Amoebas!

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Amoebas?
Definition

The **Newton polytope** of a (Laurent) polynomial $f$ is the convex hull of the exponent vectors of the monomials of $f$.

\[
\text{Newt}(x^5y^3 - 75x^4y - 10x^3y^2 - 7xy^4 - 17x^2y + 71xy - 44)
\]
Amoebas!

**Definition**

Let \( f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \).

\[
Z_f := \{(\zeta_1, \ldots, \zeta_n) \in (\mathbb{C}^*)^n \mid f(\zeta_1, \ldots, \zeta_n) = 0\}
\]

\[
\text{Amoeba}(f) := \{(\log |\zeta_1|, \ldots, \log |\zeta_n|) \in \mathbb{R}^n \mid f(\zeta_1, \ldots, \zeta_n) = 0\}
\]

The **amoeba** of the polynomial \( f \) is \( \text{Log}(Z_f) \).
Univariate example

\[ f(x_1) := x_1^2 + x_1 + 1 \]

Newt(\(f\)):

Amoeba(\(f\)):

\( \mathbb{R}^1 \setminus \text{Amoeba}(f) \):
Tentacles

[Viro, 2002]
Another one

\[ f(x, y) = x^3 + y^3 + 1 + x^2 y^3 + x^3 y^2 + 14x + 14x^2 + 14y + 14y^2 + 14x^3 y + 14xy^3 + 196xy + 196x^2 y + 196xy^2 + 196x^2 y^2 \]

[Bogaard, 2015]
Theorem (Gelfand, Kapranov, and Zelevinsky, 1990)

Given any $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$,

- $\mathbb{R}^n \setminus \text{Amoeba}(f)$ is a finite union of open convex sets.

Furthermore,

- for each vertex $\gamma$ of $\text{Newt}(f)$, there is a unique connected component of $\mathbb{R}^n \setminus \text{Amoeba}(f)$ containing a translate of the outer normal cone of $\gamma$.

- the number of compact connected components of $\mathbb{R}^n \setminus \text{Amoeba}(f)$ is $\leq |\mathbb{Z}^n \cap \text{Int}(	ext{Newt}(f))|$. 
Example

\[ f(x, y) = 1 + y^2 + x^3 - 10xy \]

Newt(\(f\)):

Amoeba(\(f\)):
Same support, different amoeba

\[ g(x, y) = 1 + y^2 + x^3 - \frac{1}{2}xy \]

Newt\((g)\) is:

Amoeba\((g)\) is:
Planar Amoebas

Proposition

Although a planar amoeba is not bounded, its area is finite. For $f \in \mathbb{C}[x, y],$

$$\text{Area}(\text{Amoeba}(f)) \leq \pi^2 \text{Area}(\text{Newt}(f)).$$

Note: This does not generalize to higher dimensions.
Let \( f(x) \) be a Laurent polynomial. All the components of \( \mathbb{R}^n \setminus \text{Amoeba}(f) \) are convex subsets in \( \mathbb{R}^k \). The components of the complement are in bijective correspondence with Laurent series expansions of the rational function \( 1/f(x) \).

There is a vector \( v \) in the outer normal cone of the vertex \( \gamma \) of \( \text{Newt}(f) \) such that the Laurent series of \( 1/f(x) \) converges absolutely for any \( x \in (\mathbb{C}^*)^n \) for which the vector \( \log(x) = (\log |x_1|, \ldots, \log |x_n|) \) lies in the affine cone \((v + \text{outer normal cone of } \gamma)\).
Thanks and
References

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Semi-definite Programming

I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky (1994)
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Oleg Viro (2002)
WHAT IS an amoeba?
*Notices of the AMS* 49(8), 916–917.

Milo Bogaard (2015)
Introduction to amoebas and tropical geometry
*Master Thesis*, Universiteit van Amsterdam
Proposition (GKZ)

Let $f(x)$ be a Laurent polynomial. All the components of $\mathbb{R}^n \setminus \text{Amoeba}(f)$ are convex subsets in $\mathbb{R}^k$. The components of the complement are in bijective correspondence with Laurent series expansions of the rational function $1/f(x)$.

Proof.

- For $f(x_1, \ldots, x_n) = \sum a_\omega x^\omega$, let $\gamma \in \text{Newt}(f)$ be any vertex.
- Write $f(x) = a_\gamma x^\gamma \left(1 + \sum_{\omega \neq \gamma} \frac{a_\omega}{a_\gamma} x^{\omega-\gamma}\right) = a_\gamma x^\gamma (1 + g(x))$.
- Using the geometric series, construct the Laurent expansion $\frac{1}{f(x)} = a_\gamma^{-1} x^{-\gamma} (1 - g(x) + g(x)^2 - \cdots)$
- Facts: this expansion is a well-defined Laurent series whose exponents lie in the affine cone $-\gamma + \mathbb{R}_{>0} \cdot (\text{Newt}(f) - \gamma) \in \mathbb{R}^n$