Tarski Circle Squaring

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Two polygons are **dissection congruent** if the first can be cut into finitely many polygons that are then rearranged in order to form the second (ignoring boundaries).
Wallace-Bolyai-Gerwien

Two shapes that are dissection congruent must have the same measure.
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Theorem (Wallace-Bolyai-Gerwien Theorem, 1807, 1833)

Any two polygons with the same measure are dissection congruent.
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**Theorem (Wallace-Bolyai-Gerwien Theorem, 1807, 1833)**

*Any two polygons with the same measure are dissection congruent.*

Dissection congruence is an equivalence relation. So, it is enough to show that any polygon is dissection congruent to a square.
Wallace-Bolyai-Gerwien Proof

Cut the polygon up into triangles.
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Change each triangle into a parallelogram and then a rectangle.
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Transform each rectangle into a square.
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Cut the polygon up into triangles.

Change each triangle into a parallelogram and then a rectangle.

Transform each rectangle into a square.

Combine each set of two squares.
Hilbert’s Third Problem

Question (Hilbert, 1900)

Are any two polytopes of the same volume in 3 dimensions dissection congruent?
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Theorem (Dehn, 1902)

NO! The cube and the regular tetrahedron are not dissection congruent.

Uses an invariant now known as the Dehn invariant.
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**Question (Hilbert, 1900)**

Are any two polytopes of the same volume in 3 dimensions dissection congruent?

**Theorem (Dehn, 1902)**

**NO!** The cube and the regular tetrahedron are not dissection congruent.

Uses an invariant now known as the Dehn invariant.

**Theorem (Sydler, 1965)**

Two polytopes in 3 dimensions are dissection congruent if and only if they share the same Dehn invariant and have the same volume.

It is open if there is an analogue of the Dehn invariant in higher dimensions.
A, B ∈ \( \mathbb{R}^n \) are called **equidecomposable** if A can be partitioned into finitely many pieces, which can be rearranged by isometries to partition B.
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Question (Tarski’s Circle Squaring Problem, 1925)

Given a disc and a square (of the same area) in ℝ^2, are they equidecomposable?
A, B ∈ \( \mathbb{R}^n \) are called **equidecomposable** if A can be partitioned into finitely many pieces, which can be rearranged by isometries to partition B.

**Question (Tarski’s Circle Squaring Problem, 1925)**

*Given a disc and a square (of the same area) in \( \mathbb{R}^2 \), are they equidecomposable?*

In \( \mathbb{R} \) and \( \mathbb{R}^2 \), there exist finitely additive isometry invariant measures extending the Lebesgue measure and defined on every subset. Note: these cannot exist in \( \mathbb{R}^n \) for \( n \geq 3 \) because of the Banach-Tarski paradox.
**Scissor Congruence**

$A, B \in \mathbb{R}^2$ are **scissor congruent** if $A$ can be cut into finitely many pieces-each of which is homeomorphic to a circle and with a boundary of finite length—which can be rearranged to form $B$(ignoring boundaries).

![Diagram of scissor congruence](image)

Corollary (Dubins-Hirsch-Karush, 1964)

The disc and the square are not scissors congruent.
Scissor Congruence

$A, B \in \mathbb{R}^2$ are **scissor congruent** if $A$ can be cut into finitely many pieces—each of which is homeomorphic to a circle and with a boundary of finite length—which can be rearranged to form $B$ (ignoring boundaries).

If you create or destroy circular concave perimeter, then you create exactly the same amount of circular convex perimeter and vice versa. So

\[
\text{circular concave perimeter} - \text{circular convex perimeter}
\]

is an invariant of scissor congruence.

**Corollary (Dubins-Hirsch-Karush, 1964)**

*The disc and the square are not scissors congruent.*
Laczkovich’s Circle Squaring

Theorem (Laczkovich, 1990)

*The square is equidecomposable with the disc of the same area!*
Laczkovich’s Circle Squaring

**Theorem (Laczkovich, 1990)**

*The square is equidecomposable with the disc of the same area!*

More generally,

**Theorem (Laczkovich, 1992)**

*If \( A, B \subseteq \mathbb{R}^k \) are bounded sets with the same positive Lebesgue measure whose boundaries have upper Minkowski dimension less than \( k \), then \( A \) and \( B \) are equidecomposable.*
"The Proof" for $\mathbb{R}^2$

First Idea: **Work in the torus**
Scale and translate $A$ and $B$ so they lie within $[0, 1)^2$. $A$ and $B$ are now equidecomposable as subsets of $\mathbb{T}^2$ if and only if they are equidecomposable in $\mathbb{R}^2$ although potentially with more pieces.
"The Proof"

Second Idea: **Random Translations**

Choose two translations \( u_1, u_2 \in T_2 \) at random. These give us an action of \( \mathbb{Z}_2 \) on \( T_2 \).

With probability 1, the translations we pick give us a free action of \( \mathbb{Z}_2 \) and we can visualize each orbit as a copy of \( \mathbb{Z}_2 \).

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"The Proof"

Second Idea: Random Translations

Choose two translations $u_1, u_2 \in \mathbb{T}^2$ at random. These give us an action of $\mathbb{Z}^2$ on $\mathbb{T}^2$.

With probability 1, the translations we pick give us a free action of $\mathbb{Z}^2$ and we can visualize each orbit as a copy of $\mathbb{Z}^2$. 
"The Proof"

We connect points in our orbits to get a graph. Now, an equidecomposition between $A$ and $B$ is just a matching problem where we match the vertices of our orbit that are in $A$ with the vertices that are in $B$ by only matching points that are a fixed bounded distance apart.
"The Proof"

For us to have a matching as described, we need big bounded size boxes in our graph to have approximately as many points from $A$ as points from $B$. (i.e. satisfy the conditions of Hall’s Matching Theorem).
"The Proof"

This is where Laczkovich does the heavy lifting and through a series of number theoretic and combinatorial lemmas, he proves a quantitative ergodic theorem that gives us what we want. (This is where upper Minkowski Dimension comes in). Then, we just apply Hall’s Matching Theorem to get a matching.
Borel Circle Squaring

Question (Wagon, 1985)

Can you solve Tarski’s Circle Squaring Problem with Borel pieces?

The only nonconstructive part of Laczkovich’s proof is the use of Hall’s Matching Theorem.
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Theorem (Marks-Unger, 2017)

*Laczkovich’s Theorem is true with Borel pieces.*