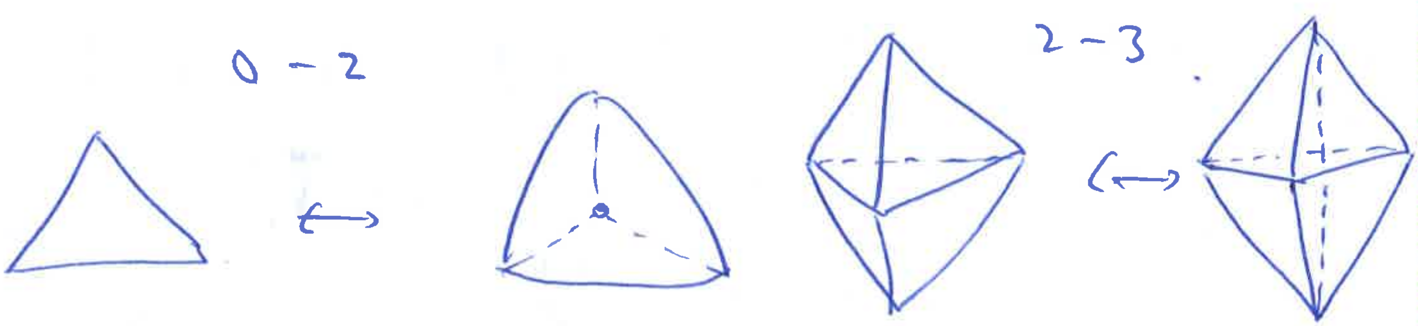
Turaev – Viro Invariants.

M closed 3-mfd. A triangulation of M consists of a finite collection of Euclidean tetrahedra $\sigma_1, \cdots, \sigma_n$ and a set of affine homeomorphisms $\{\phi_{ij}\}$ between pairs of faces of $\{\sigma_k\}$ s.t

$$M \cong \coprod_{k=1}^{n} \sigma_k \Big/ \{\phi_{ij}\}.$$

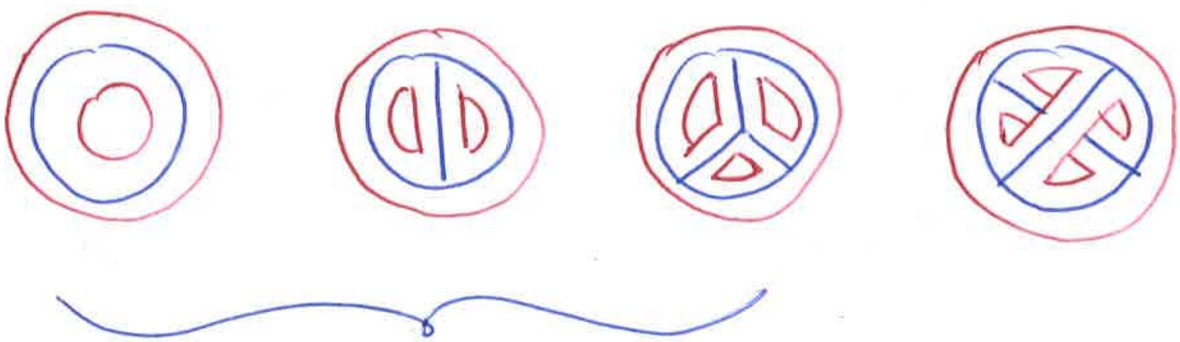
Thm (Moise, Bing) Every 3-mfd is triangulable.

Thm (Matveev, Piergallini) Any two triangulations of M are related by a sequence of 0-2 and 2-3 Pachner Moves.



Def: A <u>tri-valent ribbon graph</u> $\Gamma \subset \mathbb{R}^2$ is a 3-valent graph w/ a 2-dim thickening.

Eg



only need these to define TV-inv.

- Let $r \geq 3$, $r \in \mathbb{N}$, $I_r = \{0, 1, \dots, r-2\}$.

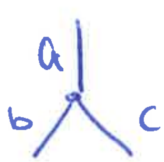
Def: A triple $(a, b, c) \in I_r^3$ is <u>r-admissible</u> if

(1) $a+b \geq c$, $b+c \geq a$, $a+c \geq b$,

(2) $a+b+c \leq 2(r-2)$ and

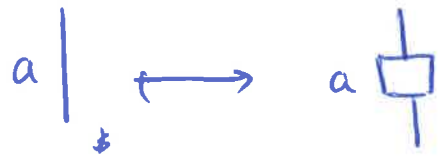
(3) $a+b+c$ is even.

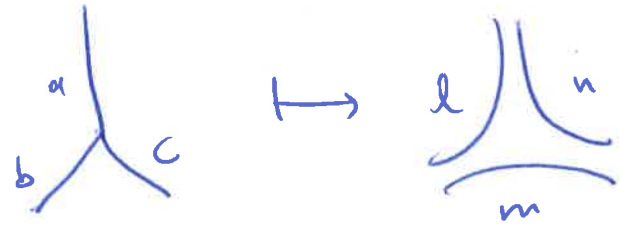
Def: An <u>r-admissible coloring</u> of Γ is an assignment of elements of I_r to edges of Γ s.t. for each vertex



, the triple (a, b, c) is r-admissible.

• Each Γ-admissible coloring c of Γ determines a complex number $|\Gamma, c|$ as follows.

1. For each edge  a-th Jones-Wenzl f_a.

2. For each vertex ,

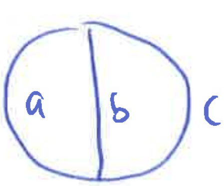
where $l = \frac{a+b-c}{2}$, $m = \frac{b+c-a}{2}$, $n = \frac{a+c-b}{2}$.

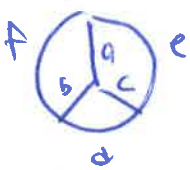
In this way, get a skein in $\mathbb{R}^2$.

3. Take Kauffman bracket of the resulting skein at $A_r = e^{\frac{\pi i}{2r}}$.

Eg:  $= \langle$  $\rangle = (-1)^a [a+1]$.

Recall: $[n] = \dfrac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}$, $[n]! = [n] \cdots [1]$, $[0]! = 1$.

 $= (-1)^{\frac{a+b+c}{2}} \dfrac{\left[\frac{a+b+c}{2}+1\right]! \left[\frac{a+b-c}{2}\right]! \left[\frac{b+c-a}{2}\right]! \left[\frac{a+c-b}{2}\right]!}{[a]![b]![c]!}$



$$= \frac{\prod_{i=1}^{4}\prod_{j=1}^{3}[Q_j - T_i]!}{[a]![b]!\cdots[f]!}\sum_{k=\max\{T_i\}}^{\min\{Q_j\}} \frac{(-1)^k [k+1]!}{\prod_{i=1}^{4}[k-T_i]! \prod_{j=1}^{3}[Q_j-k]!},$$

where

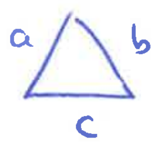
$$T_1 = \frac{a+b+c}{2}, \quad T_2 = \frac{a+e+f}{2}, \quad T_3 = \frac{b+d+f}{2}, \quad T_4 = \frac{c+d+e}{2},$$

and

$$Q_1 = \frac{a+b+d+e}{2}, \quad Q_2 = \frac{a+c+d+f}{2}, \quad Q_3 = \frac{b+c+e+f}{2}.$$

. Let $(M, \mathcal{T})$ triangulated 3-mfd, V, E, F, T resp. sets of vertices, edges, faces, tetrahedra.

A coloring $c: E \to I_r$ is r-admissible if for each $f \in F$,



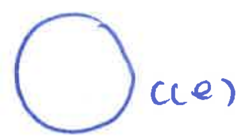
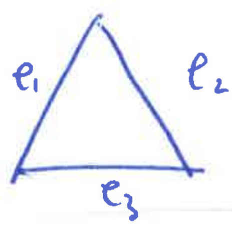
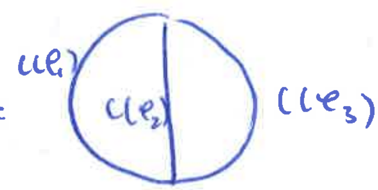
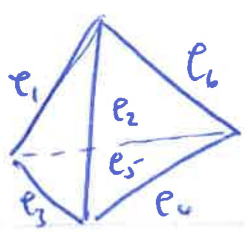
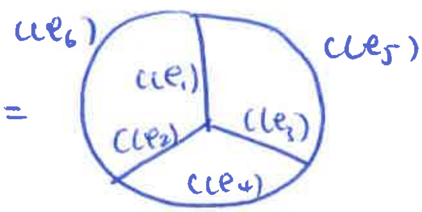
, the triple (a, b, c) is r-admissible. I.e.,

(1) $a+b \geq c$, $b+c \geq a$, $a+c \geq b$,

(2) $a+b+c \leq 2(r-2)$, and

(3) $a+b+c$ is even.

Let $\mathcal{A}_r = \mathcal{A}_r(M, \mathcal{T})$ be the set of all r-admissible colorings of $(M, \mathcal{T})$, and let $c \in \mathcal{A}_r$.

For	let
e 	$\lvert e \rvert_c =$ 
f 	$\lvert f \rvert_c =$ 
σ 	$\lvert \sigma \rvert_c =$ 

Def / Thm (Turaev – Viro) Let $(M, \mathcal{T})$ be a triangulated 3-mfd. For $r \geq 3$, let $\eta = \dfrac{-2r}{(A^2 - A^{-2})^2}$. Then

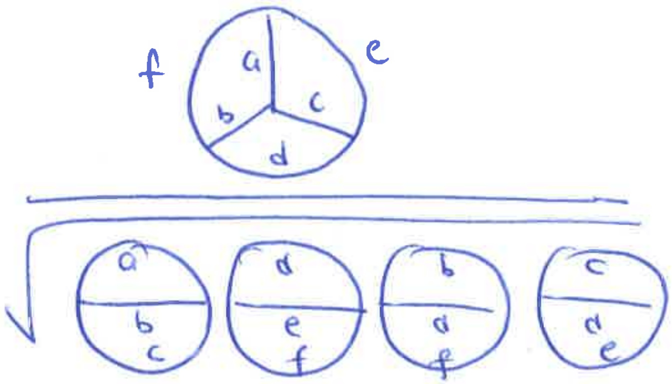
$$TV_r(M) = \eta^{-|V|} \sum_{c \in \mathcal{A}_r} \prod_{e \in E} \lvert e \rvert_c \prod_{f \in F} \lvert f \rvert_c^{-1} \prod_{\sigma \in T} \lvert \sigma \rvert_c$$

defines a real valued invariant of M, ie, is invariant under 0-2 and 2-3 Pachner Moves.

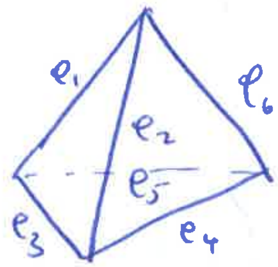
Quantum 6j - symbol:

For r-adm triples (a,b,c), (a,e,f), (b,d,f), (c,d,e),

$$\begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} =$$



For $\sigma :=$



, let $\|\sigma\|_c = \begin{vmatrix} c(e_1) & c(e_2) & c(e_3) \\ c(e_4) & c(e_5) & c(e_6) \end{vmatrix}$.

Then

$$\boxed{TV_r(M) = \eta^{-|V|} \sum_{c \in \mathcal{A}_r} \prod_{e \in E} |e|_c \prod_{\sigma \in T} \|\sigma\|_c}.$$

rm: For 3-mfd M w/ $\partial M \neq \emptyset$, can consider ideal triangulations of M, and define

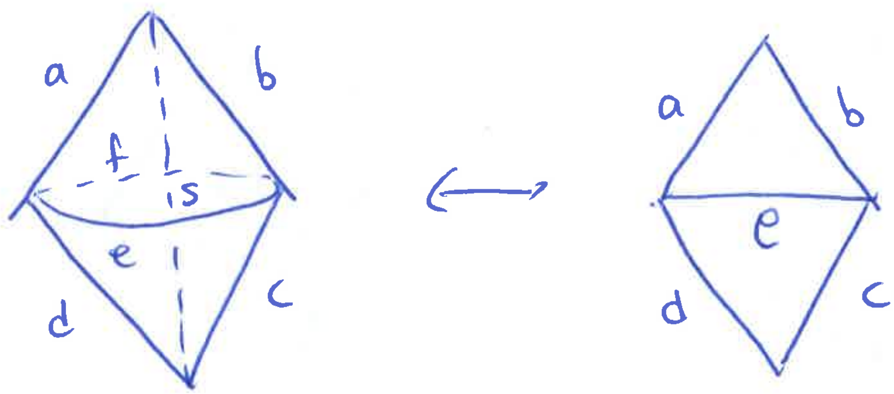
$$TV_r(M) := \sum_{c \in \mathcal{A}_r} \prod_{e \in E} |e|_c \prod_{\sigma \in T} \|\sigma\|_c .$$

Notation: $|n| = (-1)^n [n+1]$.

Orthogonality: If (a,b,e), (a,b,f), (c,d,e), (c,d,f) are r-adm, then

$$\sum_{s} |s||e| \begin{vmatrix} a & b & e \\ c & d & s \end{vmatrix} \begin{vmatrix} a & b & f \\ c & d & s \end{vmatrix} = \delta_{ef},$$

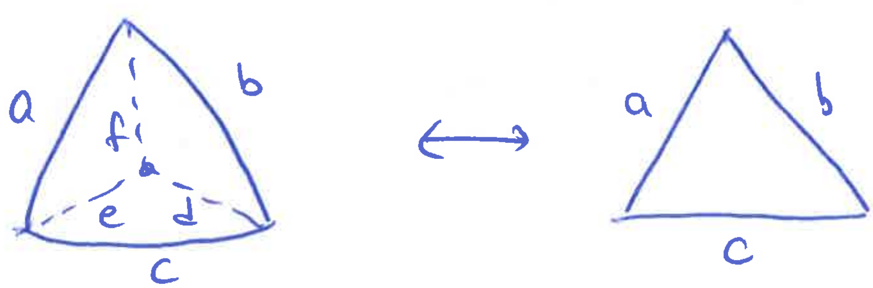
where $s \in I_r$ s.t (a,d,s), (b,c,s) r-adm.



Cor: If (a,b,c) is r-adm, then

$$\eta^{-1} \sum_{d,e,f} |d||e||f| \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} \begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = 1,$$

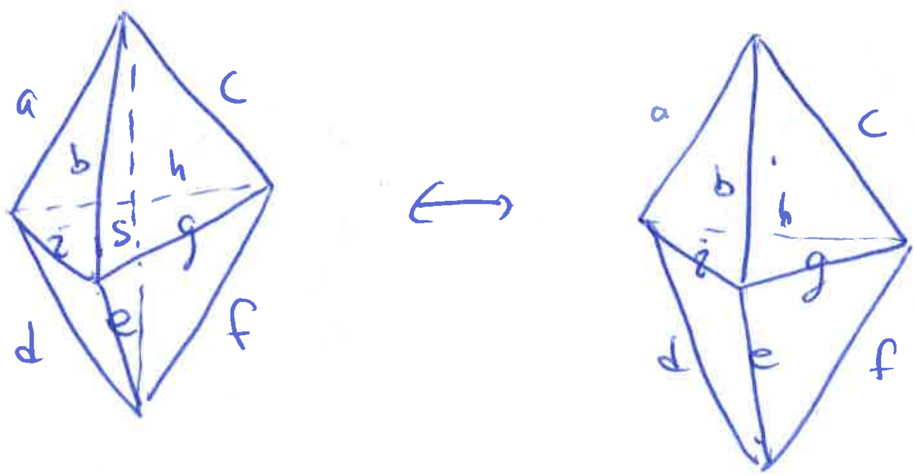
where $d,e,f \in I_r$ s.t (a,e,f) (b,d,f) (c,d,e) r-adm.



Biedenharn - Elliot Identity: If (a,b,c), (b,c,g), (a,c,h), (d,e,i), (e,f,g), (d,f,g) r-adm, then

$$\sum_s |s| \begin{vmatrix} a & b & i \\ e & d & s \end{vmatrix} \begin{vmatrix} b & c & g \\ f & e & s \end{vmatrix} \begin{vmatrix} c & a & h \\ d & f & s \end{vmatrix} = \begin{vmatrix} a & b & i \\ g & h & c \end{vmatrix} \begin{vmatrix} d & e & i \\ g & h & f \end{vmatrix},$$

where $s \in I_r$ s.t (a,d,s), (b,e,s), (c,f,s) r-adm.



Pf of Thm: Direct consequence of Cor and BE identity.

The pf of Orthogonality, Cor and BE identity will be given in the next lectures.

$\square$