The inverse spectral problem of discrete Schrödinger operators

Prerequisites: Linear algebra and Calculus

Summary: My plan is to have a group of 5-6 undergraduate students to do a project together. The project will involve advanced reading, and possibly discovering asymptotics of eigenvalues and oscillation theory of eigenfunctions for finite Jacobi matrices, and novel numerical algorithms. Original research papers are expected for successful students. At the beginning, I will lead the project. I am also looking for VAPs, postdocs and senior graduates to work together on supervising the undergraduate students. The regular period for undergraduate research is two or three semesters with weekly meetings: one semester on reading books and papers, and one or two semester on research projects.

Description: The greatest success in the inverse spectral problem theory was achieved for the Sturm-Liouville operator \(-y'' + q(x)y, x \in [0, \pi]\) [1–18]. However, the discrete Schrödinger operators are much less understood, which is one of the motivations of the my project. Denote by \(J(a, \alpha)\) the following matrix,

\[
J(a, \alpha) = \begin{bmatrix}
a_1 & 1 & 0 & \cdots & \alpha \\
1 & a_2 & 1 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\bar{a} & \ddots & \ddots & \ddots & 1 \\
\end{bmatrix},
\]

where \(a = (a_1, a_2, \cdots, a_n)\), and \(\alpha\) can be regarded as the boundary condition. When \(n \to \infty\), the matrix \(J(a, \alpha)\) becomes infinite dimensional and is usually referred as the discrete Schrödinger operator.

The inverse spectral problem is to reconstruct the diagonal sequence \(a = (a_1, a_2, \cdots, a_n)\) or boundary condition \(\alpha\) from its eigenvalues. Here is the tentative plan for undergraduate research.

I. (2 or 3 weeks) Compute the eigenvalues and eigenfunctions for specific models. For example, consider \(a_1 = a_2 = \cdots = a_n = 0\) and compute the eigenvalues and eigenfunctions for \(J(a, 1)\) and \(J(a, 0)\).

II. (3 or 4 weeks) Prove the asymptotics of the eigenvalues \(\lambda_j\) of Sturm-Liouville operators [19]:

\[
\sqrt{\lambda_j} = j + \frac{c_1}{j^2} + \frac{c_2}{j^4} + \cdots + \frac{c_n}{j^n} + O\left(\frac{1}{j^n+1}\right),
\]

where \(c_j\) depends on \(V\), in particular, \(c_1 = -\frac{1}{2\pi} \int_0^\pi V(x)dx\).

III. (3 or 4 weeks) Weyl theory for finite dimensional matrices [20].

IV. Research Projects (the motivations are from Sturm-Liouville operators):

- Construct examples to show that the eigenvalues are not enough to reconstruct the diagonal sequence \(\{a_j\}\) or boundary condition \(\alpha\).
- Tentative to prove that two families of the eigenvalues can determine the \(\{a_j\}, \alpha_1\) and \(\alpha_2\).
- Partial \(\{a_j\}\) +partial eigenvalues determine the whole sequence \(\{a_j\}\).
- The numerical algorithm to reconstruct sequence \(\{a_j\}\) by the eigenvalues. It is expected to use the Weyl theory in [20] to study those problems.

Further study:
The undergraduate research project “The inverse spectral problem for discrete Schödinger operators” is linked to several graduate projects. For example,

- The inverse spectral problem of infinitely dimensional matrices with/without continuous spectral components.
- The inverse nodal problem: reconstruct the diagonal sequence \(a = (a_1, a_2, \cdots, a_n)\) or boundary condition \(\alpha\) from the “zeros” of the eigenfunctions.
- Construct the Kolmogorov-Arnold-Moser (KAM) tori of nonlinear Schrödinger equations and nonlinear wave equations for prescribed potentials.

I believe those topics are very suitable for PhD theses.
References