Volume Conjectures for Reshetikhin-Turaev and Turaev-Viro invariants

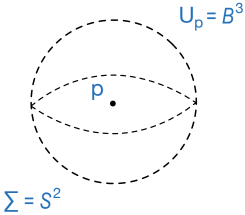
Tian Yang

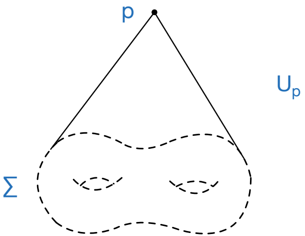
Joint with Qingtao Chen

New Developments in TQFT
QGM, Aarhus University, Denmark

- M closed pseudo 3-manifold

  For each p in M, exists neighborhood $U_p \cong$ cone over Σ.





- p is a singular point if $\Sigma \ncong S^2$.

- $\mathcal{T}$ triangulation of M

  I.e.,

$$M \cong \bigsqcup_{k=1}^{n} \Delta_k \Big/ \Phi_{ij},$$

  $\{\Delta_1, \ldots, \Delta_n\}$ Euclidean tetrahedra, and

  $\{\Phi_{ij}\}$ homeomorphisms between faces of $\{\Delta_k\}$.

- Every face is glued with another.

- Singular points must be vertices.

- Example

  (1) Triangulated closed 3-manifolds

  (2) Ideally triangulated 3-manifolds with boundary

  could be considered as

$$M \setminus \bigsqcup_{\text{vertices } v} U_v .$$

  (3) In particular, knot/link complements.

- Quantum number

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

- Quantum factorial

$$[n]! = [n][n-1]\cdots[1]$$

$$[0]! = 1$$

- Remark

  Different from other literatures $[n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$.

- $r \geqslant 3$ fixed integer
- q $2r$-th root of 1 such that q^2 is primitive r-th root of 1, i.e.,

$$q = e^{\frac{k\pi i}{r}} \quad \text{with} \quad (k, r) = 1.$$

- Example $\quad k = 1, r \in \mathbb{N} \quad$ or $\quad k = 2, r$ odd.
- Notation

$$q(k) = e^{\frac{k\pi i}{r}}$$

- Example

$$q(1) = e^{\frac{\pi i}{r}} \quad \text{and} \quad q(2) = e^{\frac{2\pi i}{r}}.$$

- $r \geqslant 3$ fixed integer

- q $2r$-th root of 1 such that q^2 is primitive r-th root of 1, i.e.,

$$q = e^{\frac{k\pi i}{r}} \quad \text{with} \quad (k, r) = 1.$$

- Example $\quad k = 1, r \in \mathbb{N} \quad$ or $\quad k = 2, r$ odd.

- Notation

$$q(k) = e^{\frac{k\pi i}{r}}$$

- Example

$$q(1) = e^{\frac{\pi i}{r}} \quad \text{and} \quad q(2) = e^{\frac{2\pi i}{r}}.$$

- Let

$$I_r = \left\{0, \frac{1}{2}, 1, \ldots, \frac{r-2}{2}\right\}.$$

- (a, b, c) is admissible if

  (1) $a + b \geqslant c$, $b + c \geqslant a$ and $c + a \geqslant b$,

  (2) $a + b + c \in \mathbb{N}$, and

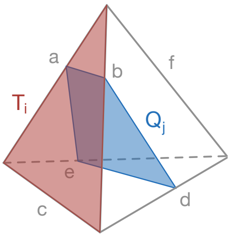
  (3) $a + b + c \leqslant r - 2$.

- Let

$$\Delta(a, b, c) = \sqrt{\frac{[a+b-c]![b+c-a]![c+a-b]!}{[a+b+c+1]!}}.$$

- Quantum $6j$-symbol

$$\begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = i^{-2(a+b+c+d+e+f)}$$
$$\times \Delta(a,b,c)\Delta(a,e,f)\Delta(b,d,f)\Delta(c,d,e)$$
$$\times \sum_{k=\max\{T_i\}}^{\min\{Q_j\}} \frac{(-1)^k [k+1]!}{\prod_{i=1}^{4}[k-T_i]! \prod_{j=1}^{3}[Q_j-k]!}$$



T_1=a+b+c T_2=a+e+f

T_3=b+d+f T_4=c+d+e

Q_1=a+b+d+e

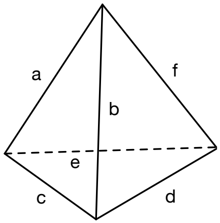
Q_2=a+c+d+f

Q_3=b+c+e+f

- Symmetries

$$\begin{vmatrix} a & b & c \\ d & e & f \end{vmatrix} = \begin{vmatrix} b & a & c \\ e & d & f \end{vmatrix} = \begin{vmatrix} a & c & b \\ d & f & e \end{vmatrix}$$

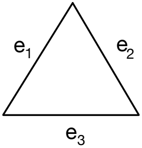
$$= \begin{vmatrix} a & e & f \\ d & b & c \end{vmatrix} = \begin{vmatrix} d & e & c \\ a & b & f \end{vmatrix} = \begin{vmatrix} d & b & f \\ a & e & c \end{vmatrix}$$



- $(M, \mathcal{T})$ triangulated pseudo 3-manifold,

  $E = \{\text{edges}\}$, $T = \{\text{tetrahedra}\}$.

- A coloring $c : E \to I_r$ is admissible if $(c(e_1), c(e_2), c(e_3))$ is admissible for each face



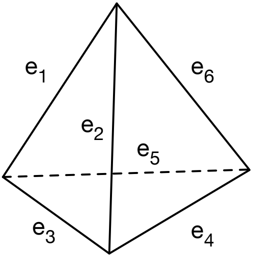
- $\mathcal{A}_r = \{\text{admissible colorings}\}$

- For $c \in \mathcal{A}_r$, let

$$|e|_c = (-1)^{2c(e)}[2c(e)+1], \quad e \in E,$$

and let

$$|\Delta|_c = \begin{vmatrix} c(e_1) & c(e_2) & c(e_3) \\ c(e_4) & c(e_5) & c(e_6) \end{vmatrix}, \quad \Delta \in T.$$



- Definition/Theorem (Turaev-Viro)

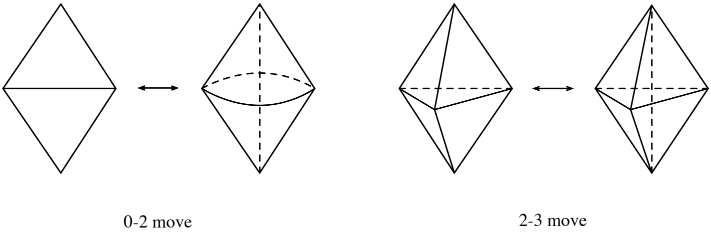
  For $(k, r) = 1$,

$$TV_r(M, q(k)) = \sum_{c \in \mathcal{A}_r} \prod_{e \in E} |e|_c \prod_{\Delta \in T} |\Delta|_c$$

  is a real-valued pseudo 3-manifold invariant.

- Theorem (Matveev, Piergallini)

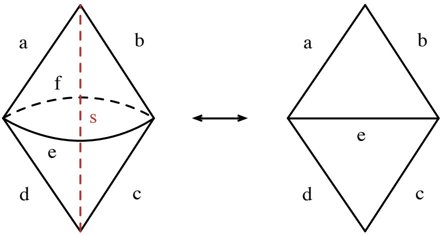
  Any two triangulations of M are related by a sequence of $0-2$ and $2-3$ Pachner moves.



- Orthogonality:

$$\sum_s |s||e| \begin{vmatrix} a & b & e \\ c & d & s \end{vmatrix} \begin{vmatrix} a & b & f \\ c & d & s \end{vmatrix} = \delta_{ef}$$

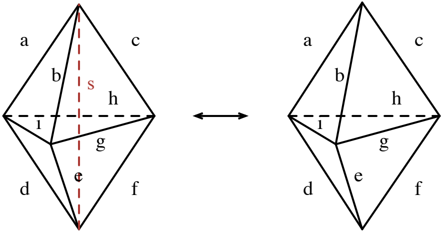
where all the triples are admissible.



- Biedenharn-Elliot Identity:

$$\sum_{s} |s| \begin{vmatrix} a & b & i \\ e & d & s \end{vmatrix} \begin{vmatrix} b & c & g \\ f & e & s \end{vmatrix} \begin{vmatrix} c & a & h \\ d & f & s \end{vmatrix} = \begin{vmatrix} a & b & i \\ g & h & c \end{vmatrix} \begin{vmatrix} d & e & i \\ g & h & f \end{vmatrix}$$

where all the triples are admissible.



- Remark

(1) For M closed, $TV_r(M,q)$ defers from Tureav-Viro's original construction by a factor a power of

$$\omega = \frac{(q - q^{-1})^2}{2r},$$

where the latter defines an invariant of 3-manifold.

(2) For M with boundary, Turaev-Viro's original construction defines a TQFT, whereas $TV_r(M,q)$ defines an invariant.

(3) In particular, $TV_r(M,q)$ defines an invariant of knots/links.

- Proposition

$$TV_r(\text{unknot}) = 1$$

$$TV_r(\text{Hopf link}) = r - 1$$

$$TV_r(\text{trefoil knot}) = \lfloor \frac{r-2}{3} \rfloor + 1$$

$$TV_r((2,4)\text{ torus link}) = \left(\lfloor \frac{r-2}{2} \rfloor + 1\right)\left(\lfloor \frac{r-1}{2} \rfloor + 1\right)$$

$$TV_r((2,6)\text{ torus link}) = \left(\lfloor \frac{r-2}{3} \rfloor + 1\right)\left(\lfloor \frac{2r-2}{3} \rfloor + 1\right)$$

- Asymptotic Expansion Conjecture (Witten)

  M closed, $TV_r(M, q(1))$ grows polynomially in r.

- Conjecture (Murakami-Murakami)

  M closed hyperbolic, $TV(M, q(1))$ and $RT(M, q(1))$ may detect the volume/complex volume using optimistic limit.

- Conjecture (Garoufalidis)

  M closed hyperbolic, $RT(M, q(1))$ may detect the complex volume using analytic continuation.

- Asymptotic Expansion Conjecture (Witten)
  M closed, $TV_r(M, q(1))$ grows polynomially in r.

- Conjecture (Murakami-Murakami)
  M closed hyperbolic, $TV(M, q(1))$ and $RT(M, q(1))$ may detect the volume/complex volume using optimistic limit.

- Conjecture (Garoufalidis)
  M closed hyperbolic, $RT(M, q(1))$ may detect the complex volume using analytic continuation.

- Conjecture 1 (Chen-Y.)

  M hyperbolic (closed, cusped or totally geodesic boundary), $TV_r(M,q(2))$ grows exponentially in r, with growth rate the volume of M. More precisely,

$$\lim_{r \text{ odd}, \, r\to\infty} \frac{2\pi}{r-2} \log TV_r(M, q(2)) = Vol(M).$$

- Volume Conjecture (Kashaev)

  K hyperbolic, $\langle K \rangle_N$ Kashaev invariant,

$$\lim_{N\to\infty} \frac{2\pi}{N} \ln |\langle K \rangle_N| = Vol(S^3 \setminus K).$$

- Conjecture 1 (Chen-Y.)

  M hyperbolic (closed, cusped or totally geodesic boundary), $TV_r(M,q(2))$ grows exponentially in r, with growth rate the volume of M. More precisely,

$$\lim_{r \text{ odd},\, r\to\infty} \frac{2\pi}{r-2} \log TV_r(M, q(2)) = Vol(M).$$

- Volume Conjecture (Kashaev, Murakami-Murakami)

  K hyperbolic, $J_N(K,q)$ colored Jones polynomial,

$$\lim_{N\to\infty} \frac{2\pi}{N} \ln |J_N(K, q(1))| = Vol(S^3 \setminus K).$$

- Evidence to Conjecture 1

  (1) Cusped

  K_{4_1}, K_{5_2} and K_{6_1} complement and sisters

  M_{2_x}, M_{3_x}, N_{2_x}, N_{3_x} in Callahan-Hildebrand-Weeks' list

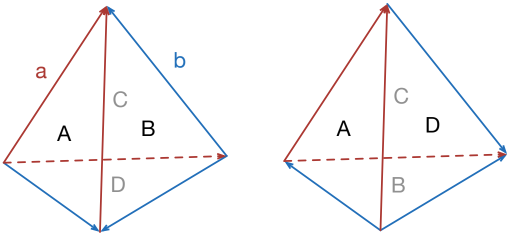
  (2) Totally geodesic boundary

  minimum volume

  (3) Closed

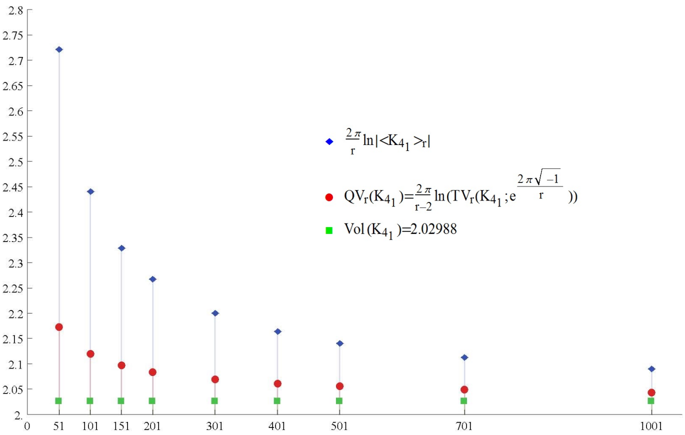
  $p-$surgery along K_{4_1} and K_{5_2} for $p = -10, \ldots, 10$

- Knot 4_1 complement

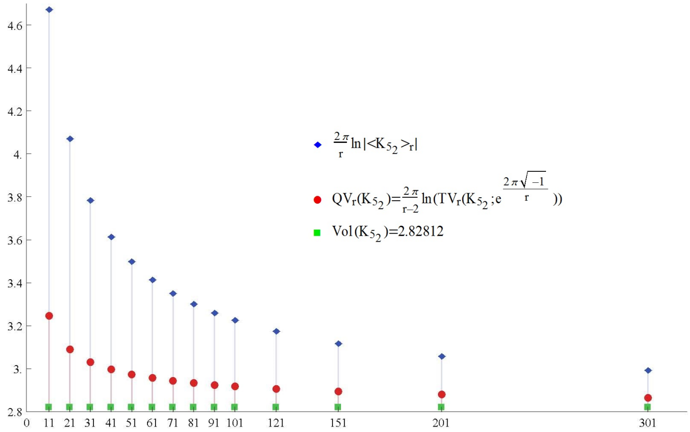


$$TV_r(S^3 \setminus K_{4_1}) = \sum_{(a,b)\in\mathcal{A}_r} |a||b| \begin{vmatrix} a & a & b \\ b & b & a \end{vmatrix} \begin{vmatrix} a & a & b \\ b & b & a \end{vmatrix}$$

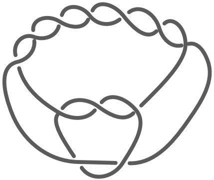
- Knot 4_1 complement



- Knot 5_2 complement



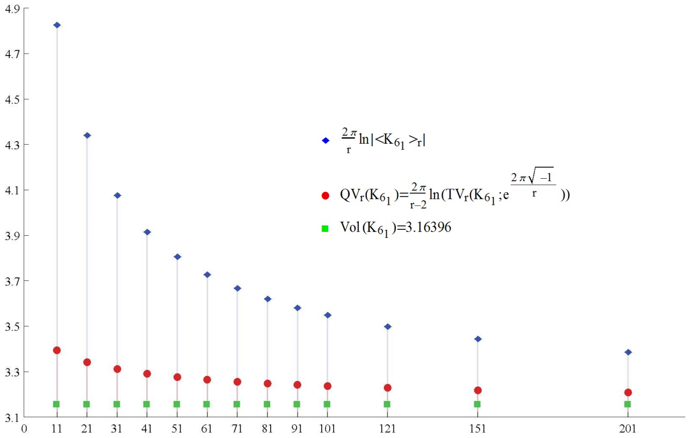
- (-2,3,7) pretzel knot complement (Knot 5_2 sister)



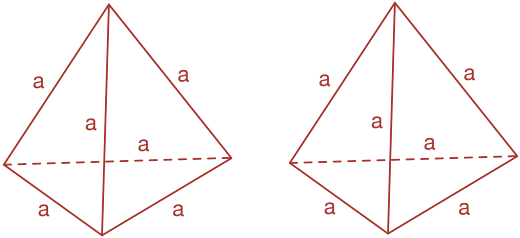
r	9	11	21
$QV_r(K_{5_2})$	3.323939608703	3.252824071268	3.095878648926
$QV_r(M_{3_6})$	3.293628656229	3.229193933374	3.095435748083

r	31	51	101
$QV_r(K_{5_2})$	3.036566821599	2.979253625182	2.923094420758
$QV_r(M_{3_6})$	3.036508195345	2.979253222913	2.923094420761

- Knot 6_1 complement

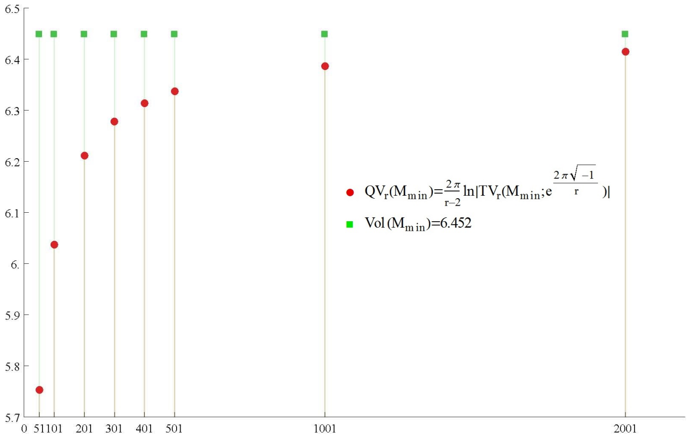


- Minimum with totally geodesic boundary



$$TV_r(M_{\min}) = \sum_{a \in \mathcal{A}_r} |a| \begin{vmatrix} a & a & a \\ a & a & a \end{vmatrix} \begin{vmatrix} a & a & a \\ a & a & a \end{vmatrix}$$

- Minimum with totally geodesic boundary



- Weeks manifold

Tetrahedron	Face 012	Face 013	Face 023	Face 123
0	2 (231)	1 (321)	0 (312)	0 (230)
1	5 (231)	4 (132)	3 (123)	0 (310)
2	6 (123)	4 (023)	3 (032)	0 (201)
3	7 (231)	5 (230)	2 (032)	1 (023)
4	8 (231)	7 (023)	2 (013)	1 (031)
5	8 (320)	6 (023)	3 (301)	1 (201)
6	7 (210)	8 (103)	5 (013)	2 (012)
7	6 (210)	8 (120)	4 (013)	3 (201)
8	7 (301)	6 (103)	5 (210)	4 (201)

- Difficult to compute directly.

- Reshetikhin-Turaev invariant

  M obtained by p-surgery along K,

$$RT_r(M,q) \sim \sum_{n=0}^{r-2} [n+1]\langle e_{n+1}\rangle_{K_p},$$

  where q primitive r-th root of 1. ($SO(3)$ invariant)

- In particular, it is defined at $q(2)$ for r odd.

- Remark

  Reshetikhin-Turaev's original definition ($SU(2)$ invariant) is at primitive $2r$-th root of 1.

  $SO(3)$ invariant is by Blanchet-Habegger-Masbaum-Vogel and Kirby-Melvin.

- Conjecture 2 (Chen-Y.)

  M hyperbolic, closed and oriented. Then

$$\lim_{r \text{ odd},\ r\to\infty} \frac{4\pi}{r-2} \log\Big(RT_r(M, q(2))\Big) = Vol(M) + iCS(M).$$

- Notation

$$Q_r(M) = 2\pi \operatorname{Log} \frac{RT_r(M, q(2))}{RT_{r-2}(M, q(2))}$$

- Equivalent to verify

$$\lim_{r \text{ odd},\ r\to\infty} Q_r(M) = Vol(M) + iCS(M).$$

$M = p$-surgery along K_{4_1}

- $p = -6, \quad Vol(M) + iCS(M) = 1.28449 - 1.34092i.$

r	51	101	151
$Q_r(M)$	$1.22717 - 1.34241i$	$1.28425 - 1.32879i$	$1.28440 - 1.33549i$

r	201	301	501
$Q_r(M)$	$1.28443 - 1.33786i$	$1.28446 - 1.33956i$	$1.28448 - 1.34043i$

- $p = -5, \quad Vol(M) + iCS(M) = 0.98137 - 1.52067i.$

r	51	101	151
$Q_r(M)$	$0.87410 - 1.50445i$	$0.98003 - 1.51521i$	$0.98130 - 1.51712i$

r	201	301	501
$Q_r(M)$	$0.98131 - 1.51865i$	$0.98134 - 1.51977i$	$0.98136 - 1.52035i$

$M = p$-surgery along K_{4_1}

- $p = 5$, $\quad Vol(M) + iCS(M) = 0.98137 + 1.52067i.$

r	51	101	151
$Q_r(M)$	$0.87410 + 1.50445i$	$0.98003 + 1.51521i$	$0.98130 + 1.51712i$

r	201	301	501
$Q_r(M)$	$0.98131 + 1.51865i$	$0.98134 + 1.51977i$	$0.98136 + 1.52035i$

- $p = 6$, $\quad Vol(M) + iCS(M) = 1.28449 + 1.34092i.$

r	51	101	151
$Q_r(M)$	$1.22717 + 1.34241i$	$1.28425 + 1.32879i$	$1.28440 + 1.33549i$

r	201	301	501
$Q_r(M)$	$1.28443 + 1.33786i$	$1.28446 + 1.33956i$	$1.28448 + 1.34043i$

$M = p$-surgery along K_{4_1}

- $p = 7, \quad Vol(M) + iCS(M) = 1.46378 + 1.19653i.$

r	51	101	151
$Q_r(M)$	$1.43670 + 1.10084i$	$1.46354 + 1.18016i$	$1.46367 + 1.18930i$

r	201	301	501
$Q_r(M)$	$1.46372 + 1.19246i$	$1.46375 + 1.19472i$	$1.46377 + 1.19588i$

- $p = 8, \quad Vol(M) + iCS(M) = 1.58317 + 1.07850i.$

r	51	101	151
$Q_r(M)$	$1.57167 + 0.96311i$	$1.58282 + 1.05821i$	$1.58304 + 1.06949i$

r	201	301	501
$Q_r(M)$	$1.58309 + 1.07343i$	$1.58313 + 1.07625i$	$1.58315 + 1.07769i$

$M = p$-surgery along K_{5_2}

- $p = -2, \quad Vol(M) + iCS(M) = 1.84359 - 4.63884i.$

r	51	75	101
$Q_r(M)$	$1.84822 - 4.59073i$	$1.84289 - 4.61357i$	$1.84317 - 4.62490i$

r	125	151	201
$Q_r(M)$	$1.84331 - 4.62978i$	$1.84339 - 4.63265i$	$1.84348 - 4.63536i$

- $p = -1, \quad Vol(M) + iCS(M) = 1.39851 - 4.86783i.$

r	51	75	101
$Q_r(M)$	$1.40943 - 4.84865i$	$1.39808 - 4.85045i$	$1.39817 - 4.85817i$

r	125	151	201
$Q_r(M)$	$1.39827 - 4.86157i$	$1.39834 - 4.86355i$	$1.39841 - 4.86542i$

$M = p$-surgery along K_{5_2}

- $p = 5, \quad Vol(M) + iCS(M) = 0.98137 - 1.52067i.$

r	51	75	101
$Q_r(M)$	$0.87410 - 1.50445i$	$0.96890 - 1.48899i$	$0.98003 - 1.51521i$

r	125	151	201
$Q_r(M)$	$0.98098 - 1.51539i$	$0.98130 - 1.51712i$	$0.98131 - 1.51865i$

- $p = 6, \quad Vol(M) + iCS(M) = 1.41406 - 1.51206i.$

r	51	75	101
$Q_r(M)$	$1.40044 - 1.46756i$	$1.41501 - 1.50631i$	$1.41339 - 1.50836i$

r	125	151	201
$Q_r(M)$	$1.41356 - 1.50968i$	$1.41372 - 1.51042i$	$1.41386 - 1.51113i$

- Summary

  (1) Turaev-Viro invariants extend to 3-manifolds with boundary using ideal triangulations.

  (2) Values at other roots of unity also have rich structures.

- Questions

(1) For M with boundary, is there a complex volumed invariant $RT_r(M,q)$ such that

$$TV_r(M,q) \sim \left|RT_r(M,q)\right|^2 \quad ?$$

(2) What are the asymptotic behaviors of $TV_r(M,q)$ and $RT_r(M,q)$ at other $q(k)$, $k \neq 1$ or 2 ?

(3) Are there physical or geometric interpretations of $RT_r(M,q(2))$?