Lecture 2: Jones Polynomial

Reference: L. Kauffman: State Models and the Jones Polynomial, Topology, 1986

Def: A <u>link</u> in S^3 is a smooth embedding

$$L : \coprod_{k} S^1 \longrightarrow S^3$$

of k disjoint S^1 into S^3.

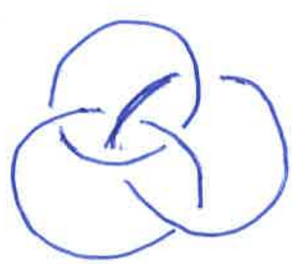
If $k=1$, it is a <u>knot</u>, denoted by K.

Eg: Trivial knot (Unknot) 

Trefoils: 

Figure 8: 

Hopf link: , Whitehead link: 

Borromean rings: 

Two links L_1 and L_2 are equivalent if they are isotopic, ie, $\exists$ smooth map

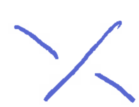
$$L : \left(\coprod_k S^1 \right) \times [0,1] \longrightarrow S^3$$

s.t. $L\Big|_{(\coprod_k S^1)\times\{0\}} = L_1$, $L\Big|_{(\coprod_k S^1)\times\{1\}} = L_2$, and

$\forall\, t \in (0,1)$, $L\Big|_{(\coprod_k S^1)\times\{t\}}$ is a link.

All knots / links above are different, why ??

Not easy, one way to answer it is to use invariant.

Def: A link diagram is a smooth four-valent graph in $\mathbb{R}^2$ w/ each vertex replaced by a over-and-under crossing 

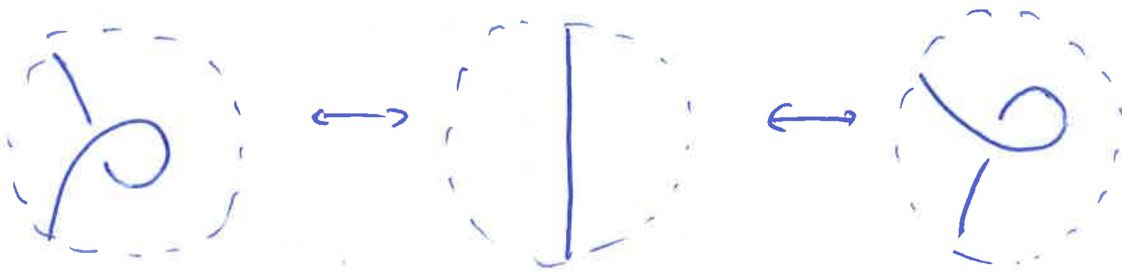
- projection of a link in $\mathbb{R}^2 \subset \mathbb{R}^3 \subset S^3$.

Q: When do two diagrams represent same link?

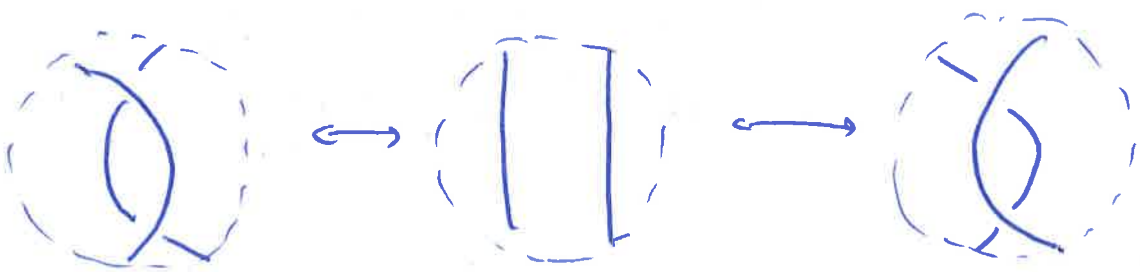
Thm (Reidemeister, 1927)

Two diagrams represent the same link iff one could be obtained from the other by a sequence of (ambient isotopies of $\mathbb{R}^2$ and) the following three local moves.

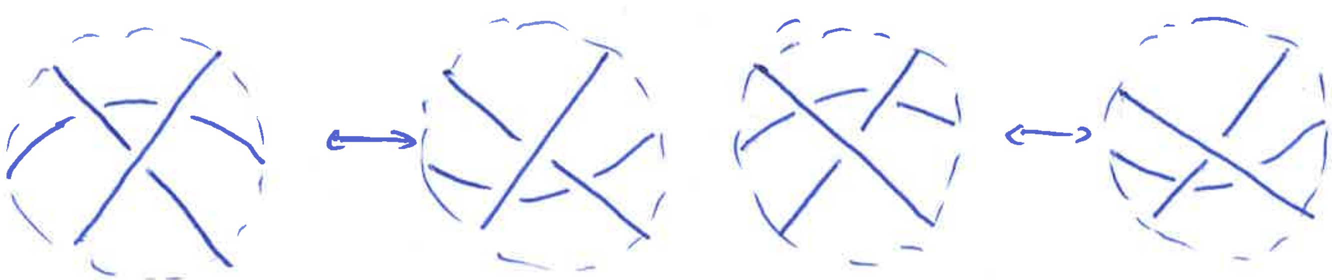
RmI:



RmII:



RmIII:



- A good proof can be found in Goldman 1986 Invent paper, 5.6 Lemma.

Kauffman bracket

Given a diagram $D = D(L)$, $\langle D \rangle$ is a Laurent polynomial in A defined by the following two rules:

① (Kauffman bracket) skein relation:

positive resolution — go along lower strand, then turn right.

negative resolution — - - - left

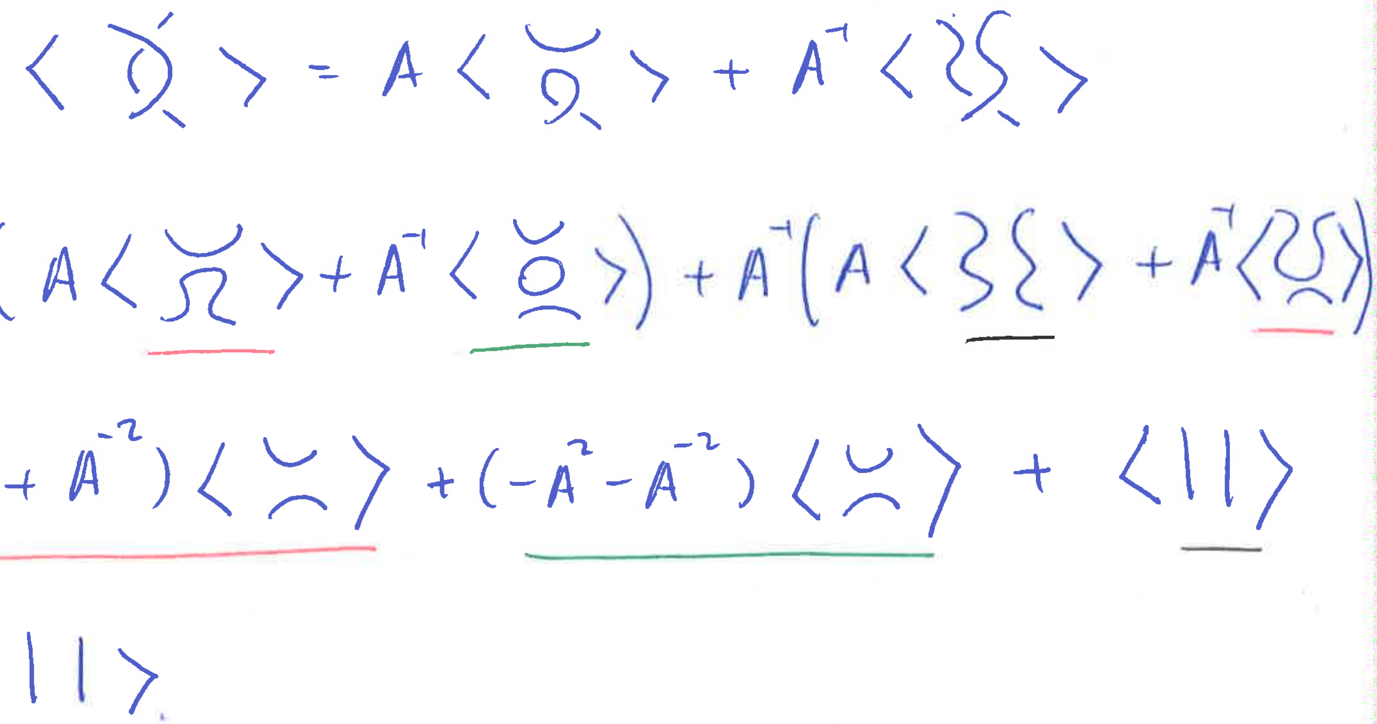
② Framing relation:

$$\langle \bigcirc \cup L \rangle = (-A^2 - A^{-2}) \langle L \rangle$$

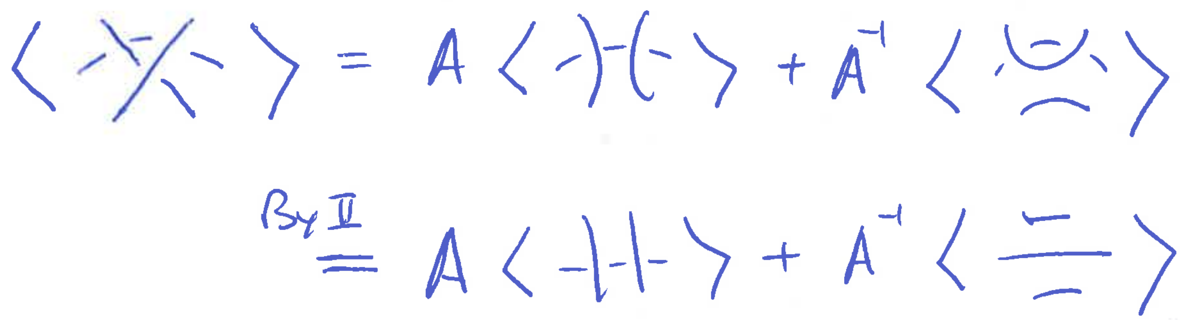
disjoint, unlinked, unknot.

Q: Is $\langle D \rangle$ invariant under RM I, II and III?

RM II:

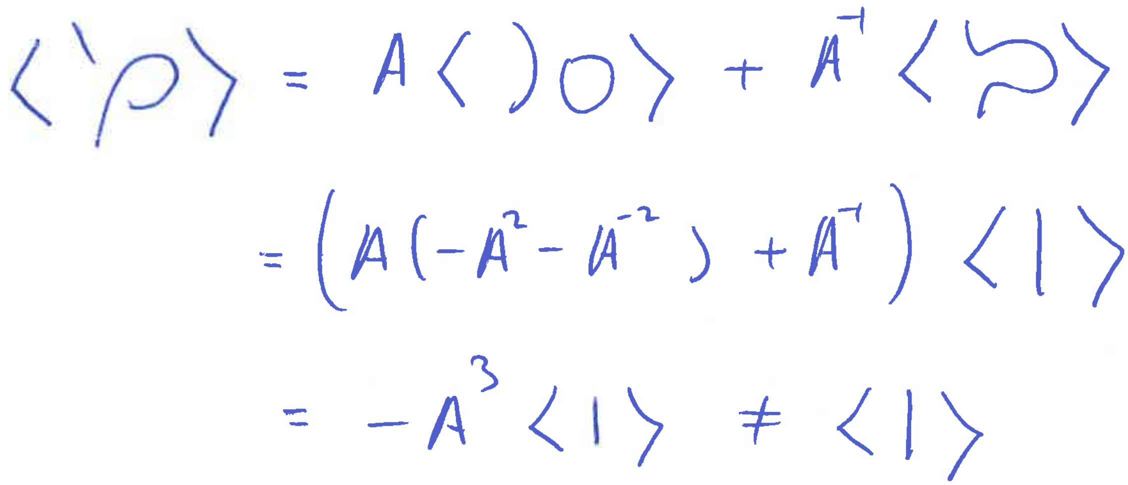


RM III:



Similarly:

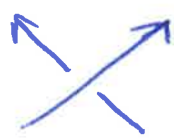
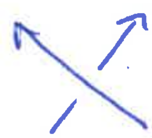
RM I:



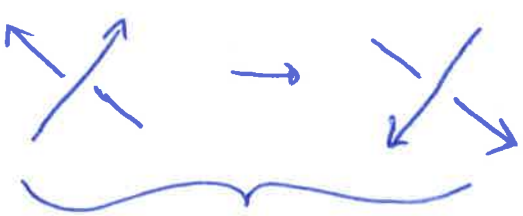
Similarly,

But not TOO bad

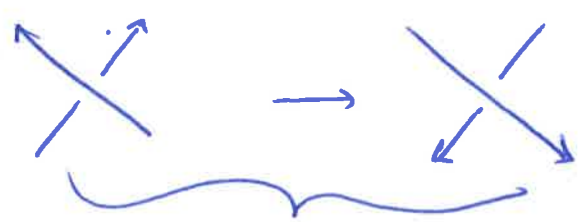
Let $D = D(L)$, and give D arbitrarily an orientation of the components.

A crossing is <u>positive</u>  if as you go along the lower arrow, the upper arrow points the <u>right</u>; and is <u>negative</u>  if otherwise.

RMK: If $D = D(K)$ is a diagram of a knot K, the being "+" or "−" is independent of orientation.



"+"



"−"

<u>Def</u>: The <u>writh number</u> of $D = D(K)$ is

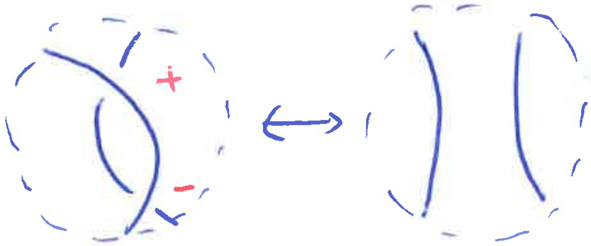
$$w(D) = \#\{\text{"+" crossings}\} - \#\{\text{"−" crossings}\}$$

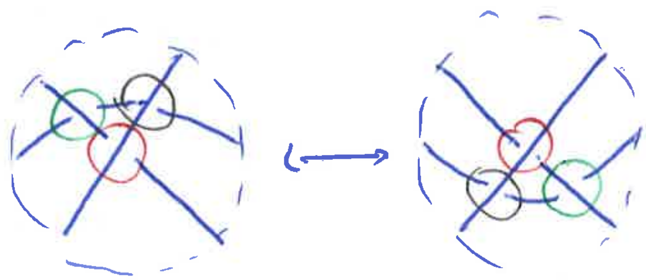
Def \ Thm (Jones) The following

$$J(K, A) = (-A^3)^{-w(D(K))} \langle D(K) \rangle$$

is invariant under RM I, II, III, ie, defines an invariant of K.

pf: If $D'(K)$ and $D(K)$ differ by a RM II or III, then $w(D') = w(D)$ and $\langle D' \rangle = \langle D \rangle$.





the two crossings have different signs.

the crossings in the circles of same color have the same sign.

If they differ by a RM I, eg,

D' D,

then $w(D') = w(D) + 1$ and

$$\langle D' \rangle = -A^3 \langle D \rangle.$$

Therefore, $(-A^3)^{-w(D')} \langle D' \rangle = (-A^3)^{-w(D)-1} (-A^3) \langle D \rangle$

$$= (-A^3)^{-w(D)} \langle D \rangle \qquad \square$$

Eg: $J\langle O, A\rangle = -A^{2} - A^{-2}$

$J\langle$  $, A\rangle = (-A^{2} - A^{-2})(-A^{16} + A^{12} + A^{4})$

$J\langle$  $, A\rangle = (-A^{2} - A^{-2})(-A^{-16} + A^{-12} + A^{-4})$.

$\Rightarrow$  $\neq$ 

HW. Show  $=$  by find a

Sequence of RM's.

Normalized Jones polynomial.

$$J'(K, A) = \frac{J(K, A)}{J(O, A)}$$

Conj (Jones). Jones polynomials detect unknot,

ie, if $J'(K, A) = J'(O, A) = 1$, then $K = O$.

Still OPEN!