak{p}$ of A. Since $k(\mathfrak{p}) \otimes_A B$ is clearly a finitely-generated $k(\mathfrak{p})$-algebra, the previous problem implies that $k(\mathfrak{p}) \otimes_A B$ is a finite $k(\mathfrak{p})$-algebra.

   $(iii) \implies (iv)$ Notice simply that for any $\mathfrak{p} \in \mathrm{Spec}(A)$,

   $$(f^*)^{-1}(\mathfrak{p}) = \mathrm{Spec}(k(\mathfrak{p}) \otimes_A B)$$

   where it is assumed that $k(\mathfrak{p}) \otimes_A B$ is a finite $k(\mathfrak{p})$-algebra. From the previous problem, it is Artinian and so from the problem before that, it is discrete and finite.

5. Let k be an algebraically closed field, X an affine variety with coordinate ring $A \neq 0$, and $\phi : X \mapsto L$ be the surjective linear map onto some subspace of k^n. Moving to coordinate rings, the induced map $\varphi : k[y_1, \ldots, y_m] \mapsto A$ is finite by construction (The construction of this map was the Noether normalization lemma). Since A is a finitely-generated k-algebra by the Hilbert basis theorem, the previous theorem applies and we get a map $\varphi : \mathrm{Spec}(A) \mapsto \mathrm{Spec}(k[y_1, \ldots, y_m])$. Since k is algebraically closed, we may identify $L \subseteq \mathrm{Spec}(k[y_1, \ldots, y_k])$ by the Nullstellensatz (L corresponds to the maximal ideals of $\mathrm{Spec}(k[y_1, \ldots, y_n])$) and similarly we may identify $X \subseteq \mathrm{Spec}(A)$. Then the map $\varphi^*|_A$ is exactly the map ϕ (plug everything in to see. This is essentially how ϕ was defined), but the map $\varphi^*|_A$ has finite fibers (from the previous problem) so that the map ϕ has finite fibers.

   From one of the previous problems, we have that the number of fibers is bounded by the cardinality of $\mathrm{Spec}(k[y_1, \ldots, y_k](\mathfrak{p}) \otimes_{k[y_1,\ldots,y_k]} A)$, but the cardinality of this set is bounded by the dimension of this vector space (the ring itself), which is less than or equal to the dimension of A as a $k[y_1, \ldots, y_n]$-vector space.

6. Let A be a Noetherian ring and $\mathfrak{q}$ a $\mathfrak{p}$-primary ideal in A. Clearly, since A is Noetherian, all chains of primary ideals from $\mathfrak{q}$ to $\mathfrak{p}$ are of finite length. By Zorn's lemma, maximal chains exist (an increasing chain of such chains corresponds to an increasing chain of ideals which is stationary since A is Noetherian. Hence, the sequence of chains is stationary and has an upper bound). Assume

   $$0 = \mathfrak{q}_0 \subset \ldots \subset \mathfrak{q}_n = \mathfrak{p},\ 0 = \mathfrak{r}_0 \subset \ldots \subset \mathfrak{r}_m = \mathfrak{p}$$

   are two maximal chains. Clearly, $\mathfrak{q}_0 \cap \mathfrak{r}_0$ is a $\mathfrak{p}$-primary ideal (and so is nonzero). By maximality of these chains, we then necessarily have that $\mathfrak{q}_0 = \mathfrak{q}_0 \cap \mathfrak{r}_0 = \mathfrak{r}_0$. From here, it is easy to see that because of the containment, if we quotient by $\mathfrak{q}_0 = \mathfrak{r}_0$, we get chains of $\mathfrak{p}$-primary ideals in the quotient of length one less. Continuing this process, we get that the lengths of the chains are the same. Therefore, all chains are bounded by the length of any maximal chain.

Chapter 9

1. Let A be a Dedekind domain and S a multiplicative subset. Since localizations preserve products and there is a correspondence between the primary ideals of $S^{-1}A$ with the contracted ideals in A and similarly with prime ideals, all primary ideals in $S^{-1}A$ can be written as a power of a prime ideal. If $S^{-1}A$ has any nonzero prime ideals, they are maximal by the correspondence and so $S^{-1}A$ has dimension one and is Noetherian since A is Noetherian. That is, $S^{-1}A$ is a Dedekind domain. If $S^{-1}A$ has no nonzero prime ideals, then it is a field, from which it is clear that $S^{-1}A = \mathrm{frac}(A)$.

   Let $S \neq A \setminus \{0\}$ be a multiplicative subset so that $S^{-1}A \neq \mathrm{frac}(A)$ and $S^{-1}A$ is a Dedekind domain from above. Then the groups of ideals I_A and $I_{S^{-1}A}$ are both defined. Since A is Noetherian, every fractional ideal in I_A has the form $x^{-1}\mathfrak{a}$ for an integral ideal of A. We may then define a map $\phi : I_A \mapsto I_{S^{-1}A}$ by

   $$\phi(x^{-1}\mathfrak{a}) = x^{-1}(S^{-1}\mathfrak{a}).$$

It is easy to check that this map is well-defined. Clearly, $(x^{-1}\mathfrak{a})(y^{-1}\mathfrak{b}) = (xy)^{-1}\mathfrak{a}\mathfrak{b}$ so that

$$\phi((x^{-1}\mathfrak{a})(y^{-1}\mathfrak{b})) = (xy)^{-1}(S^{-1}(\mathfrak{a}\mathfrak{b})) = (xy)^{-1}(S^{-1}\mathfrak{a}S^{-1}\mathfrak{b}) = (x^{-1}S^{-1}\mathfrak{a})(y^{-1}S^{-1}\mathfrak{b}) = \phi(x^{-1}S^{-1}\mathfrak{a})\phi(y^{-1}S^{-1}\mathfrak{b}).$$

That is, ϕ is a group homomorphism. Projecting with the composition $I_{S^{-1}A} \mapsto I_{S^{-1}A}/P_{S^{-1}A} = H_{S^{-1}A}$, we see that $S^{-1}(u) = (u/1)$ so that $P_A \subseteq \ker\phi$ and so there is an induced map

$$\phi : H_A = I_A/P_A \mapsto I_{S^{-1}A}/P_{S^{-1}A} = H_{S^{-1}A}.$$

This map has image equal to the image of the composition $I_A \mapsto I_{S^{-1}A} \mapsto H_{S^{-1}A}$. Since the latter map is surjective, it suffices to show the first map is surjective. However, this is clear since $S^{-1}A$ is Noetherian and so every element of $I_{S^{-1}A}$ has the form $(a/1)^{-1}(S^{-1}\mathfrak{a})$ for some $a \in A$ and $\mathfrak{a}$ an ideal of A (really, of the form $(a/s)^{-1}\mathfrak{b}$ where $a/s \in S^{-1}A$ and $\mathfrak{b}$ an ideal of $S^{-1}A$, but every ideal of $S^{-1}A$ is an extended ideal of A and it is clear that we can remove the s from the denominator since $s/1 \in S^{-1}A$ is a unit). Therefore, the induced map $\phi : H_A \mapsto H_{S^{-1}A}$ is surjective.

2. Let A be a Dedekind domain. For $f, g \in A[x]$, Let $c(f), c(g), c(fg)$ be the content of f, g, and fg respectively. Clearly, $c(fg) \subseteq c(f)c(g)$. To show that we have equality, consider the inclusion map $\phi : c(fg) \hookrightarrow c(f)c(g)$ as an A-module homomorphism. We wish to show that this map is surjective. To do so, it suffices to show that $\phi_\mathfrak{p} : c(fg)_\mathfrak{p} \hookrightarrow c(f)_\mathfrak{p}c(g)_\mathfrak{p}$ is surjective for each maximal ideal $\mathfrak{p}$, where $A_\mathfrak{p}$ is then a discrete valuation ring (that is, we just need $c(fg)_\mathfrak{p} = c(f)_\mathfrak{p}c(g)_\mathfrak{p}$ for each prime ideal $\mathfrak{p}$ of A). Note that $c(f)_\mathfrak{p} = c(\overline{f})$ where $\overline{f}$ is the image of f in $A_\mathfrak{p}$. From this, it suffices to prove the result in full generality in the case that A is a discrete valuation ring.

   Assume A is a discrete valuation ring, $f, g \in A[x]$, and $c(f), c(g), c(fg)$ are the content of f, g and fg respectively. Then there exists $a \in A$ such that we have $c(f) = (a^r)$ and $c(g) = (a^s)$ for some $r, s \geq 0$. From here, we may write

$$(f(x)/a^r)(g(x)/a^s) = (fg)(x)/a^{r+s}.$$

   We clearly have $c(f(x)/a^r) = c(g(x)/a^s) = (1)$ (since otherwise, $c(f) \neq (a^r)$ and $c(g) \neq (a^s)$). Therefore, from chapter 1, we have that $c((fg)(x)/a^{r+s}) = (1)$. That is,

$$c(fg) = (a^{r+s}) = (a^r)(a^s) = c(f)c(g).$$

   The result then follows from the reduction mentioned above.

3. Let A be a valuation ring that is not a field (that is, it has some nonzero prime ideal). Clearly, if A is a discrete valuation ring, then A is Noetherian.

   If A is a Noetherian valuation ring, then for any ideal $\mathfrak{a}$, we may write $\mathfrak{a} = (x_1, \ldots, x_n)$. From chapter 5, problem 28, we have that the (x_i) are linearly ordered and so there is a minimal element which implies $\mathfrak{a} = (a)$ where $a = a_i$ for some i. That is, A is a PID. Since a PID is a UFD and we are assuming A is not a field, there is some nonzero maximal ideal $\mathfrak{m} = (p)$, where p is then necessarily prime and irreducible. As in the case of $\mathbb{Z}$ and $k[x]$ from the text, for any element $a \in K = \mathrm{frac}(A)$, we may write uniquely write $a = p^{v(a)}y$ where the numerator and denominator of y are prime to p. Then the assignment $a \mapsto v(a)$ is a discrete valuation. It is clear that the ring of integers in this case is the localization of A at the prime ideal $\mathfrak{m}$. However, since A is a local ring (it is a valuation ring), we have $A = A_\mathfrak{m}$ so that A is the ring of integers of v. That is, A is a discrete valuation ring.

4. Let A be a local domain which is not a field and whose maximal ideal $\mathfrak{m} = (m)$ is principal and $\bigcap_{i=1}^{\infty} \mathfrak{m}^i = 0$. Similar to above, with $K = \mathrm{frac}(A)$, we may define $v : K \mapsto \mathbb{Z}$ as follows. For every $a \in A$, $a \neq 0$, there is a maximal $k > 0$ such that $a \in \mathfrak{m}^k$ (since if it is infinitely many, this implies that $a = 0$). Then let $v : A \mapsto \mathbb{Z}$ be the assignment of a to this maximal $k > 0$. For $a, b \in A$, it follows from the fact that A is an integral domain and that $\mathfrak{m} = (m)$ is principal that $v(ab) = v(a) + v(b)$ ($ab \in \mathfrak{m}^{v(a)+v(b)}$, if $ab = cm^{v(a)+v(b)+1}$, cancel powers of m to get a contradiction). Clearly, we have the inequality

$$v(a+b) \geq \min\{v(a), v(b)\}$$

   from simple containment of powers of $\mathfrak{m}$. Similar to a previous problem on valuation rings, this implies that there is an extension to a valuation $v : K^* \mapsto \mathbb{Z}$ by $v(a/b) = v(a) - v(b)$. It is apparent from the definition and cancellation in K^* that the ring of integers of v is exactly $A_\mathfrak{m} = A$. Therefore, A is a discrete valuation ring.

5. Let M be a finitely-generated A-module where A is a Dedekind domain. If M is flat, then $M_\mathfrak{m}$ is flat for every maximal ideal $\mathfrak{m}$. If $A_\mathfrak{m} \hookrightarrow K$ is the injective map of $A_\mathfrak{m}$ into $K = \text{frac}(A_\mathfrak{m})$, then $T(M_\mathfrak{m}) = \ker\phi$ where $\phi : M_\mathfrak{m} = A_\mathfrak{m} \otimes_{A_\mathfrak{m}} M_\mathfrak{m} \mapsto K \otimes_{A_\mathfrak{m}} M_\mathfrak{m}$. However, since $M_\mathfrak{m}$ is flat, this map is injective and $T(M_\mathfrak{m}) = 0$. Since torsion-free is a local property, this implies that M is torsion-free.

   Conversely, assume that M is torsion-free. Then $T(M_\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m}$ of A. Since A is a Dedekind domain, $A_\mathfrak{m}$ is a discrete valuation ring and so is a PID. Therefore, $M_\mathfrak{m}$ can be decomposed into a direct sum of a free $A_\mathfrak{m}$-module and its torsion module, which is zero. Therefore, $M_\mathfrak{m}$ is free for every $\mathfrak{m}$ which from chapter 5, problem 16 implies that M is flat.

6. Let M be a finitely-generated torsion A-module (that is, $T(M) = M$) where A is a Dedekind domain. Then for every prime ideal $\mathfrak{p}$, $M_\mathfrak{p}$ is a torsion $A_\mathfrak{p}$-module (easy to check) and $A_\mathfrak{p}$ is a discrete valuation ring and so is a PID. Since $M_\mathfrak{p}$ is a finitely-generated $A_\mathfrak{p}$-module, $M_\mathfrak{p}$ can be decomposed into a free $A_\mathfrak{p}$-module of finite rank and a finite number of cyclic torsion modules of the form $A_\mathfrak{p}/\mathfrak{p}_i^{n_i}$ for some prime ideal $\mathfrak{p}_i$ of $A_\mathfrak{p}$. Since $A_\mathfrak{p}$ is a local ring of dimension 1 (it is a discrete valuation ring), there is only one prime ideal and so $M_\mathfrak{p} \simeq A_\mathfrak{p}/\mathfrak{p}^n \simeq (A/\mathfrak{p}^n)_\mathfrak{p} = A/\mathfrak{p}^n$ (where the last equality is because $\mathfrak{p}$ is the unique maximal ideal so everything outside of $\mathfrak{p}$ is already a unit).

   For nonzero prime ideal $\mathfrak{p}_i$ of A, let $M_{\mathfrak{p}_i} = A/\mathfrak{p}_i^{n_i}$ and define $\phi : M \mapsto \oplus_i M_{\mathfrak{p}_i}$. To see that this map is bijective, it suffices to show each localization is bijective. However, since each localization is the identity map (this takes a moment to verify) it follows that ϕ is bijective.

*7.

8. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be ideals of a Dedekind ring A. We clearly see $(\mathfrak{a}\cap\mathfrak{b})+(\mathfrak{a}\cap\mathfrak{c}) \subseteq \mathfrak{a}\cap(\mathfrak{b}+\mathfrak{c})$ and $\mathfrak{a}+(\mathfrak{b}\cap\mathfrak{c}) \subseteq (\mathfrak{a}+\mathfrak{b})\cap(\mathfrak{a}+\mathfrak{c})$. From here, we may consider the inclusion maps between these ideals. To show that these inclusions are equalities, we may localize the inclusion maps and show that they are always surjective. That is, is now suffices to show the other inclusion holds for every localization of A (since localizations commute with finite intersections and sums). Since all localizations of A are discrete valuation rings, it suffices to show the opposite inclusions in this case alone.

   Let A be a discrete valuation ring and $x \in A$ such that $\mathfrak{a} = (x^r)$, $\mathfrak{b} = (x^s)$, and $\mathfrak{c} = (x^t)$. It is clear that $(x^n) \cap (x^m) = (x^{\max\{n,m\}})$ and $(x^n) + (x^m) = (x^{\min\{n,m\}})$. The result then follows from these relations and the equalities

   $$\max\{r, \min\{s,t\}\} = \min\{\max\{r,s\}, \max\{r,t\}\}, \;\; \min\{r, \max\{s,t\}\} = \max\{\min\{r,s\}, \min\{r,t\}\}$$

   by simply plugging everything into the above equations (the inequalities are easily proved case-wise).

9. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be ideals and $x_0, \dots, x_n$ be elements in a Dedekind domain A. The reduction from the text is obvious with a minute of thought. That is, the statment, "there exists $x \in A$ such that $x \equiv x_i \bmod \mathfrak{a}_i$ for all i if and only if $x_i \equiv x_j \bmod (\mathfrak{a}_i + \mathfrak{a}_j)$" is equivalent to the exactness of the sequence

   $$A \xrightarrow{\phi} \bigoplus_{i=1}^{n} A/\mathfrak{a}_i \xrightarrow{\psi} \bigoplus_{i<j} A/(\mathfrak{a}_i + \mathfrak{a}_j),$$

   where

   $$\phi(x) = (x + \mathfrak{a}_1, \dots, x + \mathfrak{a}_n), \;\; \psi(x_1 + \mathfrak{a}_1, \dots, x_n + \mathfrak{a}_n) = (x_i - x_j + \mathfrak{a}_i + \mathfrak{a}_j)_{i<j}.$$

   Since it is clear that $\operatorname{Im}\phi \subseteq \ker\psi$, it suffices as usual to show they are equal in all localizations (that is, show the localizations of the inclusion map are bijective). All localizations of A are discrete valuation rings and the localization of the direct sums are the direct sums of the localizations. Therefore, it suffices to prove the result in the case that A is a discrete valuation ring.

   Assume A is a discrete valuation ring, $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ are ideals, $x_1, \dots, x_n \in A$, and ϕ, ψ be as above. Since A is a valuation ring, the set of ideals of A is linearly ordered and so we may assume that $\mathfrak{a}_1 \subseteq \dots \subseteq \mathfrak{a}_n$. The map $\psi : \oplus_{i=1}^n A/\mathfrak{a}_i \mapsto \oplus_{i<j} A/\mathfrak{a}_i$ (since $\mathfrak{a}_i + \mathfrak{a}_j = \mathfrak{a}_i$ for $i < j$ now) is then given by

   $$\psi(x_1 + \mathfrak{a}_1, \dots, x_n + \mathfrak{a}_n) = (x_i - x_j + \mathfrak{a}_i)_{i<j}.$$

   If $(x_1 + \mathfrak{a}_1, \dots, x_n + \mathfrak{a}_n) \in \ker\psi$, then for all i, $x_i - x_n \in \mathfrak{a}_i$. That is, $x_i + \mathfrak{a}_i = x_n + \mathfrak{a}_i$. From this, we have

   $$\phi(x_n) = (x_n + \mathfrak{a}_1, \dots, x_n + \mathfrak{a}_n) = (x_1 + \mathfrak{a}_1, \dots, x_n + \mathfrak{a}_n).$$

That is, $\ker \psi \subseteq \operatorname{Im} \phi$ as desired. From this, the localizations of the inclusion map $i : \operatorname{Im} \phi \mapsto \ker \psi$ is surjective for all prime ideals of A and so bijective (since it is already injective). Therefore, the original sequence is exact, which is equivalent to the result.

Chapter 10

1. For $n \geq 1$ and a prime number p, define the group homomorphism $\alpha_n : \mathbb{Z}/p\mathbb{Z} \mapsto \mathbb{Z}/p^n\mathbb{Z}$ by $\alpha_n(1) = p^{n-1}$. Then let A be the direct sum of countable copies of $\mathbb{Z}/p\mathbb{Z}$ and $B = \oplus \mathbb{Z}/p^n\mathbb{Z}$ so that the maps α_n determine a map $\alpha : A \mapsto B$.

   It is clear that the filtration of subgroups corresponding to (p),

   $$A \supset (p)A \supset (p)^2 A \supset \ldots$$

   is exactly zero after the first group. That is, the p-adic completion of A is the inverse limit of the sequence

   $$A \leftarrow A \leftarrow A \leftarrow \ldots.$$

   Since these maps are all the identity, it follows that the p-adic completion of A is exactly A.

   From the map $\alpha : A \mapsto B$, the p-adic topology on B induces a pullback topology on A determined exactly by taking the preimage under α of open subsets in B. Notice that there is a neighborhood base of 0 in B given by sets of the form

   $$(p)^k B = \bigoplus_{n>k} p^k(\mathbb{Z}/p^n\mathbb{Z}).$$

   Therefore, the sets

   $$\alpha^{-1}((p)^k B) = \bigoplus_{n>k} \mathbb{Z}/p\mathbb{Z}$$

   form a neighborhood base of 0 in A in the pullback topology. This gives the filtration

   $$A = A_0 \supset \bigoplus_{n>1} \mathbb{Z}/p\mathbb{Z} = A_1 \supset \bigoplus_{n>2} \mathbb{Z}/p\mathbb{Z} = A_2 \supset \ldots,$$

   of which the completion of A in the pullback topology is given by the inverse limit of quotients A_0/A_k. It is clear that A/A_k is isomorphic to k copies of $\mathbb{Z}/p\mathbb{Z}$ and that the connecting maps $\theta_{n+1} : A/A_{n+1} \mapsto A/A_n$ are given by mapping the first n coordinates identically. From this, it is clear that the inverse limit is $\prod \mathbb{Z}/p\mathbb{Z}$ since there is an obvious isomorphism of sequences of the form

   $$(a_1, 0, \ldots), (a_1, a_2, 0, \ldots), (a_1, a_2, a_3, \ldots), \ldots$$

   with sequences $(a_1, a_2, a_3, \ldots)$. That is, there is a clear isomorphism from $\lim_{\leftarrow} A/A_k$ to $\prod \mathbb{Z}/p\mathbb{Z}$. Therefore, the completion of A in the pullback topology is exactly the countable direct product $\prod \mathbb{Z}/p\mathbb{Z}$.

   From the notation of proposition 10.2, it is clear that the p-adic functor is right exact if the map d^A is surjective for every exact sequence (where the inverse systems are determined by the p-adic topology). In our case, since the connecting maps are identity maps, d^A is the zero map and therefore is not surjective. From this, the p-adic functor is not right exact.

2. Let A and A_k be as in the previous problem. Then there is an exact sequence of inverse systems

   $$0 \to \{A_n\} \to \{A\} \to \{A/A_n\} \to 0.$$

   If takes a second of thought to realize that $\lim_{\leftarrow} A_n = 0$. Using the previous problem, the latter inverse limits are A and $\prod \mathbb{Z}/p\mathbb{Z}$ respectively. Therefore, we have an exact sequence

   $$0 \to 0 \to A \to \prod \mathbb{Z}/p\mathbb{Z}.$$

Since the latter map is obviously not surjective, we have that adjoining the zero map on the end will break exactness so the inverse limit functor is not right exact in this case.

Since the connecting maps in the middle inverse system from above are all the identity map, the map d^B (with notation from prop 10.2) is the zero map. Therefore, there is a short exact sequence

$$0 \to \bigoplus \mathbb{Z}/p\mathbb{Z} \to \prod \mathbb{Z}/p\mathbb{Z} \to \varprojlim{}^1 A_n \to 0.$$

From this, we have

$$\varprojlim{}^1 A_n = \left(\prod \mathbb{Z}/p\mathbb{Z}\right) / \left(\bigoplus \mathbb{Z}/p\mathbb{Z}\right).$$

3. Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, and M a finitely-generated A-module. From the Krull intersection theorem,

$$m \in \bigcap_{n \geq 1} \mathfrak{a}^n M \iff \exists a \in \mathfrak{a},\ (1-a)m = 0.$$

Notice that for any maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$, $1 - a \notin \mathfrak{m}$, since otherwise, $1 \in \mathfrak{m}$. Therefore, $m \in \ker(M \mapsto M_{\mathfrak{m}})$ for all maximal ideals $\mathfrak{m}$ that contain $\mathfrak{a}$ ($m/1 = 0$ since $(1-a)m = 0$ where $1 - a \in A \setminus \mathfrak{m}$). That is,

$$\bigcap_{n \geq 1} \mathfrak{a}^n M \subseteq \bigcap_{\mathfrak{m} \supseteq \mathfrak{a}} \ker(M \mapsto M_{\mathfrak{m}}).$$

Conversely, if

$$m \in \bigcap_{\mathfrak{m} \supseteq \mathfrak{a}} \ker(M \mapsto M_{\mathfrak{m}}),$$

then the submodule N generated by m is such that $N_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ that contain $\mathfrak{a}$. From this, it follows that $N = \mathfrak{a}N$ and we may write $m = am$ for some $a \in \mathfrak{a}$ so that $(1-a)m = 0$. From Krull's intersection theorem, this then implies that $m \in \cap \mathfrak{a}^n M$. Therefore, the other inclusion holds and we have

$$\bigcap_{n \geq 1} \mathfrak{a}^n M = \bigcap_{\mathfrak{m} \supseteq \mathfrak{a}} \ker(M \mapsto M_{\mathfrak{m}}).$$

Notice now that if $\ker(M \mapsto M_{\mathfrak{m}}) = M$ for some maximal ideal $\mathfrak{m}$, then $M_{\mathfrak{m}} = 0$ since we may write an arbitrary element as the product of $1/s$ for $s \in A \setminus \mathfrak{m}$ and $m/1$ in the image. Therefore, if $\ker(M \mapsto M_{\mathfrak{m}}) = M$ for every maximal ideal $\mathfrak{m}$ that contains $\mathfrak{a}$, then $M_{\mathfrak{m}} = 0$ for every maximal ideal that contains $\mathfrak{a}$. That is, $M = \mathfrak{a}M$. In this scenario, it is clear by considering the filtration $\mathfrak{a}^n M$ that $\widehat{M} = 0$. Since $\operatorname{Supp}(M) \cap V(\mathfrak{a}) = \emptyset$ implies $M_{\mathfrak{m}} = 0$ for every maximal ideal that contains $\mathfrak{a}$, we have

$$\operatorname{Supp}(M) \cap V(\mathfrak{a}) = \emptyset \implies \widehat{M} = 0.$$

Conversely, if $\widehat{M} = 0$, then from the equality

$$\ker(M \mapsto \widehat{M}) = \bigcap_{n \geq 1} \mathfrak{a}^n M = \bigcap_{\mathfrak{m} \supseteq \mathfrak{a}} \ker(M \mapsto M_{\mathfrak{m}}),$$

we have that $\ker(M \mapsto M_{\mathfrak{m}}) = M$ for all maximal ideals $\mathfrak{m}$ that contain $\mathfrak{a}$. As above, it is easy to see that this implies $M_{\mathfrak{m}} = 0$ for every maximal ideal containing $\mathfrak{a}$. For any prime ideal $\mathfrak{p}$ and maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$, we have $A \setminus \mathfrak{m} \subseteq A \setminus \mathfrak{p}$. Since $\ker(M \mapsto M_{\mathfrak{m}}) = M$, it is clear that the composition $\ker(M \mapsto M_{\mathfrak{m}} \mapsto M_{\mathfrak{p}}) = M$ so that $M_{\mathfrak{p}} = 0$. Therefore, $\operatorname{Supp}(M) \cap V(\mathfrak{a}) = \emptyset$. Therefore for a Noetherian ring A and a finitely-generated A module, we have

$$\widehat{M} = 0 \iff \operatorname{Supp}(M) \cap V(\mathfrak{a}) = \emptyset.$$

4. Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, and $x \in A$ not be a zero divisor. Then the A-module homomorphism $\phi_x : A \mapsto A$ defined by $\phi(a) = ax$ is injective. Since the $\mathfrak{a}$-adic completion functor is exact for finitely-generated modules over a Noetherian ring, it preserves injective functions so $\widehat{\phi}_x : \widehat{A} \mapsto \widehat{A}$ is injective. It is easy to check

at this point that the map $\widehat{\phi}_x$ is multiplication by $\widehat{x}$ in $\widehat{A}$. That is, $\widehat{\phi}_x = \phi_{\widehat{x}} : \widehat{A} \mapsto \widehat{A}$. From this, it is clear that $\widehat{x}$ is not a zero-divisor in $\widehat{A}$ since $\ker \phi_{\widehat{x}} = 0$.

Let $A = k[x, y]/(y^2 - x^2 - x^3)$. It takes a moment to see that the ideal $(y^2 - x^2 - x^3)$ is prime so that A is an integral domain. After this, $\sqrt{1+x} \in \widehat{A}$ by the binomial theorem so that

$$(y - x\sqrt{1+x})(y + x\sqrt{1+x}) = y^2 - x^2 - x^3 = 0,$$

so $\widehat{A}$ is not an integral domain.

5. Let A be Noetherian and $\mathfrak{a}, \mathfrak{b}$ be ideals of A. For an A-module M, let $M^{\mathfrak{a}}$ and $M^{\mathfrak{b}}$ be the $\mathfrak{a}$-adic and $\mathfrak{b}$-adic completions respectively. Let M be a finitely-generated A-module and consider the exact sequence

$$0 \to \mathfrak{b}^m M \to M \to M/\mathfrak{b}^m M \to 0.$$

Since A is Noetherian and each of these modules is finitely-generated, the $\mathfrak{a}$-adic functor is exact and we have the exact sequence

$$0 \to (\mathfrak{b}^m M)^{\mathfrak{a}} \to M^{\mathfrak{a}} \to (M/\mathfrak{b}^m M)^{\mathfrak{a}} \to 0.$$

We then see

$$(M/\mathfrak{b}^m M)^{\mathfrak{a}} = A^{\mathfrak{a}} \otimes_A M/\mathfrak{b}^m M = (A^{\mathfrak{a}} \otimes_A M)/(A^{\mathfrak{a}} \otimes_A \mathfrak{b}^m M) = M^{\mathfrak{a}}/\mathfrak{b}^m M^{\mathfrak{a}}.$$

Then

$$(M^{\mathfrak{a}})^{\mathfrak{b}} = \varprojlim_m M^{\mathfrak{a}}/\mathfrak{b}^m M^{\mathfrak{a}} = \varprojlim_m (M/\mathfrak{b}^m)^{\mathfrak{a}}.$$

Using that $\mathfrak{a}(M/N) = (\mathfrak{a}M + N)/N$, this gives

$$(M^{\mathfrak{a}})^{\mathfrak{b}} = \varprojlim_m \varprojlim_n M/(\mathfrak{a}^n M + \mathfrak{b}^m M) = \varprojlim_n M/((\mathfrak{a}^n + \mathfrak{b}^n)M) = M^{\mathfrak{a}+\mathfrak{b}}.$$

At this point, the inclusions $(\mathfrak{a}+\mathfrak{b})^{2n} \subseteq \mathfrak{a}^n + \mathfrak{b}^n \subseteq (\mathfrak{a}+\mathfrak{b})^n$ imply that the topologies on M induced by the filtrations $(\mathfrak{a}^n + \mathfrak{b}^n)M$ and $(\mathfrak{a}+\mathfrak{b})^n M$ are the same and so the completions are the same.

(The verification that the quotient from a few lines above is equal to the other quotient is simple written out. The combination of inverse limits is a simple result in noticing that all elements with lattice points in the first quadrant are determined by their values on the diagonal.)

6. Let A be a ring, $\mathfrak{a}$ be an ideal, and give A the $\mathfrak{a}$-adic topology. If $\mathfrak{a} \subseteq \mathfrak{J}$, then for every maximal ideal $\mathfrak{m}$, $\mathfrak{a} \subseteq \mathfrak{m}$. For $s \in A \setminus \mathfrak{m}$, $s + \mathfrak{a} \subseteq A \setminus \mathfrak{m}$ (where $s + \mathfrak{a}$ is the translate of $\mathfrak{a}$ and so is open). This is because if $s + a \in \mathfrak{m}$ for $a \in \mathfrak{a}$, this would imply that $s \in \mathfrak{m}$ since $\mathfrak{a} \subseteq \mathfrak{m}$. Therefore, every point of $A \setminus \mathfrak{m}$ has a neighborhood contained in $A \setminus \mathfrak{m}$ and so this set is open which implies $\mathfrak{m}$ is closed. That is, all maximal ideals are closed in the $\mathfrak{a}$-adic topology.

Conversely, if a maximal ideal $\mathfrak{m}$ is closed, then $A \setminus \mathfrak{m}$ is open and so for $s \in A \setminus \mathfrak{m}$, there exists $\mathfrak{a}^n$ such that $s + \mathfrak{a}^n \subseteq A \setminus \mathfrak{m}$. Then we may write $s + a = 1$ for some $a \in \mathfrak{a}^n$. This implies that $s \in A \setminus \mathfrak{a}$ since if $s \in \mathfrak{a}$, we should have $1 \in \mathfrak{a}$. The contrapositive of this is that $s \in \mathfrak{a}$ implies $s \in \mathfrak{m}$. Therefore, $\mathfrak{a} \subseteq \mathfrak{m}$. If every maximal ideal is closed, then $\mathfrak{a} \subseteq \mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ so that $\mathfrak{a} \subseteq \mathfrak{J}$.

7. Let A be a Noetherian ring, $\mathfrak{a}$ be an ideal, and $\widehat{A}$ be the $\mathfrak{a}$-adic completion of A. From chapter 3, since $\widehat{A}$ is flat, $\widehat{A}$ is faithfully flat if and only if for every A-module M, the map $M \mapsto \widehat{A} \otimes_A M$ is injective. It will be shown that this is equivalent to the condition that for every finitely-generated A-module M, the map $M \mapsto \widehat{A} \otimes_A M (= \widehat{M})$ is injective. The implication $\implies$ is obvious. For the other, let M be an A-module and consider $\phi : M \mapsto \widehat{A} \otimes_A M$. If $u = m \in \ker \phi$, let N be a finitely-generated submodule of M containing m such that $\widehat{1} \otimes m = 0$ in $\widehat{A} \otimes N$. Then ϕ restricts to a map $N \mapsto \widehat{A} \otimes N$ where N is finitely-generated. Therefore, this restriction is injective which implies that $m = 0$ so that ϕ is injective. Therefore, to show that $\widehat{A}$ is faithfully flat, it suffices to show that for every finitely-generated A-module M, the map $M \mapsto \widehat{M}$ is injective.

Notice that if A in the $\mathfrak{a}$-adic topology is a Zariski ring, then $\mathfrak{a} \subseteq \mathfrak{J}_A$ so that $1+\mathfrak{a}$ consists of units. By the Krull intersection theorem, this then implies that the map $M \mapsto \widehat{M}$ is injective for every finitely-generated A-module M. That is, $\widehat{A}$ is faithfully flat.

Assume $\widehat{A}$ is faithfully flat. Since $A/\mathfrak{m}$ is finitely-generated (since A is Noetherian, so is $A/\mathfrak{m}$), the map $A/\mathfrak{m} \mapsto \widehat{A/\mathfrak{m}}$ is injective. That is, no elements are annihilated by any element of $1+\mathfrak{a}$. Therefore, for any $1+a \in 1+\mathfrak{a}$,

$$(1+a)(1+\mathfrak{m}) = (1+a+\mathfrak{m})(1+\mathfrak{m})1+a+\mathfrak{m} \neq 0+\mathfrak{m}.$$

From this, it follows that $\mathfrak{a}^e \subseteq \mathfrak{J}_{A/\mathfrak{m}} = 0$ (since $A/\mathfrak{m}$ is a field). However, this implies that $\mathfrak{a} \subseteq \mathfrak{m}$. Since $\mathfrak{m}$ was arbitrary, this implies that $\mathfrak{a} \subseteq \mathfrak{J}_A$ so A is a Zariski ring.

8. Let A be the local ring of the origin in $\mathbb{C}^n$ (rational runctions defined at the origin), B the ring of power series in $z_1, \ldots, z_n$ that converge in some neighborhood of the origin, and C the ring of formal power series in $z_1, \ldots, z_n$ so that $A \subseteq B \subseteq C$. Let

$$\mathfrak{b} = (z_1, \ldots, z_n)$$

be an ideal of B (of elements whose limit function vanishes at 0). For an element of $B \setminus \mathfrak{b}$, the limit function is analytic and nonzero at the origin. Therefore, the reciprocal of the limit function is also analytic in some neighborhood of the origin and so has a convergent power series in this neighborhood. That is, any element of $B \setminus \mathfrak{b}$ is invertible, which implies that $\mathfrak{b}$ is the unique maximal ideal of B so that B is a local ring. Considering powers of $\mathfrak{b}$ and the consecutive quotients $B/\mathfrak{b}^n$, it is clear that the completion of B is the ring of formal power series in $z_1, \ldots, z_n$ (much in the same way that the completion of the polynomial ring at the ideal (x) is the ring of formal power series). From chapter 3, problem 17, to show that B is A-flat, it suffices now to show that C is A-flat (since we are assume B is Noetherian, C is faithfully B-flat from the last problem).

Let $(z_1, \ldots, z_n)$ be a maximal ideal of $\mathbb{C}[z_1, \ldots, z_n]$. It is easy to see that $\mathbb{C}[z_1, \ldots, z_n]_{(z_1, \ldots, z_n)} = (1 + (z_1, \ldots, z_n))^{-1}\mathbb{C}[z_1, \ldots, z_n]$. Therefore, we may consider A to be the set of rational functions, f/g, where $g(0) = 1$. It will be shown that C is the completion of A with respect to the $\mathfrak{a}$-adic topology where $\mathfrak{a} = (z_1, \ldots, z_n)$ and so is a flat A-module. Consider a coherent sequence

$$\left(\frac{f_1}{g_1} + \mathfrak{a}, \frac{f_2}{g_2} + \mathfrak{a}^2, \frac{f_3}{g_3} + \mathfrak{a}^3, \ldots\right).$$

Notice that $s/t \in \mathfrak{a}^n$ if and only if $s \in \mathfrak{a}^n$ since t is a unit. Then

$$\frac{f_n}{g_n} - \frac{f_n}{1} = \frac{f_n(1-g_n)}{g_n} \in \mathfrak{a}^n.$$

Therefore, we may replace the coherent sequence above by the sequence

$$(f_1 + \mathfrak{a}, f_2 + \mathfrak{a}^2, \ldots).$$

From this point, it is clear that $\widehat{A} \subseteq C$ (since the f_i are polynomials in $z_1, \ldots, z_n$). Conversely, it is clear that every element of C can appear as such an element. Therefore, $\widehat{A} = C$ so C is a flat A-module. It then follows that B is a flat A-module.

9. Let A be a ring with prime ideal $\mathfrak{m}$ that is $\mathfrak{m}$-adic complete and let $f \in A[x]$ be a monic polynomial of degree n such that there exists coprime monic polynomials $\overline{g}, \overline{h} \in (A/\mathfrak{m})[x]$ of degrees r and $n-r$ such that $\overline{f} = \overline{g}\overline{h}$.

It will be shown that we can inductively find $g_k, h_k \in A[x]$ such that $g_k h_k - f \in \mathfrak{m}^k[x] = \mathfrak{m}^k A[x]$ and so that the sequence $g_k + \mathfrak{m}^k$ is coherent. Let $g_1, h_1 \in A[x]$ be any lifts of $\overline{g}$ and $\overline{h}$ respectively. Assuming $g_k, h_k \in A[x]$ are defined, we know that $\overline{g}_k = \overline{g}$ and $\overline{h}_k = \overline{h}$ and so they are relatively prime. From the note at then end, for $1 \leq p \leq n$, there exists $a_p, b_p \in A[x]$ of degrees $a_p \leq n-r$ and $b_p \leq r$ such that

$$\overline{a_p}(x)\overline{g_k}(x) + \overline{b_p}(x)\overline{h_k}(x) = \overline{x^p}.$$

Since all of the above terms above are of degree $\leq n$, there exists $r_p(x) \in \mathfrak{m}[x]$ of degree $\leq n$ such that

$$x^p = a_p(x)g_k(x) + b_p(x)h_k(x) + r_p(x).$$

We may write

$$f(x) - g_k(x)h_k(x) = \sum_i m_i x^i = g_k(x)\sum_i m_i a_i(x) + h_k(x)\sum_i m_i b_i(x) + \sum_i m_i r_i(x),$$

with $m_i \in \mathfrak{m}^k$. Let

$$g_{k+1}(x) = g_k(x) + \sum_i m_i b_i(x),\ \ h_{k+1}(x) = h_k(x) + \sum_i m_i a_i(x).$$

From the above, we see

$$f - g_{k+1}h_{k+1} = \sum_i m_i r_i(x) - \sum_{i,j} m_i m_j a_i(x) b_j(x) \in \mathfrak{m}^{k+1}[x],$$

since $r_i(x) \in \mathfrak{m}[x]$. It is clear that $g_{k+1} = g_k \bmod \mathfrak{m}^k$ and $h_{k+1} = h_k \bmod \mathfrak{m}^k$ and $\deg g_k = n - r$, $\deg h_k = r$ so that these polynomials satisfy the desired conditions.

The sequences (g_k) and (h_k) are coherent. Since A is $\mathfrak{m}$-adic complete, the map $A \mapsto \widehat{A}$ is surjective and these sequences are of the form $(g + \mathfrak{m}, g + \mathfrak{m}^2, \ldots)$ and $(h + \mathfrak{m}, h + \mathfrak{m}^2, \ldots)$ respectively. From this, it is clear that $g(x)h(x) = f(x)$ by our construction of these sequences and injectivity of the map $A \mapsto \widehat{A}$.

Note: The following is proof of the result used in the above. For an integral domain A and polynomials $g, h \in A[x]$ of degrees n and m respectively, by the extended Euclidean algorithm, there are polynomials $a, b \in A[x]$ such that

$$ag + bh = 1$$

and $\deg a < m = \deg h$, $\deg b < n = \deg g$. For any polynomial f of degree $\min\{m, n\} \leq \deg f \leq m + n$, we may clearly linear combinations of g and h to lower the degree of f to strictly less than $\min\{m, n\}$. Consider now the case that f of degree strictly less than $\min\{m, n\}$. If $n < m$, we may continually multiply g by x^{m-n} and subtract from h to lower the degree of h until $\deg h < n$ (note that now h has the form $h - cg$ where $\deg c = m - n$). We may repeat the same process now until $\deg f = 0$. That is, until f is constant, in which case we may subtract a multiple of $ag + bh = 1$ so that we have written f as a sum of linear combinations of g and h with the desired powers. On the other hand, if $\deg g = \deg h$, then we may subtract a constant multiple of g from h so that $\deg h < \deg g$. Then the above process gives the desired linear combination as well.

10a. Let A be a ring with prime ideal $\mathfrak{m}$ be $\mathfrak{m}$-adic complete and $f \in A[x]$ be monic such that $\overline{f} \in (A/\mathfrak{m})[x]$ has a simple root $\alpha \in A/\mathfrak{m}$. Then we may write $\overline{f} = (\overline{x} - \alpha)\overline{h}$ where these factors are relatively prime since α is a simple root. Therefore, there is a lifting of these polynomials to $g, h \in A[x]$ such that $f = gh$ and their degrees are equal to the degrees of their projections. That is, $\overline{g}(x) = \overline{x} - \alpha$ and $\deg g = \deg(\overline{x} - \alpha) = 1$. Therefore, $g(x) = x - a$ for some $a \in A$. Then we have that $f(x) = (x - a)h(x)$ so $f(a) = 0$ and clearly, $\overline{a} = \alpha$.

10b. Notice that $\mathbb{Z}_7/7\mathbb{Z}_7 \simeq \mathbb{Z}/7\mathbb{Z}$ and so the ideal generated by 7 is maximal and $\mathbb{Z}_7$ is 7-adic complete by definition. Therefore, from the above, if suffices to show that 2 is a square in $\mathbb{Z}/7\mathbb{Z}$, but $3^2 = 9 = 2 \bmod 7$. Therefore, 2 is a square in $\mathbb{Z}_7$.

10c. Let k be a field and $f(x, y) \in k[x, y] \subseteq k[[x]][y]$ (since $k[x, y]$ is a Noetherian domain, apply Krull intersection theorem) where $k[[x]][y]$ is the completion of $k[x, y]$ with respect to the (x)-adic topology. The condition that $f(0, y)$ has a simple root a_0 is equivalent to the condition that $\overline{f} \in k[[x]][y]/(x)$ has a simple root. From above, this then implies there is some root of $f \in k[[x]][y]$,

$$f\left(x, \sum a_n x^n\right) = 0$$

where a_0 is the same a_0 as above.

11. Let A be the ring of germs of smooth functions in some neighborhood of the origin. Clearly, A is local as the functions who vanish at the origin is the unique maximal ideal $\mathfrak{m}$ (everything that doesn't vanish at the origin is invertible in A). Note that $e^{-1/x^2} \in \cap \mathfrak{m}^n$ so that A is not Noetherian (since otherwise, it is a Noetherian local ring, which would imply this intersection is empty by the Krull intersection theorem). It can be checked

via Taylor's theorem that $\mathfrak{m}^k$ is the set of all functions whose first k derivatives (including $k = 0$) vanish at the origin. From this, it is clear that in $\widehat{A}$, the Taylor series of a function converges to the function itself and by Borel's theorem, every formal power series appears as the Taylor series of some smooth function. Therefore, $\widehat{A} = \mathbb{R}[[x]]$ is Noetherian. $\widehat{A}$ is a finitely-generated A-module since $A \mapsto \widehat{A}$ is surjective by Borel's theorem again.

Note: Borel's theorem, (which I've never heard of, though I should have) states that for any sequence of real numbers, there is a smooth function whose Taylor series is given by the generating function of this sequence.

12. Let A be Noetherian and consider the sequence of maps

$$A \to A[x] \to A[[x]].$$

From chapter 2, the first map is flat and the latter map is flat since $A[[x]]$ is the completion of $A[x]$ in the (x)-adic topology (and $A[x]$ is Noetherian by the Hilbert basis theorem). Therefore, $A[[x]]$ is a flat A-algebra. From chapter 1, the map $\mathrm{Spec}(A[[x]]) \mapsto \mathrm{Spec}(A)$ is surjective. From chapter 3, this implies that $A[[x]]$ is a faithfully-flat A-module.

Chapter 11

1. Let $f \in k[x_1, \ldots, x_n]$ be an irreducible polynomial over an algebraically closed field k, $V = V(f)$ be the hypersurface defined by f, $A = A(V) = k[x_1, \ldots, x_n]/(f)$ be the coordinate ring of V, $P = (a_1, \ldots, a_n)$ be a point of V, and $\overline{\mathfrak{m}}$ be the maximal ideal corresponding to the point P (which is the image of the ideal $\mathfrak{m} = (x_i - a_i)$ in $k[x_1, \ldots, x_n]$ modulo (f)). Notice first that $A/\overline{\mathfrak{m}}$ is a finitely-generated k-algebra that is also a field. Therefore, it is a finite algebraic extension of k. Since k is algebraically closed, this implies that $A/\overline{\mathfrak{m}} = k$ and

$$A_{\overline{\mathfrak{m}}}/\overline{\mathfrak{m}}_{\overline{\mathfrak{m}}} = (A/\overline{\mathfrak{m}})_{\overline{\mathfrak{m}}} = k_{\overline{\mathfrak{m}}} = k.$$

Since A is Noetherian, $A_{\overline{\mathfrak{m}}}$ is a local Noetherian ring. From the equivalences of regularity, to show that $A_{\mathfrak{m}}$ is regular, it suffices to show that $\dim_{A_{\overline{\mathfrak{m}}}/\overline{\mathfrak{m}}_{\overline{\mathfrak{m}}}} \overline{\mathfrak{m}}_{\overline{\mathfrak{m}}}/\overline{\mathfrak{m}}^2_{\overline{\mathfrak{m}}} = \dim A_{\overline{\mathfrak{m}}}$. Notice

$$\begin{aligned} \dim_{A_{\overline{\mathfrak{m}}}/\overline{\mathfrak{m}}_{\overline{\mathfrak{m}}}} \overline{\mathfrak{m}}_{\overline{\mathfrak{m}}}/\overline{\mathfrak{m}}^2_{\overline{\mathfrak{m}}} &= \dim_k (\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2)_{\overline{\mathfrak{m}}} = \dim_k \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2, \\ \dim A_{\overline{\mathfrak{m}}} &= \dim A = \dim k[x_1, \ldots, x_n]/(f) = n - 1. \end{aligned}$$

Therefore, it simply suffices to show that $\dim_k \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = n - 1$. Since Taylor's theorem holds for arbitrary polynomial rings over a field (by writing polynomials in powers of $(x_i - a_i)$), we may write

$$f(x_1, \ldots, x_n) = \sum a_{i_1, \ldots, i_n} (x_1 - a_1)^{m_1} \ldots (x_n - a_n)^{m_n}.$$

From this, P is singular if and only if $a_{e_i} = 0$ for all $1 \leq i \leq n$ (with $e_i = (0, \ldots, 1, \ldots, 0)$ and the 1 in the i-th position) (also remember that since $P \in V$, $a_{0,\ldots,0} = 0$). That is, P is singular if and only if $f \in (x_i - a_i)^2 = \mathfrak{m}^2$. As a k-vector space, it's clear that $\mathfrak{m}/\mathfrak{m}^2$ has a basis of $\{x_1 - a_1, \ldots, x_n - a_n\}$ and so has dimension n. As a k-vector space $\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$ is the quotient of $\mathfrak{m}/\mathfrak{m}^2$ by the subspace (f) and so has dimension n if $f \in \mathfrak{m}^2$ and $n - 1$ if $f \notin \mathfrak{m}^2$. That is, $\dim_k \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = n - 1$ if and only if $f \notin \mathfrak{m}^2$ if and only if P is nonsingular. Therefore, $A_{\overline{\mathfrak{m}}}$ is a regular local ring if and only if P is nonsingular.

2. Let $(A, \mathfrak{m})$ be a Noetherian local ring containing a field k such that A is complete with respect to the $\mathfrak{m}$-adic topology and $x_1, \ldots, x_d$ be a system of parameters. Since $\mathfrak{q} = (x_1, \ldots, x_d)$ is an $\mathfrak{m}$-primary ideal, there exists $r > 0$ such that $\mathfrak{m}^r \subseteq \mathfrak{q} \subseteq \mathfrak{m}$. From this, the $\mathfrak{q}$-adic and $\mathfrak{m}$-adic topologies coincide and A is $\mathfrak{q}$-adic complete. Consider the map $\phi : k[t_1, \ldots, t_d] \mapsto A$ defined by $t_i \mapsto x_i$. Completing with respect to the $\mathfrak{q}$-adic topology gives a map $\widehat{\phi} : k[[t_1, \ldots, t_d]] \mapsto A$ defined by $t_i \mapsto x_i$. To show that $\widehat{\phi}$ is injective, the proof of 11.21 will essentially be replicated. Assume the formal power series $f \in k[[t_1, \ldots, t_d]]$ is such that $\widehat{\phi}(f) = 0$. Then we may write $f = f_s +$ higher terms, where f_s is homogeneous of degree s (s is then minimal). Since $f(x_1, \ldots, x_d) = 0$, we necessarily have that $f_s(x_1, \ldots, x_d) \in \widehat{\mathfrak{q}}^{s+1}$. Since f_s is a homogeneous polynomial with coefficients in $(k \subseteq)A$, 11.21 applies and shows that the coefficients of f_s are in $\mathfrak{m}$, but this implies the coefficients of f_s are all zero (since no unit lies in $\mathfrak{m}$ and k is all units except for zero). This is a contradiction and shows that $\widehat{\phi}$ is necessarily injective.

3. The proof follows verbatim up until the use of 11.26. At this point, we have that $\dim V \geq \dim A_{\mathfrak{m}} = \dim B_{\mathfrak{n}}$ where $A = A(V)$ is the coordinate ring, $B = k[x_1, \ldots, x_d] \subseteq A$ is obtained from the normalization lemma (which it was never shown that we can take d equal to $\dim V$), $\mathfrak{m} = (x_i - a_i)$, and $\mathfrak{n} = B \cap \mathfrak{m}$. At this point, apply lemma 11.26 again with $A = \overline{k}[x_1, \ldots, x_d]$ since this is integral over B with B integrally closed. This implies that $\dim B_{\mathfrak{n}} = \dim \overline{k}[x_1, \ldots, x_n]_{\mathfrak{n}'}$. The rest follows as in the text to show that this number is $\geq \dim V$.

4. Let $A = k[x_1, \ldots]$ be a polynomial ring over a field k in countably many indeterminants. Let $m_n = 2^n$ so that $m_{n+1} - m_n = 2^n$ and let $\mathfrak{p}_i = (x_{m_i}, \ldots x_{m_{i+1}-1})$. Then $S = A \setminus \cup \mathfrak{p}_i$ is a multiplicative subset since each $\mathfrak{p}_i$ is a prime ideal. Let $B = S^{-1}A$.

   To show that B is Noetherian, a problem from chapter 7 will be used. Notice first that by the prime ideal correspondence, the only maximal ideals of B are in bijective correspondence with the prime ideals of A that are maximal in $\cup \mathfrak{p}_i$. Let $\mathfrak{p}$ be such an ideal. If $f_1, f_2 \in \mathfrak{p}$ are such that $f_1 \in \mathfrak{p}_i$ and $f_2 \in \mathfrak{p}_j$ for $i \neq j$, then by the definitions of $\mathfrak{p}_i$, $f_1 + f_2 \notin \cup \mathfrak{p}_i$ (consider degrees) so that $f_1 + f_2 \notin \mathfrak{p}$. Therefore, we necessarily have $\mathfrak{p} \subseteq \mathfrak{p}_i$ for some i. Conversely, each $\mathfrak{p}_i$ is maximal by the same reasoning and each is contained in $\cup \mathfrak{p}_i$. Therefore, $\mathfrak{p} = \mathfrak{p}_i$ for some i. That is, the maximal ideals of B are exactly the ideals $S^{-1}\mathfrak{p}_i$. Consider polynomial ring $K[x_j]_{m_i \leq j \leq m_{i+1}-1}$ where $K = k(x_j)_{j < m_i,\ j > m_{i+1}-1}$ is the field of rational functions in all other indeterminants x_i. Then the localization $B_{S^{-1}\mathfrak{p}_i}$ is then the localization of $K[x_j]_{m_i \leq j \leq m_{i+1}-1}$ at the ideal $1 + (x_j)_{m_i \leq j \leq m_{i+1}-1}$. Since the polynomial ring is Noetherian by the Hilbert basis theorem, the localization $B_{S^{-1}\mathfrak{p}_i}$ is Noetherian as well.

   On the other hand, it is clear that for every $x \neq 0$, it is clear that there is a unique maximal ideal that contains x (consider degrees). Therefore, B satisfies properties (1) and (2) from chapter 7, problem 9. Therefore, B is Noetherian.

   To see that $\dim B = \infty$, notice that by the prime ideal correspondence with localizations, the height of the ideal $S^{-1}\mathfrak{p}_i$ is exactly $m_{i+1} - m_i = 2^i$, which shows the $\dim B$ is infinite.

5.

6. Let A be any ring. For any prime ideal $\mathfrak{p}$ and chain of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n = \mathfrak{p},$$

   there is a chain of prime ideals

$$\mathfrak{p}_0[x] \subset \mathfrak{p}_1[x] \subset \ldots \subset \mathfrak{p}_n[x] = \mathfrak{p}[x].$$

   Since $A[x]/\mathfrak{p}[x] \simeq (A/\mathfrak{p})[x]$ is not a field (even if $\mathfrak{p}$ is maximal), this shows this chain can be extended. Therefore, $1 + \dim A \leq \dim A[x]$.

   For the other inequality, let $f : A \mapsto A[x]$ be the canonical embedding. Consider a fiber of a prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$ under the map $f^* : \mathrm{Spec}(A[x]) \mapsto \mathrm{Spec}(A)$. From a problem in chapter 3, this fiber can be identified with the set $\mathrm{Spec}(k(\mathfrak{p}) \otimes_A A[x]) = \mathrm{Spec}(k[x])$, where $k = k(\mathfrak{p})$ is the residue field of A at the prime $\mathfrak{p}$. Recall that $\dim k[x] = 1$ since $k[x]$ is a PID and not a field. From this, any chain of prime ideals laying over a prime has length at most 1. For any chain of prime ideals in $A[x]$, we may contract to get a chain of prime ideals of A. We see that there are at most two primes laying above each prime in this chain so there are at most twice the number of primes plus two of the chain in $A[x]$. That is, there are at most twice as many links plus one. Taking the supremum, we get $\dim A[x] \leq 1 + 2\dim A$.

7. From the previous problem, the inequality $1 + \dim A \leq \dim A[x]$ holds almost trivially. Therefore, it suffices to show only the other inequality.

   Let $\mathfrak{q} \in \mathrm{Spec}(A[x])$ and $\mathfrak{p} = \mathfrak{q}^c = \mathfrak{q} \cap A$. For any maximal increasing chain of prime ideals ending in $\mathfrak{q}$, it is clear that $\mathfrak{p}[x]$ appears (consider ideals of constant terms of these prime ideals, insert $\mathfrak{p}[x]$ where the constant term ideals become $\mathfrak{p}$). From the previous problem, any chain of prime ideals in $A[x]$ laying over a prime in A has length at most 1 (so there are 2 prime ideals). From this and the fact that the maximal constant term ideal of this chain is $\mathfrak{p}$, it follows that the chain has the form

$$\mathfrak{q}_0 \subset \ldots \subset \mathfrak{p}[x] \subseteq \mathfrak{q}$$

   (where the last chain may be an equality). From this, it suffices to show that $\mathrm{height}\mathfrak{p}[x] \leq \mathrm{height}(\mathfrak{p}) \leq \dim A$.

   For any prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$ of height m, we may find $a_1, \ldots, a_m \in A$ such that $\mathfrak{p}$ is a minimal prime of $\mathfrak{a} = (a_1, \ldots, a_m)$ (see below). Then from chapter 4, problem 7, it follows that $\mathfrak{p}[x]$ is a minimal prime of $\mathfrak{a}[x]$.

Since $\mathfrak{a}[x]$ is generated by m elements, we have that $\text{height}(\mathfrak{p}[x]) \leq m = \text{height}(\mathfrak{p})$ by 11.16. Therefore, the result follows.

To see that we can always find such a_i, we of course use induction. The base case, $m = 0$ is trivial since any minimal prime is a minimal prime of 0. Assuming the result for a fixed m, let $\mathfrak{p}$ be of height $m+1$ so that there exists a maximal increasing chain of prime ideals

$$\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_m \subset \mathfrak{p}_{m+1} = \mathfrak{p}.$$

By inductive hypothesis, there exists $a_1, \ldots, a_m \in A$ such that $\mathfrak{p}_m$ is a minimal prime of $(a_1, \ldots, a_m)$. We necessarily have $\mathfrak{p} \not\subseteq \cup \mathfrak{q}_i$ where this union is over the minimal elements of the set of prime ideals containing $(a_1, \ldots, a_m)$ (which is finite since $A/(a_1, \ldots, a_m)$ is Noetherian), since otherwise, $\mathfrak{p} \subset \mathfrak{q}_i$ for some i and has height $\leq m$ by 11.16. Therefore, there exists $a_{m+1} \in \mathfrak{p} \backslash \cup \mathfrak{q}_i$ (in particular, $a_{m+1} \notin \mathfrak{p}_m$). Let $\mathfrak{a} = (a_1, \ldots, a_{m+1})$. For any prime ideal $\mathfrak{q}$ containing $(a_1, \ldots, a_{m+1})$, $\mathfrak{q}$ contains $(a_1, \ldots, a_m)$. Writing out a primary decomposition and taking radicals, we have that $\mathfrak{q}$ contains the intersection of minimal primes of $(a_1, \ldots, a_m)$ and so contains one of these minimal primes. Since these each have height $\leq m$ and $\mathfrak{q}$ properly contains one of them (it contains a_{m+1}), $\text{height}(\mathfrak{q}) \geq i+1$. Since no prime ideals of the same height can contain one another, if $\text{height}(\mathfrak{q}) = i+1$, there is no containment with $\mathfrak{q}$ and $\mathfrak{p}_{m+1}$. If $\text{height}(\mathfrak{q}) > i+1$, then if there is a containment, it can only be $\mathfrak{p}_{m+1} \subseteq \mathfrak{q}$. Therefore, $\mathfrak{p}_{m+1}$ is a minimal prime of $\mathfrak{a} = (a_1, \ldots, a_{m+1})$ as desired.