Additivity of higher rho invariants and nonrigidity of topological manifolds

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Abstract

Let $X$ be a closed oriented topological manifold of dimension $n$. The structure group $S^{TOP}(X)$ is the abelian group of equivalence classes of all pairs $(f, M)$ such that $M$ is a closed oriented manifold and $f: M \to X$ is an orientation-preserving homotopy equivalence. The main purpose of this article is to prove that the higher rho invariant is a group homomorphism from the structure group $S^{TOP}(X)$ of $X$ to the analytic structure group $K_n(C^*_{L,0}(\tilde{X})^\Gamma)$ of $X$. Here $\tilde{X}$ is the universal cover of $X$, $\Gamma = \pi_1 X$, and $C^*_{L,0}(\tilde{X})^\Gamma$ is a certain $C^*$-algebra. We then apply this result to study non-rigidity of topological manifolds. More precisely, we give a lower bound for the free rank of a version of reduced structure group $\tilde{S}_n(X)$ of $X$, by the number of torsion elements in $\pi_1 X$. Furthermore, we introduce a notion of homological higher rho invariant, which can be used to construct many elements in the structure group of a closed oriented topological manifold, even when the fundamental group of the manifold is torsion free.

1 Introduction

Let $X$ be a closed oriented topological manifold of dimension $n$. The structure group $S^{TOP}(X)$ is the abelian group of equivalence classes of all pairs $(f, M)$ such that $M$ is a closed oriented manifold and $f: M \to X$ is an orientation-preserving homotopy equivalence. The main result of this article is to prove that the higher rho invariant defines a group homomorphism from $S^{TOP}(X)$ to $K_n(C^*_{L,0}(\tilde{X})^\Gamma)$, where $\Gamma = \pi_1 X$, $\tilde{X}$ is the universal cover of $X$ and $C^*_{L,0}(\tilde{X})^\Gamma$ is a certain geometric $C^*$-algebra. See Section

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4.6 for the precise definition of the higher rho invariant. An essential step of the proof is a new description of the topological structure group in terms of smooth manifolds with boundary (see Section 3, especially Section 3.4, for more details). This new description uses more objects and an equivalence relation broader than $h$-cobordism, which allows us to replace the topological manifolds in the usual definition of $\mathcal{S}^{\text{TOP}}(X)$ by smooth manifolds with boundary. Such a description has analytic advantages and leads to a transparent group structure. Indeed, the group structure is given by disjoint union.

Recall that there are two different ways for a self-homotopy $h$ of $X$ to act on $\mathcal{S}^{\text{TOP}}(X)$ (cf. Section 8). One action is functorial, and induces a group isomorphism, denoted by $\alpha_h$; the other action is not functorial, and induces a set bijection. Let $\tilde{\mathcal{S}}_n(X)$ be the quotient group of $\mathcal{S}^{\text{TOP}}(X)$ modulo the subgroup generated by elements of the form $\theta - \alpha_h(\theta)$ for all $\theta \in \mathcal{S}^{\text{TOP}}(X)$ and all orientation-preserving self-homotopy equivalences $h$ of $X$ (see Definition 8.1). We call $\tilde{\mathcal{S}}_n(X)$ a reduced structure group of $X$. Similarly, there is another version of reduced structure group $\tilde{\mathcal{S}}^{\text{TOP}}(X)$ of $X$ induced by the other action. We apply our main theorem, combining with the work in [61], to give a lower bound of the free rank of $\tilde{\mathcal{S}}_n(X)$. There is strong evidence which suggests that an analogue holds for $\tilde{\mathcal{S}}^{\text{TOP}}(X)$ as well.

In the case where the strong Novikov conjecture holds for $\Gamma = \pi_1 X$, we can define a homological higher rho invariant. We call this invariant the Novikov rho invariant for obvious reasons to be explained in Section 7. The Novikov rho invariant can be used to construct many elements in $\mathcal{S}^{\text{TOP}}(X)$, even when $\Gamma = \pi_1 X$ is torsion free. It is equivalent to the higher rho invariant, if the Baum-Connes conjecture holds for $\Gamma$.

We point out that, in the odd dimensional case, the higher rho invariant for signature operators in the literature (cf. [23, Section 3] [37, Remark 4.6] [67]) is twice of the higher rho invariant of this paper, cf. Remark 6.7 and Theorem 6.9 below.

The higher rho invariant map on the structure set of a smooth manifold was first introduced by Higson and Roe [23]. Zenobi extended the higher rho invariant map (as a map of sets) to topological manifolds [67]. In the cyclic cohomology setting, Lott studied the higher eta invariant (a close relative of the higher rho invariant) under certain conditions [32].

In the topological setting, the first author studied a different higher rho invariant under certain conditions [56]. Those invariants can be defined for certain manifolds, not just elements in structure groups. The higher rho invariant defined in the current paper can also be generalized to those settings. For example, if $M$ is a manifold and $\mathbb{Z}/2$ acts freely and homologically trivially in the sense of [55], then this involution has an invariant living in the analytic structure set of $M$ (which away from the prime 2 is realized). This plays an important role in the bordism of homologically trivial actions.

Our approach to the higher rho invariant is very much inspired by the work of Higson and Roe on analytic surgery long exact sequence for smooth manifolds and structure sets of smooth manifolds [21, 22, 23]. In their work, Higson and Roe proved that in the smooth setting, the higher rho invariant establishes a set theoretic commutative diagram between the smooth surgery sequence and the analytic surgery sequence. Our main result implies that in the topological setting, the higher rho invariant can be used to construct a commutative diagram of abelian groups between the topological
surgery sequence and the analytic surgery sequence (see Section 6). There are other equivalent ways of studying the topological surgery sequence. Our approach in the current paper is closer to those of Wall [54] and Quinn [39], and is more geometric in nature. If we were to take a more algebraic approach by using Ranicki’s algebraic surgery long exact sequence [42], then many of the discussions in Section 3 can be avoided. In particular, if we use Ranicki’s algebraic surgery long exact sequence, then the techniques from [21, 22, 23] can be adapted more directly to the topological setting. On the other hand, our geometric approach appears to be more intuitive. We remark that the rational additivity of higher rho invariant for finite fundamental groups was proved by Crowley and Macko [12].

The paper is organized as follows. In Section 2, we recall some standard definitions from coarse geometry. In Section 3, we introduce a definition of the structure group of topological manifolds based on ideas of Wall and ideas from control topology. One essential advantage of our definition is that every element of the structure group can be represented by a smooth manifold with boundary, which is crucial for the construction of our higher rho invariant. Another essential advantage is that the group structure of the structure group becomes obvious. Indeed, the group structure can be described as disjoint union. We prove this alternative definition of structure group is canonically equivalent to the classical definition. In Section 4, we define the higher rho invariant map $\rho: \mathcal{S}_{\text{TOP}}(X) \to K_n(C_{L,0}^*(\tilde{X})^F)$, and prove that it is a group homomorphism. The fact that the higher rho invariant map is well-defined is proved in Section 5. In Section 6, we compare the geometric surgery long exact sequence to the analytic surgery long exact sequence. In particular, the geometric surgery long exact sequence maps naturally to the analytic surgery long exact sequence, and they fit into a commutative diagram of exact sequences (cf. Diagram (9)). In Section 7, we introduce the Novikov rho invariant, which is a homological version of the higher rho invariant. In Section 8, we apply the main results of the paper to study the non-rigidity problem of topological manifolds. We give a lower bound of the free-rank of $\mathcal{S}_n(X)$, under certain mild conditions. In Section 9, we outline how to adapt the methods in this paper to handle signature operators arising from Lipschitz structures on topological manifolds.

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2 Preliminaries

In this section, we briefly recall some standard definitions. We refer the reader to [16, 44, 63] for more details.

Let $X$ be a proper metric space. That is, every closed ball in $X$ is compact. An $X$-module is a separable Hilbert space equipped with a $*$-representation of $C_0(X)$, the algebra of all continuous functions on $X$ which vanish at infinity. An $X$-module is called nondegenerate if the $*$-representation of $C_0(X)$ is nondegenerate. An $X$-module
is said to be standard if no nonzero function in $C_0(X)$ acts as a compact operator.

**Definition 2.1.** Let $H_X$ be a $X$-module and $T$ a bounded linear operator acting on $H_X$.

(i) The propagation of $T$ is defined to be $\sup\{d(x, y) \mid (x, y) \in \text{Supp}(T)\}$, where $\text{Supp}(T)$ is the complement (in $X \times X$) of the set of points $(x, y) \in X \times X$ for which there exist $f, g \in C_0(X)$ such that $gTf = 0$ and $f(x) \neq 0$, $g(y) \neq 0$;

(ii) $T$ is said to be locally compact if $fT$ and $Tf$ are compact for all $f \in C_0(X)$;

(iii) $T$ is said to be pseudo-local if $[T, f]$ is compact for all $f \in C_0(X)$.

**Definition 2.2.** Let $H_X$ be a standard nondegenerate $X$-module and $B(H_X)$ the set of all bounded linear operators on $H_X$.

(i) The Roe algebra of $X$, denoted by $C^*(X)$, is the $C^*$-algebra generated by all locally compact operators with finite propagations in $B(H_X)$.

(ii) $C^*_L(X)$ is the $C^*$-algebra generated by all bounded and uniformly norm-continuous functions $f : [0, \infty) \to C^*(X)$ such that

\[
\text{propagation of } f(t) \to 0, \text{ as } t \to \infty.
\]

(iii) $C^*_{L,0}(X)$ is the kernel of the evaluation map

\[
\text{ev} : C^*_L(X) \to C^*(X), \quad \text{ev}(f) = f(0).
\]

In particular, $C^*_{L,0}(X)$ is an ideal of $C^*_L(X)$.

(iv) If $Y$ is a subspace of $X$, then $C^*_L(Y; X)$ (resp. $C^*_{L,0}(Y; X)$) is defined to be the closed subalgebra of $C^*_L(X)$ (resp. $C^*_{L,0}(X)$) generated by all elements $f$ such that there exist $c_t > 0$ satisfying $\lim_{t \to \infty} c_t = 0$ and $\text{Supp}(f(t)) \subset \{(x, y) \in X \times X \mid d((x, y), Y \times Y) \leq c_t\}$ for all $t$.

**Remark 2.3.** Similarly, we can also define maximal versions of $C^*_L(X)$, $C^*_{L,0}(X)$, $C^*_L(Y; X)$, $C^*_{L,0}(Y; X)$, cf. [16].

Now in addition we assume that a discrete group $\Gamma$ acts properly on $X$ by isometries. In particular, if the action of $\Gamma$ is free, then $X$ is simply a $\Gamma$-covering of $X/\Gamma$.

Now let $H_X$ be a $X$-module equipped with a covariant unitary representation of $\Gamma$. If we denote the representation of $C_0(X)$ by $\varphi$ and the representation of $\Gamma$ by $\pi$, this means

\[
\pi(\gamma)(\varphi(f)v) = \varphi(f^\gamma)(\pi(\gamma)v),
\]

where $f \in C_0(X)$, $\gamma \in \Gamma$, $v \in H_X$ and $f^\gamma(x) = f(\gamma^{-1}x)$. In this case, we call $(H_X, \Gamma, \varphi)$ a covariant system.

**Definition 2.4 ([66]).** A covariant system $(H_X, \Gamma, \varphi)$ is called admissible if

(1) the $\Gamma$-action on $X$ is proper and cocompact;
(2) $H_X$ is a nondegenerate standard $X$-module;

(3) for each $x \in X$, the stabilizer group $\Gamma_x$ acts on $H_X$ regularly in the sense that the action is isomorphic to the action of $\Gamma_x$ on $l^2(\Gamma_x) \otimes H$ for some infinite dimensional Hilbert space $H$. Here $\Gamma_x$ acts on $l^2(\Gamma_x)$ by translations and acts on $H$ trivially.

We remark that for each locally compact metric space $X$ with a proper and cocompact isometric action of $\Gamma$, there exists an admissible covariant system $(H_X, \Gamma, \varphi)$. Also, we point out that the condition (3) above is automatically satisfied if $\Gamma$ acts freely on $X$. If no confusion arises, we will denote an admissible covariant system $(H_X, \Gamma, \varphi)$ by $H_X$ and call it an admissible $(X, \Gamma)$-module.

**Remark 2.5.** For each $(X, \Gamma)$ above, there always exists an admissible $(X, \Gamma)$-module $H$. In particular, $H \oplus H$ is an admissible $(X, \Gamma)$-module for every $(X, \Gamma)$-module $H$.

**Definition 2.6.** Let $X$ be a locally compact metric space with a proper and cocompact isometric action of $\Gamma$. If $H_X$ is an admissible $(X, \Gamma)$-module, we denote by $\mathbb{C}[X]^\Gamma$ the $\ast$-algebra of all $\Gamma$-invariant locally compact operators with finite propagations in $\mathcal{B}(H_X)$. We define $C^\ast(X)^\Gamma$ to be the completion of $\mathbb{C}[X]^\Gamma$ in $\mathcal{B}(H_X)$.

Similarly, we can also define $C^\ast_t(X)^\Gamma$, $C^\ast_{L,0}(X)^\Gamma$, $C^\ast_L(Y; X)^\Gamma$, $C^\ast_{L,0}(Y; X)^\Gamma$.

**Remark 2.7.** Up to isomorphism, $C^\ast(X) = C^\ast(X, H_X)$ does not depend on the choice of the standard nondegenerate $X$-module $H_X$. The same holds for $C^\ast_L(X)$, $C^\ast_{L,0}(X)$, $C^\ast_L(Y; X)$, $C^\ast_{L,0}(Y; X)$ and their $\Gamma$-equivariant versions.

**Remark 2.8.** Note that we can also define maximal versions of all $\Gamma$-equivariant $C^\ast$-algebras above. For example, we define the maximal $\Gamma$-invariant Roe algebra $C^\ast_{\text{max}}(X)^\Gamma$ to be the completion of $\mathbb{C}[X]^\Gamma$ under the maximal norm:

$$\|a\|_{\text{max}} = \sup_{\phi} \{\|\phi(a)\| \mid \phi : \mathbb{C}[X]^\Gamma \to \mathcal{B}(H') \text{ a } \ast\text{-representation}\}.$$  

Similarly, we can define the maximal versions of $C^\ast_L(X)^\Gamma$, $C^\ast_{L,0}(X)^\Gamma$, $C^\ast_L(Y; X)^\Gamma$, and $C^\ast_{L,0}(Y; X)^\Gamma$. See for example [61] for a more detailed discussion.

## 3 Structure groups of topological manifolds

In this section we give a definition of the structure group of topological manifolds, based on ideas of Wall and of controlled topology. It is very similar in spirit to the definition given by Ranicki in [42], but has some advantages for our analytic purposes. One essential advantage of our definition is that every element of the structure group can be represented by a smooth manifold with boundary, which is crucial for the construction of our higher rho invariant. Another essential advantage is that the group structure of the structure group becomes obvious. Indeed, the group structure can be described as disjoint union.

Given an oriented closed topological manifold $X$, the structure set $\mathcal{S}^\text{TOP}(X)$ is defined to be the equivalence classes of orientation-preserving homotopy equivalences
\( f: M \to X \). Here \( M \) is an oriented closed topological manifold. Two orientation-preserving homotopy equivalences \( f: M \to X \) and \( g: N \to X \) are equivalent if there exists an h-cobordism \((W; M, N)\) with an orientation-preserving homotopy equivalence \( F: (W; M, N) \to (X \times I; X \times \{0\}, X \times \{1\}) \) such that \( F|_M = f \) and \( F|_N = g \). It is known that \( S^{\text{TOP}}(X) \) has a group structure, cf. [5, 31, 42].

More generally, let \( X \) be a (not necessarily oriented) closed topological manifold and \( Y \) be a topological space. We call a continuous map \( \varphi: Y \to X \) a control map of \( Y \) if there exist proper continuous homotopy equivalences \( f: (M; w_M) \to (X; w) \) and \( \psi: (N; w_N) \to (X; w) \) such that the diameter of the set \( \Phi(\{1\}) = \{\Phi(H_s(z, t)) \mid 0 \leq s \leq 1\} \) goes uniformly (i.e. independent of \( z \in Y \)) to zero, as \( t \to \infty \);

\( \Phi: Z \times [1, \infty) \to X \times [1, \infty) \) and \( \Psi: Y \times [1, \infty) \to X \times [1, \infty) \),

\( F: Y \times [1, \infty) \to Z \times [1, \infty) \) and \( G: Z \times [1, \infty) \to Y \times [1, \infty) \)

satisfying the following conditions:

(1) \( \Phi \circ F = \Psi \);

(2) \( F|_{Y \times \{1\}} = f \), \( \Phi|_{Z \times \{1\}} = \varphi \), and \( \Psi|_{Y \times \{1\}} = \psi \);

(3) there exists a proper continuous homotopy \( \{H_s\}_{0 \leq s \leq 1} \) between

\( H_0 = F \circ G \) and \( H_1 = \text{Id}: Z \times [1, \infty) \to Z \times [1, \infty) \)

such that the diameter of the set \( \Phi(H(z, t)) = \{\Phi(H_s(z, t)) \mid 0 \leq s \leq 1\} \) goes uniformly (i.e. independent of \( z \in Z \)) to zero, as \( t \to \infty \);

(4) there exists a proper continuous homotopy \( \{H'_s\}_{0 \leq s \leq 1} \) between

\( H'_0 = G \circ F \) and \( H'_1 = \text{Id}: Y \times [1, \infty) \to Y \times [1, \infty) \)

such that the diameter of the set \( \Psi(H'(y, t)) = \{\Psi(H'_s(y, t)) \mid 0 \leq s \leq 1\} \) goes uniformly to zero, as \( t \to \infty \);

3.1 A definition of the structure group

In this subsection, we introduce a definition of the structure group of a topological manifold based on ideas of Wall and of controlled topology. Let \( X \) be a closed topological manifold. Fix a metric on \( X \) that agrees with the topology of \( X \). Note that such a metric always exists.

**Definition 3.1.** Let \( Y \) be a topological space. We call a continuous map \( \varphi: Y \to X \) a control map of \( Y \).

**Definition 3.2.** Let \( Y \) and \( Z \) two compact Hausdorff spaces equipped with continuous control maps \( \psi: Y \to X \) and \( \varphi: Z \to X \). A continuous map \( f: Y \to Z \) is said to be an infinitesimally controlled homotopy equivalence over \( X \), if there exist proper continuous maps

\( \Phi: Z \times [1, \infty) \to X \times [1, \infty) \) and \( \Psi: Y \times [1, \infty) \to X \times [1, \infty) \),

\( F: Y \times [1, \infty) \to Z \times [1, \infty) \) and \( G: Z \times [1, \infty) \to Y \times [1, \infty) \)

satisfying the following conditions:

(1) \( \Phi \circ F = \Psi \);

(2) \( F|_{Y \times \{1\}} = f \), \( \Phi|_{Z \times \{1\}} = \varphi \), and \( \Psi|_{Y \times \{1\}} = \psi \);

(3) there exists a proper continuous homotopy \( \{H_s\}_{0 \leq s \leq 1} \) between

\( H_0 = F \circ G \) and \( H_1 = \text{Id}: Z \times [1, \infty) \to Z \times [1, \infty) \)

such that the diameter of the set \( \Phi(H(z, t)) = \{\Phi(H_s(z, t)) \mid 0 \leq s \leq 1\} \) goes uniformly (i.e. independent of \( z \in Z \)) to zero, as \( t \to \infty \);

(4) there exists a proper continuous homotopy \( \{H'_s\}_{0 \leq s \leq 1} \) between

\( H'_0 = G \circ F \) and \( H'_1 = \text{Id}: Y \times [1, \infty) \to Y \times [1, \infty) \)

such that the diameter of the set \( \Psi(H'(y, t)) = \{\Psi(H'_s(y, t)) \mid 0 \leq s \leq 1\} \) goes uniformly to zero, as \( t \to \infty \);
We will also need the following notion of restrictions of homotopy equivalences gaining infinitesimal control on parts of spaces. Suppose $M$ is a topological manifold with boundary $\partial M$. We define the space of $M$ attached with a cylinder by

$$CM = M \cup_{\partial M} (\partial M \times [1, \infty)).$$

Suppose $(M, \partial M, \varphi)$ and $(N, \partial N, \psi)$ are two manifold pairs equipped with continuous maps $\varphi: M \to X$ and $\psi: N \to X$. Let $f: (N, \partial N) \to (M, \partial M)$ be a homotopy equivalence with $\varphi \circ f = \psi$. Suppose $g: (M, \partial M) \to (N, \partial N)$ is a homotopy inverse of $f$. Note that $\psi \circ g \neq \varphi$ in general. Let $\{h_s\}_{0 \leq s \leq 1}$ be a homotopy between $f \circ g$ and $\text{Id}: (M, \partial M) \to (M, \partial M)$. Similarly, let $\{h'_s\}_{0 \leq s \leq 1}$ be a homotopy between $g \circ f$ and $\text{Id}: (N, \partial N) \to (N, \partial N)$.

**Definition 3.3.** With the above notation, we say that on the boundary $f$ restricts to an infinitesimally controlled homotopy equivalence $f|_{\partial N}: \partial N \to \partial M$ over $X$, if there exist proper continuous maps $\Phi: CM \to X \times [1, \infty)$ and $\Psi: CN \to X \times [1, \infty)$, $F: CN \to CM$ and $G: CM \to CN$,

a proper continuous homotopy $\{H_s\}_{0 \leq s \leq 1}$ between

$$H_0 = F \circ G \text{ and } H_1 = \text{Id}: CM \to CM$$

and a proper continuous homotopy $\{H'_s\}_{0 \leq s \leq 1}$ between

$$H'_0 = G \circ F \text{ and } H'_1 = \text{Id}: CN \to CN$$

satisfying the following conditions:

1. $\Phi|_M = \varphi$, $\Psi|_N = \psi$, $F|_N = f$, $G|_M = g$, $H_s|_M = h_s$, and $H'_s|_N = N$;
2. $\Phi \circ F = \Psi$;
3. the diameter of the set $\Phi(H(a, t)) = \{\Phi(H_s(a, t)) \mid 0 \leq s \leq 1\}$ goes uniformly to zero, for all $(a, t) \in \partial M \times [1, \infty)$, as $t \to \infty$;
4. the diameter of the set $\Psi(H'(b, t)) = \{\Psi(H'_s(b, t)) \mid 0 \leq s \leq 1\}$ goes uniformly to zero, for all $(b, t) \in \partial N \times [1, \infty)$ as $t \to \infty$;

In the following, we adopt the notion of manifold $k$-ads from Wall’s book [54, Chapter 0]. For example, a manifold 1-ad is a manifold with boundary.

Let $w: \pi_1(X) \to \mathbb{Z}/2$ be the orientation character of $X$. We define $\mathcal{S}_n(X; w)$ to be the set of equivalence classes as follows.

**Definition 3.4 (Objects for the definition of $\mathcal{S}_n(X; w)$).** An object consists of the following data:
Figure 1: Object \( \theta = (M, \partial M, \varphi, N, \partial N, \psi, f) \) for the definition of \( S_n(X; w) \), where red curves mean infinitesimally controlled homotopy equivalence, and dark curves mean homotopy equivalence.

1. two manifold 1-ads \((M, \partial M)\) and \((N, \partial N)\) with \( \dim M = \dim N = n \), where \( \partial M \) (resp. \( \partial N \)) is the boundary of \( M \) (resp. \( N \)),

2. continuous maps \( \varphi : M \to X \) and \( \psi : N \to X \) so that \( \varphi^*(w) \) and \( \psi^*(w) \) describe the orientation characters of \( M \) and \( N \) respectively,

3. a homotopy equivalence of manifold 1-ads \( f : (N, \partial N) \to (M, \partial M) \) such that \( \varphi \circ f = \psi \). Moreover, on the boundary \( f \) restricts to an infinitesimally controlled homotopy equivalence \( f|_{\partial N} : \partial N \to \partial M \) over \( X \).

Remark 3.5. In fact, for the case where \( X \) is non-orientable, we need to be slightly more careful with how the orientation character \( w : \pi_1(X) \to \mathbb{Z}/2 \) is used in the Definition 3.4. The more appropriate way, which ensures functoriality, is to replace the orientation character by a principal \( \mathbb{Z}/2 \)-bundle. We refer the reader to a paper of Farrell and Hsiang [14, Section 3] for more details.

If \( \theta = (M, \partial M, \varphi, N, \partial N, \psi, f) \) is an object, then we denote by \(-\theta\) to be the same object except that the fundamental classes of \( M \) and \( N \) switch sign. Here the notion of fundamental class refers to the fundamental class defined in homology with local coefficients by using the orientation character. For two objects \( \theta_1 \) and \( \theta_2 \), we write \( \theta_1 + \theta_2 \) to be the disjoint union of \( \theta_1 \) and \( \theta_2 \). This sum operation is clearly commutative and associative, and admits a zero element: the object with \( M \) (hence \( N \)) empty. We denote the zero element by 0.

**Definition 3.6** (Equivalence relation for the definition of \( S_n(X; w) \)). Let

\[ \theta = (M, \partial M, \varphi, N, \partial N, \psi, f) \]

be an object from Definition 3.4. We write \( \theta \sim 0 \) if the following conditions are satisfied.
Figure 2: Equivalence relation $\theta \sim 0$ for the definition of $\mathcal{S}_n(X; w)$, where red curves indicate infinitesimally controlled homotopy equivalence, and dark curves indicate homotopy equivalence. In the picture, $V$ (resp. $W$) is a solid with boundary $\partial V$ (resp. $\partial W$).

1. There exists a manifold 2-ad $(W, \partial W)$ of dimension $(n + 1)$ with a continuous map $\Phi: W \to X$ so that $\Phi^*(w)$ describes the orientation character of $W$. Here $\partial W = M \cup_{\partial M} \partial_2 W$, and in particular $\partial M = \partial(\partial_2 W)$. In other words, $W$ is a manifold with corners, and its boundary is the union of $M$ and $\partial_2 W$ (two manifolds with boundary) which are glued together along their common boundary $\partial M = \partial(\partial_2 W)$.

2. Similarly, we have a manifold 2-ad $(V, \partial V)$ of dimension $(n + 1)$ with a continuous map $\Psi: V \to X$ so that $\Psi^*(w)$ describes the orientation character of $V$. Moreover, $\partial V = N \cup_{\partial N} \partial_2 V$.

3. There is a homotopy equivalence of 2-ads $F: (V, \partial V) \to (W, \partial W)$ such that $\Psi \circ F = \Phi$. Moreover, $F$ restricts to $f$ on $N$, and $F$ restricts\(^1\) to an infinitesimally controlled homotopy equivalence $F|_{\partial_2 V}: \partial_2 V \to \partial_2 W$ over $X$.

We further write $\theta_1 \sim \theta_2$ if $\theta_1 + (-\theta_2) \sim 0$. It is not difficult to check that $\sim$ is an equivalence relation. We denote the set of equivalence classes by $\mathcal{S}_n(X; w)$. Note that $\mathcal{S}_n(X; w)$ is an abelian group with the sum operation being disjoint union.

\(^1\)Here we are using an obvious generalization of Definition 3.3 to the case of manifold 2-ads or manifold $n$-ads.
3.2 Surgery long exact sequence

In this subsection, we give a description of the surgery long exact sequence based on ideas of Wall. This will be used later to identify our definition $\mathcal{S}_n(X; w)$ of the structure group agrees with the classical definition $\mathcal{S}^{TOP}(X; w)$.

First let us recall the following geometric definition of $L$-groups due to Wall.

**Definition 3.7 (Objects for the definition of $L_n(\pi_1X; w)$).** An object consists of the following data:

1. two manifold 1-ads $(M, \partial M)$ and $(N, \partial N)$ with $\dim M = \dim N = n$, where $\partial M$ (resp. $\partial N$) is the boundary of $M$ (resp. $\partial N$),
2. continuous maps $\varphi: M \to X$ and $\psi: N \to X$ so that $\varphi^*(w)$ and $\psi^*(w)$ describe the orientation characters of $M$ and $N$ respectively,
3. a degree one normal map of the 1-ads $f: (N, \partial N) \to (M, \partial M)$ such that $\varphi \circ f = \psi$.

Moreover, on the boundary $f|_{\partial N}: \partial N \to \partial M$ is a homotopy equivalence.

**Definition 3.8 (Equivalence relation for the definition of $L_n(\pi_1X; w)$).** Let $\theta = (M, \partial M, \varphi, N, \partial N, \psi, f)$ be an object from Definition 3.7 above. We write $\theta \sim 0$ if the following conditions are satisfied.

1. There exists a manifold 2-ad $(W, \partial W)$ of dimension $(n + 1)$ with a continuous map $\Phi: W \to X$ so that $\Phi^*(w)$ describes the orientation character of $W$. Here $\partial W = M \cup_{\partial M} \partial_2 W$, and in particular $\partial M = \partial(\partial_2 W)$.
Figure 4: Equivalence relation $\theta \sim 0$ for the definition of $L_n(\pi_1 X; w)$, where dark curves indicate homotopy equivalence, and blue curves indicate degree one normal map.

(2) Similarly, we have a manifold 2-ad $(V, \partial V)$ of dimension $(n + 1)$ with a continuous map $\Psi : V \to X$ so that $\Psi^*(w)$ describes the orientation character of $V$. Moreover, $\partial V = N \cup_{\partial N} \partial_2 V$.

(3) There is a degree one normal map of manifold 2-ads $F : (V, \partial V) \to (W, \partial W)$ such that $\Psi \circ F = \Phi$. Moreover, $F$ restricts to $f$ on $N$, and $F|_{\partial_2 V} : \partial_2 V \to \partial_2 W$ is a homotopy equivalence over $X$.

We denote the set of equivalence classes by $L_n(\pi_1 X; w)$. Note that $L_n(\pi_1 X; w)$ is an abelian group under disjoint union. It is a theorem of Wall that the above definition of $L$-groups is equivalent to the algebraic definition of $L$-groups [54, Chapter 9].

**Theorem 3.9** ([54, Chapter 9]). Let $\Gamma = \pi_1 X$. For all $n \geq 5$, $L_n(\pi_1 X; w)$ is naturally isomorphic to the standard algebraic definition of $L_n(\Gamma; w)$.

Now we shall also introduce a controlled version of Wall’s $L$-group definition, which will be identified with $H_*(X; \mathbb{L}_{\bullet})$. Here $\mathbb{L}_{\bullet}$ is an $\Omega$-spectrum of simplicial sets of quadratic forms and formations over $\mathbb{Z}$ such that $\mathbb{L}_0 \simeq G/TOP$.

**Definition 3.10** (Objects for the definition of $N_n(X; w)$). An object consists of the following data:
Figure 5: Object \( \theta = (M, \partial M, \varphi, N, \partial N, \psi, f) \) for the definition of \( N_n(X; w) \), where blue curves indicate degree one normal map, and red curves indicate infinitesimally controlled homotopy equivalence.

(1) two manifold 1-ads \((M, \partial M)\) and \((N, \partial N)\) with \( \dim M = \dim N = n \), where \( \partial M \) (resp. \( \partial N \)) is the boundary of \( M \) (resp. \( \partial N \)),

(2) continuous maps \( \varphi: M \to X \) and \( \psi: N \to X \) so that \( \varphi^*(w) \) and \( \psi^*(w) \) describe the orientation characters of \( M \) and \( N \) respectively,

(3) a degree one normal map of manifold 1-ads \( f: (N, \partial N) \to (M, \partial M) \) such that \( \varphi \circ f = \psi \). Moreover, on the boundary \( f|_{\partial N}: \partial N \to \partial M \) is an \textit{infinitesimally controlled} homotopy equivalence over \( X \).

\textbf{Definition 3.11} (Equivalence relation for the definition of \( N_n(X; w) \)). Let

\( \xi = (M, \partial M, \varphi, N, \partial N, \psi, f) \)

be an object from Definition 3.10 above. We write \( \xi \sim 0 \) if the following conditions are satisfied.

(1) There exists a manifold 2-ad \((W, \partial W)\) of dimension \((n+1)\) with a continuous map \( \Phi: W \to X \) so that \( \Phi^*(w) \) describes the orientation character of \( W \). Moreover, \( \partial W = M \cup_{\partial M} \partial_2 W \).

(2) Similarly, we have a manifold 2-ad \((V, \partial V)\) of dimension \((n+1)\) with a continuous map \( \Psi: V \to X \) so that \( \Psi^*(w) \) describes the orientation character of \( V \). Moreover, \( \partial V = N \cup_{\partial N} \partial_2 V \).

(3) There is a degree one normal map of 2-ads \( F: (V, \partial V) \to (W, \partial W) \) such that \( \Psi \circ F = \Phi \). Moreover, \( F \) restricts to \( f \) on \( N \), and \( F|_{\partial_2 V}: \partial_2 V \to \partial_2 W \) is an \textit{infinitesimally controlled} homotopy equivalence over \( X \).
We denote by $\mathcal{N}_n(X;w)$ the set of equivalence classes from Definition 3.10. Note that $\mathcal{N}_n(X;w)$ is an abelian group with the sum operation being disjoint union.

Now to form the surgery long exact sequence, let us introduce the following relative $L$-groups.

**Definition 3.12** (Objects for the definition of $L_n(\pi_1 X, X; w)$). An object

$$\theta = (M, \partial_\pm M, \varphi, N, \partial_\pm N, \psi, f)$$

consists of the following data (see Figure 7):

![Diagram](image_url)
Figure 7: An object \( \theta = (M, \partial \pm M, \varphi, N, \partial \pm N, \psi, f) \) in \( L_n(\pi_1 X, X; w) \), where red dots indicate infinitesimally controlled homotopy equivalence, blue curves indicate degree one normal map, and dark curves indicate homotopy equivalence.

(1) two manifold 2-ads \((M, \partial \pm M)\) and \((N, \partial \pm N)\) with \(\text{dim } M = \text{dim } N = n\), with \(\partial M = \partial + M \cup \partial - M\) (resp. \(\partial N = \partial + N \cup \partial - N\)) the boundary of \(M\) (resp. \(N\)). In particular, \(\partial(\partial + M) = \partial(\partial - M)\) and \(\partial(\partial + N) = \partial(\partial - N)\).

(2) continuous maps \(\varphi: M \to X\) and \(\psi: N \to X\) so that \(\varphi^*(w)\) and \(\psi^*(w)\) describe the orientation characters of \(M\) and \(N\) respectively,

(3) a degree one normal map of manifold 2-ads \(f: (N, \partial N) \to (M, \partial M)\) such that \(\varphi \circ f = \psi\),

(4) the restriction \(f|_{\partial_+ N}: \partial_+ N \to \partial_+ M\) is a homotopy equivalence over \(X\),

(5) the restriction \(f|_{\partial_- N}: \partial_- N \to \partial_- M\) is a degree one normal map over \(X\),

(6) the homotopy equivalence \(f|_{\partial_\pm N}\) restricts to an \textit{infinitesimally controlled} homotopy equivalence \(f|_{\partial(\partial_\pm N)}: \partial(\partial_\pm N) \to \partial(\partial_\pm M)\) over \(X\).

**Definition 3.13** (Equivalence relation for the definition of \(L_n(\pi_1 X, X; w)\)). Let \(\theta = (M, \partial \pm M, \varphi, N, \partial \pm N, \psi, f)\) be an object from Definition 3.12 above. We write \(\theta \sim 0\) if the following conditions are satisfied.
Figure 8: Equivalence relation $\theta \sim 0$ for the definition of $L_n(\pi_1 X, X; w)$, where red curves indicate infinitesimally controlled homotopy equivalence, blue curves indicate degree one normal map, and dark curves indicate homotopy equivalence.

(1) There exists a manifold 3-ad $(W, \partial W)$ of dimension $(n + 1)$ with a continuous map $\Phi : W \to X$ so that $\Phi^*(w)$ describes the orientation character of $W$. Here $\partial W$ is the union of $M$, $\partial_2 W$ and $\partial_3 W$. Moreover, we have decompositions $\partial M = \partial_+ M \cup \partial_- M$, $\partial(\partial_2 W) = \partial\partial_2 W \cup \partial\partial_2 - W$, and $\partial(\partial_3 W) = \partial\partial_3 W \cup \partial\partial_3 - W$ such that

$$
\partial_+ M = \partial\partial_2 + W, \quad \partial_- M = \partial\partial_2 - W \quad \text{and} \quad \partial\partial_2 - W = \partial\partial_3 + W.
$$

Furthermore, we have

$$
\partial_+ M \cap \partial_- M = \partial\partial_2 + W \cap \partial\partial_2 - W = \partial\partial_3 + W \cap \partial\partial_3 - W.
$$

(2) Similarly, we have a manifold 3-ad $(V, \partial V)$ of dimension $(n + 1)$ with a continuous map $\Psi : V \to X$ so that $\Psi^*(w)$ describes the orientation character of $V$. Moreover, $\partial V = N \cup \partial_2 V \cup \partial_3 V$ satisfies similar conditions as $W$.

(3) There is a degree one normal map of manifold 3-ads $F : (V, \partial V) \to (W, \partial W)$ such that $\Phi \circ F = \Psi$. Moreover, $F$ restricts to $f$ on $N \subset \partial V$.

(4) $F|_{\partial_2 V} : \partial_2 V \to \partial_2 W$ is a homotopy equivalence over $X$.

(5) $F|_{\partial_2 V}$ restricts to an infinitesimally controlled homotopy equivalence

$$
F|_{\partial_2 - V} : \partial\partial_2 - V \to \partial\partial_2 - W
$$

over $X$. 

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We denote by $L_n(\pi_1 X, X; w)$ the set of equivalence classes from Definition 3.12. Note that $L_n(\pi_1 X, X; w)$ is an abelian group with the sum operation being disjoint union.

There is natural group homomorphism
\[
i_* : \mathcal{N}_n(X; w) \to L_n(\pi_1 X; w)
\]
by forgetting control. Moreover, every element
\[
\theta = (M, \partial M, \varphi, N, \partial N, \psi, f) \in L_n(\pi_1 X; w)
\]
naturally defines an element in $L_n(\pi_1 X, X; w)$ by letting $\partial_- M = \emptyset$. We denote the corresponding natural homomorphism by
\[
j_* : L_n(\pi_1 X; w) \to L_n(\pi_1 X, X; w).
\]
Moreover, for every element $\theta = (M, \partial_{\pm} M, \varphi, N, \partial_{\pm} N, \psi, f) \in L_n(\pi_1 X, X; w)$, we see that
\[
\theta_- = \{\partial_- M, \partial (\partial_- M), \varphi, \partial_- N, \partial (\partial_- N), \psi, f\}
\]
defines an element in $\mathcal{N}_{n-1}(X; w)$. We call $\theta_-$ the $(\neg)$-boundary of $\theta$.

Various groups defined above fit into the following long exact sequence. The proof is essentially identical to Theorem 9.6 in Chapter 9 of Wall’s book [54].

**Theorem 3.14.** We have the following long exact sequence:
\[
\cdots \to \mathcal{N}_n(X; w) \xrightarrow{i_*} L_n(\pi_1 X; w) \xrightarrow{j_*} L_n(\pi_1 X, X; w) \xrightarrow{\partial_\ast} \mathcal{N}_{n-1}(X; w) \to L_{n-1}(\pi_1 X; w) \to \cdots
\]
where the map $\partial_\ast$ is defined as follows. For every $\theta = \{M, \partial_{\pm} M, \varphi, N, \partial_{\pm} N, \psi, f\}$ in $L_n(\pi_1 X, X; w)$, we have
\[
\partial_\ast(\theta) := \theta_- = \{\partial_- M, \partial (\partial_- M), \varphi, \partial_- N, \partial (\partial_- N), \psi, f\}
\]
that is, the $(\neg)$-boundary of $\theta$.

**Proof.** The fact that the map $\partial_\ast$ is well-defined follows immediately from the definition. It remains to prove the exactness of the sequence.

An element in $\mathcal{N}_{n-1}(X; w)$ maps to zero in $L_{n-1}(\pi_1 X)$ if and only if its image is cobordant to the empty set in $L_{n-1}(\pi_1 X)$. However, such a cobordism precisely defines an element in $L_n(\pi_1 X, X; w)$. This proves the exactness at $\mathcal{N}_{n-1}(X)$.

Note that $\partial_\ast j_* = 0$ by definition. On the other hand, if an element $\theta \in L_n(\pi_1 X, X; w)$ maps to zero in $\mathcal{N}_{n-1}(X; w)$, then we can take a cobordism of $\partial_\ast(\theta)$ to the empty set, and glue it to $\theta$ along $\partial_\ast(\theta)$. The resulting new element is cobordant to $\theta$, and its $(\neg)$-boundary is empty, so it is the image by $j_*$ of some element in $L_n(\pi_1 X; w)$. This proves the exactness at $L_n(\pi_1 X, X; w)$.

Finally, $j_* i_* = 0$, for if $\xi = \{M, \partial M, \varphi, N, \partial N, \psi, f\} \in \mathcal{N}_n(X; w)$, then it is cobordant to zero in $L_n(\pi_1 X, X; w)$. Indeed, a cobordism of $\xi$ to the empty set is provided by $\xi \times I$ where $I$ is the unit interval. More precisely, $\xi \times I$ consists of the following data.
In other words, precisely, we have the following data:

Conversely, suppose an element \( \theta = \{ M, \partial M, \varphi, N, \partial N, \psi, f \} \in L_n(\pi_1X; w) \) maps to zero in \( L_n(\pi_1X, X; w) \), that is, \( \theta \) is cobordant to the empty set in \( L_n(\pi_1X, X; w) \). More precisely, we have the following data:

(i) \( W = M \times I \) with continuous map \( \Phi = \varphi: W \rightarrow M \rightarrow X \), where \( p_1: W \rightarrow M \) is the obvious projection map. We have \( \partial W = M \cup \partial_2W \cup \partial_3W \) with \( \partial_2W = \partial M \times I \) and \( \partial_3W = M \).

(ii) There is a similar picture for \( (V, \partial V) \) with \( \partial V = N \cup \partial_2V \cup \partial_3V \), where \( \partial_2V = \partial N \times I \) and \( \partial_3V = N \).

(iii) A degree one normal map \( F = f \times \text{Id}: (V, \partial V) \rightarrow (W, \partial W) \). Note that \( F \) clearly restricts to \( f \) on \( N \subset \partial V \).

(iv) Moreover, \( F|_{\partial_2V}: \partial_2V = \partial N \times I \rightarrow \partial_2W = \partial M \times I \) is a homotopy equivalence. This is because \( f: \partial N \rightarrow \partial M \) is an infinitesimally controlled homotopy equivalence, thus in particular a homotopy equivalence.

(v) \( F|_{\partial_2-V}: \partial \partial_2-V = \partial N \rightarrow \partial \partial_2-W = \partial M \) is an infinitesimally controlled homotopy equivalence over \( X \).

Conversely, suppose an element \( \theta = \{ M, \partial M, \varphi, N, \partial N, \psi, f \} \in L_n(\pi_1X; w) \) maps to zero in \( L_n(\pi_1X, X; w) \), that is, \( \theta \) is cobordant to the empty set in \( L_n(\pi_1X, X; w) \). More precisely, we have the following data:

(1) There exists a manifold 3-ad \( (W, \partial W) \) of dimension \( (n+1) \) with a continuous map \( \Phi: W \rightarrow X \) so that \( \Phi^*(w) \) describes the orientation character of \( W \). Moreover, \( \partial W \) is the union of \( M, \partial_2W \) and \( \partial_3W \).

(2) We have decompositions \( \partial M = \partial_+M = \varnothing, \partial \partial_2W = \partial \partial_2+W \cup \partial \partial_2-W \), and \( \partial \partial_3W = \partial \partial_3+W \) with \( \partial \partial_3-W = \varnothing \) such that

\[
\partial_+M = \partial \partial_2+W \text{ and } \partial_2-W = \partial \partial_3+W.
\]

Moreover, we have \( \partial \partial_2+W \cap \partial \partial_2-W = \varnothing \).

(3) Similarly, we have a manifold 3-ad \( (V, \partial V) \) of dimension \( (n+1) \) with a continuous map \( \Psi: V \rightarrow X \) so that \( \Psi^*(w) \) describes the orientation character of \( V \). Moreover, \( \partial V = N \cup \partial_2V \cup \partial_3V \) satisfies similar conditions as \( W \).

(4) There is a degree one normal map of quadruples \( F: (V, \partial V) \rightarrow (W, \partial W) \) such that \( \Phi \circ F = \Psi \). Moreover, \( F \) restricts to \( f \) on \( N \subset \partial V \).

(5) \( F|_{\partial_2V}: \partial_2V \rightarrow \partial_2W \) is a homotopy equivalence over \( X \).

(6) \( F|_{\partial_2-V}: \partial \partial_2-V \rightarrow \partial \partial_2-W \) is an infinitesimally controlled homotopy equivalence over \( X \).

In other words, \( F: (V, \partial V) \rightarrow (W, \partial W) \) provides a cobordism between \( \theta \) and

\[
\eta = (\partial_3W, \partial_3W, \Phi|_{\partial_2W}, \partial_3V, \partial \partial_3V, \Psi|_{\partial_2V}, F|_{\partial_3W}).
\]

Note that \( \eta \) is an element in \( \mathcal{N}_n(X; w) \). This finishes the proof. \( \square \)
It is a classical theorem that $\mathcal{N}_n(X; w)$ is naturally identified to $H_n(X; \mathbb{L}_\bullet)$ with $\mathbb{L}_\bullet$ an $\Omega$-spectrum of simplicial sets of quadratic forms and formations over $\mathbb{Z}$ such that $\mathbb{L}_0 \simeq G/TOP$ (cf. [15, 62]).

**Theorem 3.15** (cf. [15, 62]). For all $n \geq 5$, we have a natural isomorphism

$$\mathcal{N}_n(X; w) \cong H_n(X; \mathbb{L}_\bullet).$$

### 3.3 Identification of various structure groups

In this subsection, we shall identify the three a priori different definitions of structure groups $\mathcal{S}_n^{TOP}(X; w), \mathcal{S}_n(X; w)$ and $L_{n+1}(\pi_1 X, X; w)$, where $X$ is a closed topological manifold of dimension $n \geq 5$.

Note that there is a natural group homomorphism $c_*: \mathcal{S}_n(X; w) \to L_{n+1}(\pi_1 X, X; w)$ by mapping $\theta = \{M, \partial M, \varphi, N, \partial N, \psi, f\} \mapsto \theta \times I$

where $\theta \times I$ consists of the following data:

1. a manifold 2-ad $(M \times I, \partial_\pm(M \times I))$ with $\partial_+(M \times I) = M = \partial_-(M \times I)$; in particular, we have $\partial \partial_+(M \times I) = \partial M = \partial \partial_-(M \times I)$;

2. similarly, another manifold 2-ad $(N \times I, \partial_\pm(N \times I))$ with $\partial_+(N \times I) = N = \partial_-(N \times I)$;

3. a continuous map $\tilde{\varphi} := \varphi \circ p_1: M \times I \xrightarrow{p_1} M \xrightarrow{\varphi} X$ such that $(\varphi \circ p_1)^*(w)$ describes the orientation character of $M \times I$, where $p_1$ is the canonical projection map from $M \times I$ to $M$; similarly, a continuous map $\tilde{\psi} := \psi \circ p_2: N \times I \xrightarrow{p_2} N \xrightarrow{\psi} X$ such that $(\psi \circ p_2)^*(w)$ describes the orientation character of $N \times I$, where $p_2$ is the canonical projection map from $N \times I$ to $N$;

4. a degree one normal map of manifold 2-ads

$$\tilde{f} := f \times \text{Id}: (N \times I, \partial_\pm(N \times I)) \to (M, \partial_\pm(M \times I))$$

such that $\tilde{\varphi} \circ \tilde{f} = \tilde{\psi}$;

5. the restriction $\tilde{f}|_{\partial_+(N \times I)}: \partial_+(N \times I) = N \to \partial_+(M \times I) = M$ is a homotopy equivalence over $X$;

6. the restriction $\tilde{f}|_{\partial_-(N \times I)}: N \to M$ is a degree one normal map over $X$; here we recall that every homotopy equivalence naturally provides a degree one normal map;

7. the homotopy equivalence $f|_{\partial_+(N \times I)}$ restricts to an *infinitesimally controlled* homotopy equivalence $f|_{\partial \partial_+(N \times I)}: \partial \partial_+(N \times I) = \partial N \to \partial \partial_+(M \times I) = \partial M$ over $X$. 

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Note that there is also a natural group homomorphism

\[ r_* : L_{n+1}(\pi_1 X, X; w) \rightarrow \mathcal{S}_n(X; w) \]

by taking the (+)-boundary of an element, that is,

\[ \theta = \{ M, \partial_{\pm} M, \varphi, N, \partial_{\pm} N, \psi, f \} \mapsto \theta_+ = \{ \partial_+ M, \partial_{\partial_+} M, \varphi, \partial_+ N, \partial_{\partial_+} N, \psi, f \}. \]

It follows from definition that the homomorphisms \( c_* \) and \( r_* \) are well-defined.

**Proposition 3.16.** The homomorphisms \( c_* \) and \( r_* \) are inverses of each other. In particular, we have \( \mathcal{S}_n(X; w) \cong L_{n+1}(\pi_1 X, X; w) \).

**Proof.** First, it is clear from the definition that \( r_* \circ c_* = 1 \).

Conversely, if \( \theta = \{ M, \partial_{\pm} M, \varphi, N, \partial_{\pm} N, \psi, f \} \in L_{n+1}(\pi_1 X, X; w) \), then \( c_* \circ r_* (\theta) \) is cobordant to \( \theta \) in \( L_{n+1}(\pi_1 X, X; w) \). Indeed, consider the element

\[ (\theta \times I) \bigcup_{(\theta_+ \times I) \times \{0\} \subset \theta \times \{1\}} (\theta_+ \times I \times I) \]

where \((\theta_+ \times I) \times \{0\}\) is glued to the subset \((\theta_+ \times I) \subset \theta \times \{1\}\). This produces a cobordism between \( c_* \circ r_* (\theta) = (\theta_+ \times I) \times \{1\} \) and \( \theta \times \{0\} \). In other words, we have \( c_* \circ r_* = 1 \). This finishes the proof. \( \square \)

Now we shall use the surgery long exact sequence to prove that \( \mathcal{S}_n(X; w) \) is naturally isomorphic to the classical definition \( \mathcal{S}^{\text{TOP}}(X; w) \).

Note that there is obvious natural map

\[ \iota_* : \mathcal{S}^{\text{TOP}}(X; w) \rightarrow \mathcal{S}_n(X; w) \]

by

\[ [\varphi : M \rightarrow X] \mapsto \theta = \{ M, \partial M = \emptyset, \varphi, X, \partial X = \emptyset, \text{Id}, f = \varphi \}. \]

It follows immediately from the definition that \( \iota_* \) is a well-defined map of sets.

For notational simplicity, we will work with the case where \( X \) is oriented, that is, the orientation character \( w : \pi_1 X \rightarrow \mathbb{Z}/2 \) is trivial. The same argument also works for the general case. Now recall that, for \( n = \dim X \geq 5 \), we have the following geometric surgery long exact sequence

\[ \cdots \rightarrow L_{n+i+1}(\pi_1 X) \rightarrow \mathcal{S}_i^{\text{TOP}}(X \times D^i) \rightarrow \mathcal{N}_i^{\text{TOP}}(X \times D^i) \rightarrow L_{n+i}(\pi_1 X) \rightarrow \cdots \rightarrow L_{n+1}(\pi_1 X) \rightarrow \mathcal{S}^{\text{TOP}}(X) \rightarrow \mathcal{N}^{\text{TOP}}(X) \rightarrow L_n(\pi_1 X) \]

where we have

(a) \( D^i \) is the \( i \)-dimensional Euclidean unit ball;

(b) \( \mathcal{S}_i^{\text{TOP}} \) is the rel \( \partial \) version\(^2\) of structure set;

\(^2\) rel \( \partial \) means “relative to boundary”.
(c) $\mathcal{N}^{\text{TOP}}$ is the set of normal invariants, and $\mathcal{N}^{\text{TOP}}_\partial$ is its rel $\partial$ version;

(d) and the map $L_{n+1}(\pi_1 X) \to \mathcal{S}^{\text{TOP}}(X)$ is a natural action of $L_{n+1}(\pi_1 X)$ on $\mathcal{S}^{\text{TOP}}(X)$.

Moreover, all items starting from $L_{n+1}(\pi_1 X)$ to the left, in the sequence (1), are abelian groups, and all arrows to the left of $L_{n+1}(\pi_1 X)$ are group homomorphisms.

Similar to the definition of $\iota_*: \mathcal{S}^{\text{TOP}}(X) \to \mathcal{S}_n(X)$, there is a natural map $\beta_*: \mathcal{S}^{\text{TOP}}_\partial(X \times D^i) \to \mathcal{S}_{n+i}(X)$ for all $i \geq 1$, by mapping $\{\varphi: (M, \partial M) \to (X \times D^i, X \times S^{i-1})\} \in \mathcal{S}^{\text{TOP}}_\partial(X \times D^i)$ to $\theta = \{M, \partial M, p \circ \varphi, X \times D^i, X \times S^{i-1}, p, f = \varphi\} \in \mathcal{S}_{n+i}(X)$ where $p: X \times D^i \to X$ is the projection of $X \times D^i$ onto $X$. Recall that, for $i \geq 1$, there is a natural group structure on $\mathcal{S}^{\text{TOP}}_\partial(X \times D^i)$, which is geometrically defined by “stacking”. With respect to this group structure, the map $\beta_*$ is in fact a group homomorphism.

**Lemma 3.17.** For $i \geq 1$, the map $\beta_*: \mathcal{S}^{\text{TOP}}_\partial(X \times D^i) \to \mathcal{S}_{n+i}(X)$ is a group homomorphism.

**Proof.** Note that the group structure on $\mathcal{S}^{\text{TOP}}_\partial(X \times D^i)$ is given by stacking, that is, by gluing two elements along the appropriate parts of their boundaries (cf. [5], [12, Definition 2.4]). It is easy to see that the stacking procedure gives a cobordism between the resulting new element and the disjoint union of the two elements that we started with. This finishes the proof. \qed

Similarly, for $i \geq 0$, we have a natural group homomorphism $\alpha_*: \mathcal{N}^{\text{TOP}}_\partial(X \times D^i) \to \mathcal{N}_{n+i}(X)$ which is defined essentially the same way. In fact, it is known that $\alpha_*$ is an isomorphism for all $i \geq 0$.

It follows from definition that, the geometric surgery long exact sequence (1) and the long exact sequence from Theorem 3.14, together with the maps $\alpha_*$, $\beta_*$ and $\iota_*$, fit into the following commutative diagram:

$$
\begin{array}{cccccccc}
\cdots & \to & \mathcal{N}^{\text{TOP}}_\partial(X \times I) & \to & L_{n+1}(\pi_1 X) & \to & \mathcal{S}^{\text{TOP}}(X) & \to & \mathcal{N}^{\text{TOP}}(X) & \to & L_n(\pi_1 X) \\
& & \alpha_* \cong & & \iota_* & & \alpha_* \cong & & \iota_* & & \\
\cdots & \to & \mathcal{N}_{n+1}(X) & \to & L_{n+1}(\pi_1 X) & \to & \mathcal{S}_n(X) & \to & \mathcal{N}_n(X) & \to & L_n(\pi_1 X).
\end{array}
$$

By using the action of $L_{n+1}(\pi_1 X)$ on $\mathcal{S}^{\text{TOP}}(X)$ and the proof of the standard five lemma, we obtain the following proposition.

**Proposition 3.18.** If $\dim X = n \geq 5$, then the map $\iota_*: \mathcal{S}^{\text{TOP}}(X) \to \mathcal{S}_n(X)$ is a bijection of sets.
Since $\mathcal{S}_n(X)$ is an abelian group, this naturally induces an abelian group structure on $\mathcal{S}^{TOP}(X)$. On the other hand, for any oriented closed topological manifold $X$ with $\dim X \geq 5$, $\mathcal{S}^{TOP}(X)$ carries an abelian group structure by Siebenmann’s periodicity theorem, which makes the geometric surgery long exact sequence (1) into an exact sequence of abelian groups everywhere. We prove that these two abelian group structures on $\mathcal{S}^{TOP}(X)$ coincide.

**Proposition 3.19.** The map $\iota_* : \mathcal{S}^{TOP}(X) \rightarrow \mathcal{S}_n(X)$ is a group homomorphism.

**Proof.** Recall that Siebenmann’s periodicity theorem states that

$$\mathcal{S}^{TOP}(X) \hookrightarrow \mathcal{S}^{TOP}_\partial (X \times D^4)$$

is an injection. The abelian group structure of $\mathcal{S}^{TOP}_\partial (X \times D^4)$ induces an abelian group structure on $\mathcal{S}^{TOP}(X)$. Furthermore, we have the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{S}^{TOP}(X) & \xrightarrow{CW} & \mathcal{S}^{TOP}_\partial (X \times D^4) \\
\iota_* \downarrow & & \downarrow \beta_* \\
\mathcal{S}_n(X) & \xrightarrow{\times \mathbb{CP}^2} & \mathcal{S}_{n+4}(X)
\end{array}$$

(2)

where the map $CW$ is the periodicity map given by Cappell and Weinberger, $\times \mathbb{CP}^2$ is the map induced by taking product with $\mathbb{CP}^2$. By Lemma 3.17, the map $\beta_*$ is a group homomorphism. It follows that $\iota_*$ is also a group homomorphism. \(\square\)

To summarize, we have the following theorem.

**Theorem 3.20.** If $\dim X = n \geq 5$, then there are natural group isomorphisms:

$$\mathcal{S}^{TOP}(X) \cong \mathcal{S}_n(X) \cong L^{n+1}(\pi_1 X, X).$$

### 3.4 Structure group by smooth or PL representatives

In this subsection, we show that the elements in our definition of the structure group always have smooth representatives.

Let $X$ be a closed topological manifold. Consider the smooth and PL (piecewise-linear) versions of the long exact sequence from Theorem 3.14, and denote them by

$$\cdots \rightarrow \mathcal{N}^C_{n+1}(X; w) \xrightarrow{i} L^C_{n+1}(\pi_1 X; w) \xrightarrow{j} \mathcal{S}^C_{n}(X; w) \xrightarrow{\partial} \mathcal{N}^C_{n}(X; w) \rightarrow \cdots$$

(3)

and

$$\cdots \rightarrow \mathcal{N}^{PL}_{n+1}(X; w) \xrightarrow{i} L^{PL}_{n+1}(\pi_1 X; w) \xrightarrow{j} \mathcal{S}^{PL}_{n}(X; w) \xrightarrow{\partial} \mathcal{N}^{PL}_{n}(X; w) \rightarrow \cdots$$

(4)

respectively, where the various groups are defined as follows.

---

*The commutativity of the diagram is implicit in the proofs of [5]. See also [57].*
Definition 3.21. An element \( \theta = (M, \partial M, \varphi, N, \partial N, f) \in S^\infty_n(X; w) \) (resp. \( S^{PL}_n(X; w) \)) consists of the following data:

1. \( \theta \) is an element of \( S_n(X; w) \) (cf. Definition 3.4);
2. \( M \) and \( N \) are smooth (resp. PL) manifolds with boundary, and the map \( f : N \to M \) is smooth (resp. PL).

But we point out that \( X \) is only a topological manifold, and the control maps \( \varphi \) and \( \psi \) are only assumed to be continuous. The groups \( N^\infty_n(X; w) \), \( L^\infty_n(\pi_1 X; w) \), \( N^{PL}_n(X; w) \) and \( L^{PL}_n(\pi_1 X; w) \) are defined similarly.

Recall that \( L^\infty_n(\pi_1 X; w) \) remains the same in all smooth, PL, and topological categories. Moreover, our definition of \( N_n(X; w) \) is also independent of which category we work in. Since there is an obvious map from the smooth version to the PL version, which is in turn mapped to the topological version, the following proposition is an immediate consequence of the five lemma.

Proposition 3.22. For \( n \geq 5 \), we have natural isomorphisms

\[
S^\infty_n(X; w) \cong S^{PL}_n(X; w) \cong S_n(X; w).
\]

In particular, every element in \( S_n(X; w) \) has a smooth (resp. PL) representative (cf. Definition 3.21 above).

Remark 3.23. The reader should not confuse \( S^\infty_n(X; w) \) with the smooth structure set of a smooth manifold. The group \( S^\infty_n(X; w) \) still characterizes topological manifold structures on the topological manifold \( X \). The novelty here is that we allow manifolds with boundary in the definition of \( S^\infty_n(X; w) \). Similar remarks apply to \( S^{PL}_n(X; w) \) as well.

This type of argument would fail if we restrict ourselves to only closed manifolds as in the classical definition of \( S^{TOP}_n(X) \). Moreover, we point out that our definition \( S_n(X) \) continues to make sense even if \( X \) is not a manifold. One essential point here is that with our new definition, we are forcing structure groups to be functorial, which they do not seem to be in the smooth case. In fact, there is no group structure for the smooth structure set.

3.5 Piecewise linear control

In this subsection, we will give another definition of the structure group using a different type of control that, we will see, can be used to obtain infinitesimal control.

Note that our definition of \( S_n(X) \) only depends on the homotopy type of \( X \). In other words, \( S_n(X) \) is isomorphic to \( S_n(X') \) for every pair of homotopy equivalent spaces \( X \) and \( X' \). Recall that every topological manifold of dimension \( \geq 5 \) admits a handle decomposition, cf. [31]. It follows that every topological manifold is homotopy equivalent to a finite \( CW \)-complex, and therefore a finite simplicial complex. On the other hand, every finite simplicial complex is homotopy equivalent to a smooth manifold with boundary. Indeed, after being embedded into a Euclidean space, a finite
simplicial complex is homotopy equivalent to a regular neighborhood, which is a smooth
manifold with boundary\(^4\). To summarize, every topological manifold of dimension \(\geq 5\)
is homotopy equivalent to a smooth manifold with boundary. So from now on, without
loss of generality, let us assume \(X\) is a smooth manifold with boundary. In particular,
let us fix a triangulation of \(X\) throughout this subsection.

Remark 3.24. In the above discussion, when we homotope a topological manifold \(Z\) to
a smooth manifold, say \(X\), the dimension of \(X\) is larger than that of \(Z\) in general. However,
we point out that the objects in the definition of \(S_n(Z)\) and \(S_n(X)\) are still of dimension
\(n\), regardless of the dimension of \(X\) or \(Z\).

In the following, we work in the PL category. In particular, all objects are equipped
with a triangulation and all morphisms are assumed to be simplicial. We refer the
reader to [48, 49, 50] for more details on PL transversality.

Definition 3.25. Let \(Y\) and \(Z\) be a pair of PL manifolds equipped with certain trian-
gulations. A homotopy equivalence \(h: Y \rightarrow Z\) is said to be PL controlled over
\(X\) via the control map \(\varphi: Z \rightarrow X\) if the following is satisfied.

\[(1) \quad \varphi\text{ is transversal to the triangulation of } X. \quad \text{That is, the map } \varphi: Z \rightarrow X \text{ is}
\]

transversal to every simplex \(\Delta^k\) in the triangulation of \(X\). In particular, the
inverse image of each simplex \(\Delta^k\) (in the triangulation of \(X\)) is a manifold
\(k\)-ad.

\[(2) \quad h \text{ restricts to a homotopy equivalence from } (\varphi \circ h)^{-1}(\Delta^k)
\]

to \(\varphi^{-1}(\Delta^k)\) for every simplex \(\Delta^k\) of \(X\). More precisely, there exists a homotopy inverse \(g: Z \rightarrow Y\) of \(h\) such that

\[(i) \quad \text{the homotopy } H: h \circ g \simeq \text{Id restricts to a homotopy on } \varphi^{-1}(\Delta^k);
\]

\[(ii) \quad \text{the homotopy } H': g \circ h \simeq \text{Id restricts to a homotopy on } (\varphi \circ h)^{-1}(\Delta^k).
\]

Remark 3.26. Note that, in the above definition, for each simplex \(\Delta^k\) in \(X\), the
homotopy equivalences \(h, g\) and the homotopies \(H\) and \(H'\) all respect the appropriate
manifold ad structure on the inverse image \(\varphi^{-1}(\Delta^k)\). In particular, near various bound-
aries of \(\varphi^{-1}(\Delta^k)\), the map \(h, g, H\) and \(H'\) have appropriate product structures. For
example, the inverse image \(K = \varphi^{-1}(\Delta^1)\) of an 1-simplex \(\Delta^1\) is a manifold 1-ad, that
is, a manifold with boundary \(\partial K\). In this case, the restrictions of \(h, g, H\) and \(H'\)
on \(\varphi^{-1}(\Delta^1)\) maps \(\partial K\) to \(\partial K\), and have product structure near \(\partial K\). We refer the reader
to [54, Chapter 0] for more details on the notion of manifold \(m\)-ads.

Now similar to Section 3.4, we can define a new surgery long exact sequence by
using PL-control instead of infinitesimal control (Definition 3.3) :

\[\cdots \rightarrow N^{PL+}_{n+1}(X; w) \xrightarrow{i} L^{PL+}_{n+1}(\pi_1 X; w) \xrightarrow{j_*} S^{PL+}_n(X; w) \xrightarrow{\partial_*} N^{PL+}_n(X; w) \rightarrow \]

(5)

where the superscript \(PL^+\) stands for PL-representatives with PL-control. More pre-
cisely, for example, for elements in \(S^n_{PL}(X; w)\), we replace infinitesimal control with
PL-control, and denote the new group by \(S^{PL+}_n(X; w)\).

\(^4\)Note that, in general, the dimension of this smooth manifold with boundary is larger than the
dimension of the original topological manifold we started with.
Remark 3.27. The definition of PL-control we gave works well only when $X$ is a PL manifold. We point out that, when $X$ is a PL manifold with boundary, to define $S^{PL+}_n(X; w)$, every element $\theta = (M, \partial M, \varphi, N, \partial N, \psi, f)$ is assumed to be disjoint from the boundary of $X$. That is, $\varphi(M) \cap \partial X = \emptyset$, $\psi(N) \cap \partial X = \emptyset$, and all other relevant data intersect with $\partial X$ at empty set. Similar remarks apply to $N^{PL+}_n(X; w)$ and $L^{PL+}_n(\pi_1 X; w)$.

Remark 3.28. Using the ideas of Quinn from [40], we can generalize the above construction to the case where $X$ is an arbitrary finite polyhedron. In that case, to define PL control, the conditions are not on inverse images of simplices, but rather of their dual cones. In this more general setting, the covariant functoriality of $S^{PL+}_n(X; w)$ becomes clear.

In the following proposition, we prove that the above notion of PL-control in fact implies infinitesimal control, at least after we modify the control map $\varphi: Z \to X$. More precisely, we can keep the map $h: Y \to Z$ unchanged, and homotope the control map $\varphi$ to another control map $\bar{\varphi}$ so that $h$ becomes infinitesimally controlled with respect to $\bar{\varphi}$.

Proposition 3.29. Let $X$ be an $n$-dimensional PL manifold with boundary equipped with a triangulation. Suppose $h: Y \to Z$ is a PL controlled homotopy equivalence over $X$ via the control map $\varphi: Z \to X$. Then there exists a control map $\bar{\varphi}: Z \to X$ such that

1. $\bar{\varphi}$ is homotopic to $\varphi$;
2. $h$ restricts to a proper homotopy equivalence
   $$h: \tilde{\psi}^{-1}U \to \bar{\varphi}^{-1}U$$
   for all open subsets $U \subset X$, where $\tilde{\psi} = \bar{\varphi} \circ h$.

Proof. The proof is by induction and uses a “dual cone” picture as described, for example, in [40, Section 6].

Suppose $K$ is a simplicial complex. We take the first barycentric subdivision of $K$. For every simplex $\sigma$ in $K$, we define the dual cone $D(\sigma)$ to be the union of all simplices of the subdivision which intersect $\sigma$ in exactly the barycenter of $\sigma$. Now the key idea of the proof is to crush all the nontrivial changes in topology of $Z$ and $Y$, and the homotopy equivalence $h$ to small parts. More precisely, we have the following induction construction. Let $X^{(k)}$ be the $k$-skeleton of $X$, that is, $X^{(k)}$ is the union of all simplices in $X$ of dimensions $\leq k$.

(i) Initial step. First, consider $Z^{(1)} = \varphi^{-1}(X^{(1)})$ the inverse image of $X^{(1)}$. Note that $\varphi^{-1}(X^{(1)})$ has product structure near $Z^{(0)} = \varphi^{-1}(X^{(0)})$. Let us denote by $Z^{(0)}_\varepsilon$ for such a small open neighborhood (with product structure) of $Z^{(0)}$ in $Z^{(1)}$. Let $Z^{(1)}_\varepsilon = Z^{(1)} - Z^{(0)}_\varepsilon$ be the complement of $Z^{(0)}_\varepsilon$ in $Z^{(1)}$.

We define a new control map $\varphi_1: Z^{(1)} = \varphi^{-1}(X^{(1)}) \to X$ by mapping each component of $Z^{(1)}_\varepsilon$ to the barycenter of the corresponding 1-simplex in $X$, and
stretching out $Z^{(0)}_\varepsilon$ (which is of product structure) accordingly. Intuitively, we see that the nontrivial changes of topology from $Z^{(0)}$ to $Z^{(1)}$ are all pushed to the barycenters of 1-simplices in $X$. In particular, for all open subsets $V \subset X^{(1)}$, $h$ restricts to a proper homotopy equivalence $h: \psi^{-1}_1(V) \to \varphi^{-1}_1(V)$, where $\psi_1 = \varphi_1 \circ h$.

![Figure 9: the original map $\varphi$](image1)

![Figure 10: the new map $\varphi_1$](image2)

(ii) **Induction step.** Suppose we have defined the control map

$$\varphi_k: Z^{(k)} = \varphi^{-1}(X^{(k)}) \to X^{(k)}.$$  

Now let us extend $\varphi_k$ to a control map $\varphi_{k+1}: Z^{(k+1)} = \varphi^{-1}(X^{(k+1)}) \to X^{(k+1)}$. Intuitively, for each simplex $\Delta^{k+1}$ of $X$, we shall define $\varphi_{k+1}$ so that most of $\varphi^{-1}(\Delta^{k+1})$ is mapped to the barycenter of $\Delta^{k+1}$, and the remaining part of $\varphi^{-1}(\Delta^{k+1})$ (which again has an appropriate product structure) is stretched out accordingly.

More precisely, note that $Z^{(k+1)}$ has product structure near $Z^{(k)}$. Let us denote by $Z^{(k)}_\varepsilon$ for such a small open neighborhood (with product structure) of $Z^{(k)}$ in $Z^{(k+1)}$. Let $Z^{(k+1)}_\varepsilon = Z^{(k+1)} - Z^{(k)}_\varepsilon$ be the complement of $Z^{(k)}_\varepsilon$ in $Z^{(k+1)}$. We define
a new control map $\varphi_2: Z^{(2)} = \varphi^{-1}(X^{(2)}) \rightarrow X$ by mapping each component of $Z^{(2)}_e$ to the barycenter of the corresponding $(k + 1)$-simplex in $X$, and stretching out $Z^{(1)}_e$ (which is of product structure) accordingly. It is clear that this process keeps $\varphi_k$ unchanged on $Z^{(k)}$.

In the end, we obtain a new control map $\bar{\varphi} = \varphi_n: Z \rightarrow X$, where $n$ is the dimension of $X$. It is clear from the construction that $\bar{\varphi}$ is homotopic to $\varphi$. Moreover, $h$ restricts to a proper homotopy equivalence

$$h: \bar{\psi}^{-1}U \rightarrow \bar{\varphi}^{-1}U$$

for all open subsets $U \subset X$, where $\bar{\psi} = \varphi \circ h$. This finishes the proof.

**Definition 3.30.** Let $Y$ and $Z$ be a pair of PL manifolds equipped with triangulations. A homotopy equivalence $h: Y \rightarrow Z$ is said to be PL infinitesimally controlled over $X$ via a control map $\bar{\varphi}: Z \rightarrow X$ if $h$ is PL controlled over $X$ via $\bar{\varphi}$ and $h$ restricts to a proper homotopy equivalence

$$h: \bar{\psi}^{-1}U \rightarrow \bar{\varphi}^{-1}U$$

for all open subsets $U \subset X$, where $\bar{\psi} = \varphi \circ h$.

The following is an immediate corollary of Proposition 3.29.

**Corollary 3.31.** Suppose $\theta = (M, \partial M, \varphi, N, \partial N, \psi, f)$ is an element in $S^\text{PL}_{m}(X)_{PL}$. Then there exists a control map $\bar{\varphi}: M \rightarrow X$ such that $\bar{\varphi}$ is homotopic to $\varphi$ and on the boundary $f$ restricts to a PL infinitesimally controlled homotopy equivalence $f|_{\partial N}: \partial N \rightarrow \partial M$.

In order to more directly apply the discussion above to the geometrically controlled category (Section 4.4), let us state the PL infinitesimal control in terms of triangulations. We borrow the notation from Definition 3.3. In our current situation, we can choose proper simplicial maps

$$\Phi: CM \rightarrow X \times [1, \infty) \quad \text{and} \quad \Psi: CN \rightarrow X \times [1, \infty),$$

$$F: CN \rightarrow CM \quad \text{and} \quad G: CM \rightarrow CN,$$

and a proper simplicial homotopy $\{H_s\}_{0 \leq s \leq 1}$ between

$$H_0 = F \circ G \quad \text{and} \quad H_1 = \text{Id}: CM \rightarrow CM$$

and a proper simplicial homotopy $\{H'_s\}_{0 \leq s \leq 1}$ between

$$H'_0 = G \circ F \quad \text{and} \quad H'_1 = \text{Id}: CN \rightarrow CN$$

such that the following are satisfied:
(1) $\Phi = \bar{\varphi} \times \text{Id}: \partial M \times [1, \infty) \to \partial M \times [1, \infty)$, where $\bar{\varphi}$ is the new controlled map obtained from $\varphi$ as in Proposition 3.29 above; $\Psi = \bar{\psi} \times \text{Id}: \partial N \times [1, \infty) \to \partial N \times [1, \infty)$, where $\bar{\psi} = \varphi \circ f$; and $F = f \times \text{Id}: \partial N \times [1, \infty) \to \partial M \times [1, \infty)$ with commutative diagram

\[
\begin{array}{c}
\partial N \times [1, \infty) \\
\downarrow \bar{\psi} \times \text{Id} \\
X \times [1, \infty) \\
\downarrow \varphi \times \text{Id} \\
\end{array}
\xrightarrow{\text{Id}}
\begin{array}{c}
\partial M \times [1, \infty) \\
\end{array}
\xrightarrow{f \times \text{Id}}
\]

(2) $X \times [1, \infty)$ is equipped with a triangulation of bounded geometry (cf. Definition 4.12) such that the sizes of simplices uniformly go to zero, as we approach infinity along the cylindrical direction; for example, this is can be achieved by the standard subdivision in Section 4.2.

(3) $f \times \text{Id}$ is PL infinitesimally controlled. More precisely, the homotopy

$H: F \circ G \simeq \text{Id}: CM \to CM$

restricts to a homotopy on $\Phi^{-1}(\Delta)$, for every simplex $\Delta$ in $X \times [1, \infty)$. The homotopy

$H': G \circ F \simeq \text{Id}: CN \to CN$

restricts to a homotopy on $\Psi^{-1}(\Delta)$, for every simplex $\Delta$ in $X \times [1, \infty)$.

(4) all maps $f \times \text{Id}, G, H$ and $H'$ are geometrically controlled over the cone $CX$ (see Definition 4.32) in the sense of Section 4.4 below.

Recall that the surgery long exact sequence built using PL transversality is equivalent to the surgery long exact sequence built using block bundles, cf. [41], also [4] and [34]. By Proposition 3.29, we have the following commutative diagram

\[
\begin{array}{cccccccc}
\cdots & \xrightarrow{\alpha_{k+1}} & N^{PL+}_{k+1}(X; w) & \xrightarrow{\beta_k} & L^{PL+}_{k+1}(\pi_1 X; w) & \xrightarrow{\lambda_k} & S^{PL+}_k(X; w) & \xrightarrow{\alpha_k} & N^{PL+}_k(X; w) & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots & \xrightarrow{\beta_{k+1}} & N^{PL}_{k+1}(X; w) & \xrightarrow{\lambda_k} & L^{PL}_{k+1}(\pi_1 X; w) & \xrightarrow{\alpha_k} & S^{PL}_k(X; w) & \xrightarrow{\beta_k} & N^{PL}_k(X; w) & \cdots \\
\end{array}
\]

Let us first prove that all vertical maps are isomorphisms, when $k \geq \dim X + 5$. We will then show how to handle the general case.

**Proposition 3.32.** If $k \geq \dim X + 5$, then $\lambda_k: S^{PL+}_k(X; w) \to S^{PL}_k(X; w)$ is an isomorphism.

**Proof.** Let us first prove that the maps

$\alpha_k: N^{PL+}_k(X; w) \to N^{PL}_k(X; w)$

and

$\beta_k: L^{PL+}_k(\pi_1 X; w) \to L^{PL}_k(\pi_1 X; w)$
are isomorphisms. This for example can be proven by the same techniques from [54, Chapter 9]. The reason for the assumption \( k \geq \dim X + 5 \) comes from the fact that, in order to apply the techniques from [54, Chapter 9], the fibers of the control maps such as \( \varphi: M \to X \) need to be at least 5-dimensional. So when \( k \geq \dim X + 5 \), we have that \( \alpha_k \) and \( \beta_k \) are isomorphisms. Now the proof is finished by applying the five lemma.

Now let us consider the case of \( S_k(X) \), where \( k < \dim X + 5 \). Recall that the surgery long exact sequence in (3) is periodic with periodicity 4. This periodicity is realized by taking the product of an element \( \theta \in \mathcal{N}_k^{PL}(X; w) \) with \( \mathbb{CP}^2 \), similarly for \( L_k^{PL}(\pi_1 X; w) \) and \( S_k^{PL}(X; w) \) as well. For our purpose, we can choose \( \mathbb{CP}^{2\ell} \) instead of \( \mathbb{CP}^2 \), so that \( k + 4 \ell \geq \dim X + 5 \). This provides a periodic isomorphism of periodicity \( 4\ell \). In any case, we have the following isomorphisms:

\[
S_k(X) \cong S_k^{PL}(X) \cong S_{k+4\ell}(X) \cong S_{k+4\ell}^{PL}(X),
\]

as long as \( k + 4 \ell \geq \dim X + 5 \). As a consequence, our definition of the higher rho invariant map (Definition 4.39)

\[
\rho: S_n(X) \to K_n(C^*_{L,0}(\tilde{X})^\Gamma)
\]

is in fact the composition

\[
S_n(X) \cong S_{n+4\ell}^{PL}(X) \overset{\rho}{\longrightarrow} K_{n+4\ell}(C^*_{L,0}(\tilde{X})^\Gamma) \cong K_n(C^*_{L,0}(\tilde{X})^\Gamma).
\]

However, by Remark 3.33 below, we see that such an identification procedure is harmless. From now on, when no possible confusion can arise, we will not distinguish between \( S_n(X) \) and \( S_{n+4\ell}^{PL}(X) \).

**Remark 3.33.** Suppose \( Y_1^m \) and \( Y_2^n \) are complete Riemannian manifolds of dimension \( m \) and \( n \). Let \( D_{Y_1} \), \( D_{Y_2} \) and \( D_{Y_1 \times Y_2} \) be the signature operator on \( Y_1 \), \( Y_2 \) and \( Y_1 \times Y_2 \) respectively. Then the signature operator \( D_{Y_1 \times Y_2} = D_{Y_1} \boxtimes D_{Y_2} \), if \( m \cdot n \) is even, and \( D_{Y_1 \times Y_2} = 2(D_{Y_1} \boxtimes D_{Y_2}) \) if \( m \cdot n \) is odd. Here \( D_{Y_1} \boxtimes D_{Y_2} \) is the external product of \( D_{Y_1} \) and \( D_{Y_2} \). See for example [47, Lemma 6]. Recall that the signature of \( \mathbb{CP}^{2\ell} \) is equal to 1 for any \( \ell > 0 \). In particular, it follows that taking product with \( \mathbb{CP}^{2\ell} \) does not affect the definition of our higher rho invariant (Definition 4.39) below.

**Remark 3.34.** In fact, with extra work, we can even use the periodic isomorphism induced by multiplying by \( \mathbb{CP}^2 \) to prove

\[
S_{k+4}(X) \cong S_{k+4}^{PL}(X),
\]

when \( k = \dim X \). In this case, the fiber dimension is only 4 in general. The smooth surgery in dimension 4 does not quite work as nicely as the case where dimension is \( \geq 5 \). However, we can get around this by taking connected sums with sufficiently many copies of \( S^2 \times S^2 \), where \( S^2 \) is the standard 2-sphere.
Remark 3.35. Everything in this subsection has an obvious equivalent counterpart in terms of smooth representatives with PL-control. For example, we also have a long exact sequence

$$\cdots \to \mathcal{N}_{n+1}^{C^\infty}(X; w)_{PL} \xrightarrow{\iota} L_{n+1}^{C^\infty}(\pi_1 X; w)_{PL} \xrightarrow{j_\ast} S_n^{C^\infty}(X; w)_{PL} \xrightarrow{\partial_2} \mathcal{N}_n^{C^\infty}(X; w)_{PL} \to$$

where the subscript $PL$ stands for PL-control. That is, for example, for elements in $S_n^{C^\infty}(X; w)$, we replace infinitesimal control with PL-control.

Remark 3.36. If we take the viewpoint of Ranicki’s algebraic surgery long exact sequence, then most of the discussion in this section (Section 3) can be avoided, and the techniques from [21, 22, 23] can be adapted more directly to deal with Ranicki’s algebraic definition of $S^{TOP}(X)$.

4 Additivity of higher rho invariants

In this section, we define the higher $\rho$-invariant for elements in $S_n(X)$ using our new description of the structure group, where $X$ is a closed oriented topological manifold of dimension $n$. Furthermore, we prove that the higher $\rho$-invariant defines a group homomorphism from $S_n(X)$ to $K_n(C^*_\ast(X; w))$, where $X$ is a universal cover of $X$ and $\Gamma = \pi_1 X$.

4.1 A hybrid $C^*$-algebra

In this subsection, we introduce a certain hybrid $C^*$-algebra that is useful for the definition of higher rho invariant.

Suppose $\Gamma$ is a countable discrete group. Let $Y$ be proper metric space equipped with a proper $\Gamma$-action.

Definition 4.1. We define $C^*_\ast(Y)^\Gamma$ to be the $C^*$-subalgebra of $C^*(Y)$ generated by elements $\alpha \in C^*(Y)$ of the following form: for any $\varepsilon > 0$, there exists a $\Gamma$-invariant $\Gamma$-cocompact subset $K \subseteq Y$ such that the propagations of $\alpha \chi_{(Y-K)}$ and $\chi_{(Y-K)} \alpha$ are both less than $\varepsilon$. Here $\chi_{(Y-K)}$ is the characteristic function on $Y - K$.

Definition 4.2. We define $C^*_{L,0,c}(Y)^\Gamma$ to be the $C^*$-subalgebra of $C^*_\ast(Y)$ generated by elements $\alpha \in C^*_{L,0}(Y)$ of the following form: for any $\varepsilon > 0$, there exists a $\Gamma$-invariant $\Gamma$-cocompact subset $K \subseteq Y$ such that the propagations of $\alpha(t) \chi_{(Y-K)}$ and $\chi_{(Y-K)} \alpha(t)$ are both less than $\varepsilon$, for all $t \in [0, \infty)$.

Let $X \times [1, \infty)$ be as before. We denote the universal cover of $X$ by $\tilde{X}$, and write $\Gamma = \pi_1 X$. Consider the $C^*$-algebra $C^*_{L,0,c}(\tilde{X} \times [1, \infty))^\Gamma$. It is obvious that, for any $r \geq 1$, the $C^*$-algebra $\mathcal{A}_r = C^*_{L,0,c}(\tilde{X} \times [1, r]; \tilde{X} \times [1, \infty))^\Gamma$ is a two-sided closed ideal of $C^*_{L,0,c}(\tilde{X} \times [1, \infty))^\Gamma$.

Definition 4.3. Let $\mathcal{A}$ be the norm closure of the union

$$\bigcup_{r \geq 1} \mathcal{A}_r = \bigcup_{r \geq 1} C^*_{L,0,c}(\tilde{X} \times [1, r]; \tilde{X} \times [1, \infty))^\Gamma.$$
Note that \( \mathcal{J} \) is also a two-sided closed ideal of \( C^*_\alpha(\tilde{X} \times [1, \infty)) \). Recall that
\[
K_i(C^*_\alpha(\tilde{X} \times [1, r]; \tilde{X} \times [1, \infty))) = K_i(C^*_\alpha(\tilde{X} \times [1, r])) = K_i(C^*_\alpha(\tilde{X}))
\]
for \( i = 0, 1 \). It follows that \( K_i(\mathcal{J}) = K_i(C^*_\alpha(\tilde{X})) \).

**Proposition 4.4.** The inclusion of \( \mathcal{J} \subset C^*_\alpha(\tilde{X} \times [1, \infty)) \) induces an isomorphism at level of \( K \)-theory. That is, we have
\[
K_i(C^*_\alpha(\tilde{X} \times [1, \infty))) \cong K_i(\mathcal{J}) = K_i(C^*_\alpha(\tilde{X}))
\]
for \( i = 0, 1 \).

**Proof.** For notational simplicity, let us write
\[
\mathcal{A} = C^*_\alpha(\tilde{X} \times [1, \infty)) \Gamma
\]
We have the following short exact sequence of \( C^* \)-algebras:
\[
0 \rightarrow \mathcal{J} \rightarrow \mathcal{A} \xrightarrow{\alpha} \mathcal{A}/\mathcal{J} \rightarrow 0.
\]
To prove the proposition, it suffices to show that
\[
K_i(\mathcal{A}/\mathcal{J}) = 0
\]
for \( i = 0, 1 \). This can be proven by an Eilenberg swindle argument as follows.

We prove the odd case, that is, \( K_1(\mathcal{A}/\mathcal{J}) = 0 \). The even case is completely similar. Suppose \( \bar{\alpha} \) is an invertible element in \( \mathcal{A}/\mathcal{J} \), where \( \mathcal{A}^+ \) is the unitization of \( \mathcal{A} \). Let \( \alpha \) be a lift of \( \bar{\alpha} \) in \( \mathcal{A}^+ \). Without loss of generality, let assume that \( \alpha - 1 \in \mathcal{A} \). For each \( n \in \mathbb{N} \), we define an element \( \alpha_n \in \mathcal{A}^+ \) as follows:
\[
\alpha_n(t) = \begin{cases} 
1 & \text{if } 0 \leq t \leq n, \\
\alpha(t - n) & \text{if } t \geq n.
\end{cases}
\]
We define
\[
\beta(t) = \bigoplus_{n=0}^{\infty} (1 + \chi_n(\alpha_n(t) - 1)\chi_n),
\]
where \( \chi_n \) is the characteristic function on the set \( \tilde{X} \times [n, \infty) \).

We claim that \( \beta \) is an element in \( \mathcal{A}^+/\mathcal{J} \). Indeed, recall that, for any \( \varepsilon > 0 \), there exists a positive integer \( N \) such that: (1) the propagation of \( \alpha(t) \) is \( < \varepsilon \), for all \( t \geq N \); (2) the propagation of \( \alpha(t) |_{\tilde{X} \times [N, \infty)} \) is \( < \varepsilon \), for all \( t \geq 0 \). It follows that, for any \( \varepsilon > 0 \), we have that
\[
(i) \text{ the propagation of } \beta(t) \text{ is } < \varepsilon \text{ for all } t \geq 2N;
\]
\[
(ii) \text{ the propagation of } \beta(t) |_{\tilde{X} \times [N, \infty)} \text{ is } < \varepsilon \text{, for all } t \geq 0.
\]

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This proves that $\beta \in \mathcal{A}^+$. Let us denote by $\bar{\beta}$ the image of $\beta$ in $\mathcal{A}^+ / \mathcal{I}$. We show that $\bar{\beta}$ is in fact invertible in $\mathcal{A}^+ / \mathcal{I}$. Let $\omega \in \mathcal{A}^+$ be the lift of $(\bar{\alpha})^{-1}$. Define

$$
\mu(t) = \bigoplus_{n=0}^{\infty} (1 + \chi_n(\omega_n(t) - 1)\chi_n),
$$

where $\omega_n$ and $\chi_n$ are defined similarly as above. Note that the operators $1 - \beta(t)\mu(t)$ and $1 - \mu(t)\beta(t)$ are supported in $\tilde{\mathcal{X}} \times [1, n + 1]$, for all $t \in [0, n]$. It follows that $1 - \beta\mu$ and $1 - \mu\beta$ are in the closure of $\bigcup_{r \geq 1} \mathcal{I}_r$. In particular, $1 - \beta\mu$ and $1 - \mu\beta$ are in $\mathcal{I}$. Therefore, $\beta$ is an invertible element in $\mathcal{A}^+ / \mathcal{I}$.

Similarly, let us define

$$
\gamma(t) = \bigoplus_{n=1}^{\infty} (1 + \chi_n(\alpha_n(t) - 1)\chi_n).
$$

The same argument from above also show that $\gamma \in \mathcal{A}^+$. Denote by $\bar{\gamma}$ the image of $\gamma$ in $\mathcal{A}^+ / \mathcal{I}$. Then $\bar{\gamma}$ is also an invertible element in $\mathcal{A}^+ / \mathcal{I}$.

Note that $\bar{\gamma}(t - 1) = \bar{\beta}(t)$. Moreover, $\bar{\gamma}$ and $\bar{\beta}$ are connected by a path of invertible elements $\bar{\gamma}_s$, $0 \leq s \leq 1$, where $\bar{\gamma}_s(t) = \bar{\gamma}(t - s)$. Therefore, we have

$$
[\bar{\gamma}] = [\bar{\beta}] \in K_1(\mathcal{A} / \mathcal{I}).
$$

It follows that

$$
[\bar{\beta}] = [\bar{\alpha}] \oplus [\bar{\gamma}] = [\bar{\alpha}] \oplus [\bar{\beta}] \in K_1(\mathcal{A} / \mathcal{I}),
$$

which implies that $[\bar{\alpha}] = 0$. This finishes the proof. \[\square\]

We also introduce a hybrid version for localization $C^*$-algebras.

**Definition 4.5.** We define $C^*_{L,c}(Y)^\Gamma$ to be the $C^*$-subalgebra of $C^*_L(Y)$ generated by elements $\alpha \in C^*_L(Y)$ of the following form: for any $\varepsilon > 0$, there exists a $\Gamma$-invariant $\Gamma$-cocompact subset $K \subseteq Y$ such that the propagations of $\alpha(t)x_{Y-K}$ and $x_{Y-K}\alpha(t)$ are both less than $\varepsilon$ for all $t \in [0, \infty)$.

However, the analogue of Proposition 4.4 does not hold for $C^*_{L,c}(Y)^\Gamma$. In fact, the following lemma shows that the $K$-theory groups of $C^*_{L,c}(Y)^\Gamma$ always vanish.

**Lemma 4.6.** We have $K_i(C^*_{L,c}(\tilde{X} \times [1, \infty))^\Gamma) = 0$, for $i = 0, 1$.

**Proof.** It is easy to see that

$$
K_i(C^*_{L,c}(\tilde{X} \times [1, \infty))^\Gamma) = K_i(C^*_L(\tilde{X} \times [1, \infty))^\Gamma).
$$

The latter is always zero. This finishes the proof. \[\square\]
As a consequence, we have the following immediate corollary.

**Corollary 4.7.** We have $K_i(C^*_c(\tilde{X} \times [1, \infty))^\Gamma) \cong K_{i+1}(C^*_{L,0,c}(\tilde{X} \times [1, \infty))^\Gamma)$.

**Proof.** It follows from applying the results above to the $K$-theory long exact sequence of

$0 \rightarrow C^*_{L,0,c}(\tilde{X} \times [1, \infty))^\Gamma \rightarrow C^*_{L,c}(\tilde{X} \times [1, \infty))^\Gamma \rightarrow C^*_c(\tilde{X} \times [1, \infty))^\Gamma \rightarrow 0$.

\[\blacksquare\]

### 4.2 Simplicial complexes and refinements

In this subsection, let us describe a refinement procedure for a given triangulation $M$. This refinement procedure produces a particular subdivision of $M$, denoted by $\text{Sub}(M)$, such that all successive refinements $\text{Sub}^n(M) := \text{Sub}(\text{Sub}^{n-1}(M))$ have uniform bounded geometry, that is, uniform with respect to $n \in \mathbb{N}$. The following discussion is taken from [24].

Let us first recall the notion of typed simplicial complexes.

**Definition 4.8 (cf. [3, 28]).** Suppose $M$ is a simplicial complex of dimension $n$. Let $M^0$ be the set of vertices of $M$. A *type* on $M$ is a map $\varphi : M^0 \rightarrow \{0, 1, \ldots, n\}$ such that for any simplex $\omega \in M$, the images by $\varphi$ of the vertices of $\omega$ are pairwise distinct. A simplicial complex equipped with a type is said to be *typed*.

Given any simplicial complex $M$ of dimension $n$, we denote its barycentric subdivision by $Y$. Then $Y$ admits a type. Indeed, $Y$ is the set of totally ordered subsets of $M$, that is,

$Y^k = \{(\sigma_0, \ldots, \sigma_k) \mid \sigma_j \in X$ and $\sigma_i$ is a face of $\sigma_{i+1}\}$.

The dimension function, which maps each barycenter of a simplex of $M$ to the dimension of that simplex, is a type on $Y$.

Now suppose $M$ is a typed simplicial complex of dimension $n$. In particular, this gives a consistent way of ordering the vertices of each simplex in $X$ according to the type map. Therefore, each $k$-simplex of $M$ can be canonically identified with the standard $k$-simplex $\Delta^k$. Now to define our refinement procedure, it suffices to describe certain subdivisions of the standard simplices so that the number of simplices containing any given vertex remains uniformly bounded for all successive subvisions.

One way to achieve this is by the so-called standard subdivision [59, Appendix II.4]. In the following, we briefly recall the construction of standard subdivision, and refer the reader to [59, Appendix II.4] for more details.

Let $\sigma = [v_0, v_1, \ldots, v_k]$ be a standard simplex with its vertices given in the order shown. Set $v_{ij} = \frac{1}{2}v_i + \frac{1}{2}v_j$, $i \leq j$; in particular, $v_{ii} = v_i$. These are the vertices of the standard subdivision of $\sigma$, denoted $\text{Sub}(\sigma)$. Define a partial ordering on these vertices by setting

$v_{ij} \leq v_{kl}$ if $k \leq i$ and $j \leq l$. 

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Now the simplices of Sub(\(\sigma\)) are all those formed from the \(v_{ij}\) which are in increasing order. Moreover, each simplex in Sub(\(\sigma\)) naturally inherits an ordering of vertices from the above partial ordering of \(v_{ij}\). It is not difficult to verify that Sub(\(\sigma\)) carries a natural type by mapping \(v_{ij} \mapsto (j - i)\).

To summarize, given a typed simplicial complex \(M\) of dimension \(n\), we apply the above standard subdivision procedure (consistently) to each \(n\)-simplex of \(M\). We call the resulting simplicial complex the standard subdivision of \(M\), denoted by Sub(\(M\)). Note that Sub(\(M\)) is also typed.

### 4.3 Hilbert-Poincaré complexes

In this subsection, we recall the definition of Hilbert-Poincaré complexes, which is fundamental for studying higher signatures of topological spaces. We refer to [21] for more details.

Let \(A\) be a unital \(C^*\)-algebra. Consider a chain complex of Hilbert modules over \(A\):

\[
E_0 \xleftarrow{b_1} E_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n
\]

where the differentials \(b_j\) are bounded adjointable operators. The \(j\)-th homology of the complex is the quotient space obtained by dividing the kernel of \(b_j\) by the image of \(b_{j+1}\). Note that, since the differentials need not to have closed range, the homology spaces are not necessarily Hilbert modules themselves.

**Definition 4.9.** An \(n\)-dimensional Hilbert-Poincaré complex (over a \(C^*\)-algebra \(A\)) is a complex of finitely generated Hilbert \(A\)-modules

\[
E_0 \xleftarrow{b_1} E_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n
\]

together with adjointable operators \(T : E_p \to E_{n-p}\) such that

1. if \(v \in E_p\), then \(T^*v = (-1)^{(n-p)p}Tv\);
2. if \(v \in E_p\), then \(Tb^*(v) + (-1)^p bT(v) = 0\);
3. \(T\) is a chain homotopy equivalence\(^5\) from the dual complex

\[
E_n \xleftarrow{b_n} E_{n-1} \xleftarrow{b_{n-1}} \cdots \xleftarrow{b_1} E_0
\]

to the complex \((E, b)\).

Now we will associate to each \(n\)-dimensional Hilbert-Poincaré complex an index class, called signature, in the \(K\)-theory group \(K_n(A)\).

\(^5\)To be precise, by item (2), we need to impose appropriate signs so that \(T\) becomes a genuine chain map. However, we will follow the usual convention and leave it as is, with the understanding that appropriate signs are employed.
Definition 4.10. Let \((E, b, T)\) be an \(n\)-dimensional Hilbert-Poincaré complex. We denote \(l\) to be the integer such that

\[
n = \begin{cases} 
2l & \text{if } n \text{ is even}, \\
2l + 1 & \text{if } n \text{ is odd}.
\end{cases}
\]

Define \(S : E \to E\) to be the bounded adjointable operator such that

\[
S(v) = i^{p(p-1)+l} T(v)
\]

for \(v \in E_p\). Here \(i = \sqrt{-1}\).

It is not hard to verify that \(S = S^*\) and \(bS + Sb^* = 0\). Moreover, if we define \(B = b + b^*\), then the self-adjoint operators \(B \pm S : E \to E\) are invertible [21, Lemma 3.5].

Definition 4.11. (i) Let \((E, b, T)\) be an odd-dimensional Hilbert-Poincaré complex. Its signature is the class in \(K_1(A)\) of the invertible operator

\[
(B + S)(B - S)^{-1} : E_{ev} \to E_{ev}
\]

where \(E_{ev} = \oplus_p E_{2p}\).

(ii) If \((E, b, T)\) is an even-dimensional Hilbert-Poincaré complex, then its signature is the class in \(K_0(A)\) determined by the formal difference \([P_+] - [P_-]\) of the positive projections of \(B + S\) and \(B - S\).

4.4 Geometrically controlled Poincaré complexes

In this subsection, we recall the definition of geometrically controlled Poincaré complexes [22]. They are Hilbert-Poincaré complexes in the geometrically controlled category.

Definition 4.12. A simplicial complex \(M\) is of bounded geometry if there is a positive integer \(k\) such that each of the vertices of \(M\) lies in at most \(k\) different simplices of \(M\).

Definition 4.13. Let \(X\) be a proper metric space. A complex vector space \(V\) is geometrically controlled over \(X\) if it is provided with a basis \(B \subset V\) and a function \(c : B \to X\) with the following property: for every \(R > 0\), there is an \(N < \infty\) such that if \(S \subset X\) has diameter less than \(R\) then \(c^{-1}(S)\) has cardinality less than \(N\). We call such \(V\) a geometrically controlled \(X\)-module from now on.

Note that each geometrically controlled vector space \(V\) over \(X\) is assigned with a basis \(B\). There is a natural completion of \(V\) into a Hilbert space \(\overline{V}\) in which the basis \(B\) of \(V\) becomes an orthonormal basis of \(\overline{V}\).

Let \(V_f^* = \text{Hom}_f(V, \mathbb{C})\) be the vector space of finitely supported linear functions on \(V\). Then \(V_f^*\) is identified with \(V\) under the inner product on \(\overline{V}\).
Definition 4.14. A linear map $T: V \to W$ is geometrically controlled over $X$ if

1. $V$ and $W$ are geometrically controlled;
2. the matrix coefficients of $T$ with respect to the given bases of $V$ and $W$ are uniformly bounded;
3. and there is a constant $K > 0$ such that the $(v,w)$-matrix coefficients is zero whenever $d(c(v), c(w)) > K$. The smallest such $K$ is called the propagation of $T$.

It is easy to see that a geometrically controlled linear map $T: V \to W$ has a natural dual

$$T^*: W^*_f \to V^*_f$$

which is canonically identified with a geometrically controlled linear map, still denoted by $T^*$,

$$T^*: W \to V.$$

Definition 4.15. A chain complex

$$E_0 \xleftarrow{b_1} E_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n$$

is called a geometrically controlled chain complex over $X$ if each $E_p$ is a geometrically controlled $X$-module, and each $b_p$ is a geometrically controlled linear map.

Definition 4.16. Let $f_1, f_2: (E, b) \to (E', b')$ be geometrically controlled chain maps between two geometrically controlled chain complexes $(E, b)$ and $(E', b')$. We say $f_1$ and $f_2$ are geometrically controlled homotopic to each other, if there exists a geometrically controlled linear map $h: (E_*, b) \to (E'_*, b')$ such that

$$f_1 - f_2 = b'h + hb.$$

In this case, $h$ is called a geometrically controlled chain homotopy between $f_1$ and $f_2$.

Now we give the definition of geometrically controlled Poincaré complexes.

Definition 4.17. An $n$-dimensional geometrically controlled Poincaré complex (with control respect to $X$) is a complex of geometrically controlled $X$-modules

$$E_0 \xleftarrow{b_1} E_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n$$

together with geometrically controlled linear maps $T: E_p \to E_{n-p}$ and $b: E_p \to E_{p-1}$ such that

1. if $v \in E_p$, then $T^*v = (-1)^{(n-p)}Tv$;
2. if $v \in E_p$, then $Tb^*(v) + (-1)^p bT(v) = 0$;
(3) \( T \) is a geometrically controlled chain homotopy equivalence from the dual complex
\[
E_n \xleftarrow{b_n^*} E_{n-1} \xleftarrow{b_{n-1}^*} \cdots \xleftarrow{b_1^*} E_0
\]
to the complex \((E, b)\). Here we have identified the finitely supported dual \( E_f^* \) with \( E \).

**Example 4.18.** Our typical example of a geometrically controlled Poincaré complex comes from a triangulation of a closed smooth manifold (more generally a triangulation of a complete Riemannian manifold without boundary, e.g. the manifold \( CM \) from above), cf. [22, Section 3 & 4].

We introduce the following notion of geometrically controlled homotopy equivalences of geometrically controlled Poincaré complexes.

**Definition 4.19.** Given two \( n \)-dimensional geometrically controlled Poincaré complexes \((E, b, T)\) and \((E', b', T')\), A geometrically controlled homotopy equivalence between them consists of the following data:

1. two geometrically controlled chain maps \( f : (E, b) \to (E', b') \) and \( g : (E', b') \to (E, b) \);
2. \( g \circ f \) and \( f \circ g \) are geometrically controlled homotopic to the identity;
3. \( fTf^* \) is geometrically controlled homotopic to \( T' \), where \( f^* \) is the dual of \( f \):

\[
\begin{array}{cccccc}
E'_n & \xleftarrow{b'_n} & E'_{n-1} & \xleftarrow{b'_{n-1}} & \cdots & \xleftarrow{b'_1} & E'_0 \\
\downarrow f^* & & \downarrow f^* & & \downarrow f^* &
\end{array}
\]

\[
E_n \xleftarrow{b_n} E_{n-1} \xleftarrow{b_{n-1}} \cdots \xleftarrow{b_1} E_0.
\]

**Remark 4.20.** In the above definition, it is automatic that \( gT'g^* \) is also geometrically controlled homotopic to \( T \). Indeed, we have \( gT'g^* \simeq g(fTf^*)g^* = (gf)T(f^*g^*) \simeq T \).

There is an obvious equivariant theory of geometrically controlled Poincaré complexes. We shall omit the details, and refer the reader to [22, Section 3] for further reading.

### 4.5 Analytically controlled Poincaré complexes

In this subsection, we recall the definition of analytically controlled Poincaré complexes [22]. In particular, we review a natural way to pass from the geometrically controlled category to the analytically controlled category, cf. [22, Section 3].

Recall from Section 2 that an \( X \)-module is a separable Hilbert space \( H \) equipped with a \(*\)-representation of \( C_0(X) \), the algebra of all continuous functions on \( X \) which vanish at infinity. To distinguish from geometrically controlled \( X \)-modules, we call such \( H \) an analytically controlled \( X \)-module from now on.
Definition 4.21. Let $H_1$ and $H_2$ be two analytically controlled $X$-modules. A linear map $T: H_1 \to H_2$ is said to be analytically controlled, if $T$ is the norm limit of locally compact and finite propagation bounded operators.

Remark 4.22. In this paper, we have chosen to work with signature operators arising from triangulations of manifolds. This is the bounded case, where all operators are bounded. If one wants to work with unbounded signature operators arising from $L^2$-de Rham complexes of Riemannian manifolds, then one needs a slightly different notion of analytical controls. See [21, Section 5] for more details.

The notion of geometrically controlled homotopy equivalence of geometrically controlled chain complexes naturally passes to the following notion of analytically controlled homotopy equivalence of analytically controlled chain complexes.

Definition 4.23. A chain complex

$$E_0 \xleftarrow{b_1} E_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n$$

is called a analytically controlled chain complex over $X$ if each $E_p$ is an analytically controlled $X$-module, and each $b_p$ is an analytically controlled morphism.

Definition 4.24. Let $f_1, f_2: (E, b) \to (E', b')$ be analytically controlled chain maps between two analytically controlled chain complexes $(E, b)$ and $(E', b')$. We say $f_1$ and $f_2$ are analytically controlled homotopic to each other, if there exists an analytically controlled linear map $h: (E_*, b) \to (E'_{*+1}, b')$ such that

$$f_1 - f_2 = b' h + h b.$$ 

Now we introduce the notion of analytically controlled Poincaré complexes.

Definition 4.25. An $n$-dimensional analytically controlled Poincaré complex (with control respect to $X$) is a complex of analytically controlled $X$-modules

$$E_0 \xleftarrow{b_1} E_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n$$

together with analytically controlled linear maps $T: E_p \to E_{n-p}$ and $b: E_p \to E_{p-1}$ such that

1. if $v \in E_p$, then $T^* v = (-1)^{(n-p)p} T v$;
2. if $v \in E_p$, then $T b^* (v) + (-1)^p b T(v) = 0$;
3. and $T$ is an analytically controlled chain homotopy equivalence from the dual complex

$$E_n \xleftarrow{b_n^*} E_{n-1} \xleftarrow{b_{n-1}^*} \cdots \xleftarrow{b_1^*} E_0$$

to the complex $(E, b)$.

The following theorem is a rephrasing of a theorem of Higson and Roe [22, Theorem 3.14].
Theorem 4.26 ([22, Theorem 3.14]). Every geometrically controlled Poincaré complex naturally defines an analytically controlled Poincaré complex, by $\ell^2$-completion.

We introduce the following notion of analytically controlled homotopy equivalences of analytically controlled Poincaré complexes.

Definition 4.27. Given two $n$-dimensional analytically controlled Poincaré complexes $(E, b, T)$ and $(E', b', T')$, an analytically controlled homotopy equivalence between them consists of the following data:

1. two analytically controlled chain maps $f: (E, b) \to (E', b')$ and $g: (E', b') \to (E, b)$;

2. $g \circ f$ and $f \circ g$ are analytically controlled homotopic to the identity;

3. $fTf^*$ is analytically controlled homotopic to $T'$, where $f^*$ is the adjoint of $f$.

For an analytically controlled Poincaré complex, its signature index naturally lies in the $K$-theory of the Roe algebra $C^*(X)$.

Definition 4.28. (i) Let $(E, b, T)$ be an odd-dimensional analytically controlled Poincaré complex. Its signature is the class in $K_1(C^*(X))$ of the invertible operator

$$(B + S)(B - S)^{-1}: E_{ev} \to E_{ev}$$

where $E_{ev} = \bigoplus_p E_{2p}$.

(ii) If $(E, b, T)$ is an even-dimensional analytically controlled Poincaré complex, then its signature is the class in $K_0(C^*(X))$ determined by the formal difference $[P_+] - [P_-]$ of the positive projections of $B + S$ and $B - S$.

The following simpler notion of analytically controlled homotopy equivalence will also be useful later.

Definition 4.29. Let $(E, b)$ be an analytically controlled chain over $X$. An operator homotopy of analytically controlled Poincaré duality operators on $(E, b)$ is a norm continuous family of operators $T_s$, $s \in [0, 1]$, such that each $(E, b, T_s)$ is an analytically controlled Poincaré complex.

Lemma 4.30 (cf. [21, Lemma 4.6]). If a Poincaré duality operator $T$ on an analytically controlled Poincaré complex $(E, b)$ is operator homotopic to $-T$ through a path of analytically controlled duality operator $T_s$, then the path

$$(B + S)(B - S_s)^{-1}$$

is a norm-continuous path of invertible elements connecting $(B + S)(B - S)^{-1}$ to the identity.

There is an obvious equivariant theory of analytically controlled Poincaré complexes. We shall omit the details, and refer the reader to [22, Section 2] for further reading.
Remark 4.31. If one prefers the exposition in terms of Hilbert $C^*$-modules, there is a natural way to make sense of everything in this subsection by using Roe algebras in Section 2. More precisely, we fix an ample and nondegenerate analytically controlled $X$-module $H$. Let $C^*(X)$ be the norm closure of locally compact and finite propagation bounded linear operators from $H$ to $H$. That is, $C^*(X)$ is the Roe algebra of $X$ associated to $H$. Now suppose $H'$ is any other analytically controlled $X$-module. We define $E(H, H')$ to be the norm closure of locally compact and finite propagation bounded linear operators from $H$ to $H'$. Then clearly $E(H, H')$ carries a natural right Hilbert $C^*(X)$-module structure. It is not difficult to see that such a language will give an equivalent description of the discussion in this subsection.

4.6 Higher rho invariant

In this subsection, we define the higher rho invariant for elements in our new description of structure group. By Proposition 3.22, without loss of generality, it suffices to construct the higher $\rho$ invariant and prove its additivity for smooth or PL representatives in $S_n(X)$. So throughout this subsection, we will be working in the PL category, unless otherwise specified.

Let $\theta = (M, \partial M, \varphi, N, \partial N, \psi, f)$ be an element of $S_n(X)$. By the discussion of Section 3.5, without loss of generality, we can assume $\theta$ consists of the following data:

1. two triangulated PL manifolds with boundary $(M, \partial M)$ and $(N, \partial N)$ with $\dim M = \dim N = n$;

2. the control map $\varphi : M \to X$ is PL transverse to the triangulation of $X$,

3. a PL homotopy equivalence $f : (N, \partial N) \to (M, \partial M)$ such that $\varphi \circ f = \psi$. Moreover, on the boundary $f$ restricts to a PL infinitesimally controlled homotopy equivalence $f|_{\partial N} : \partial N \to \partial M$ over $X$. See the discussion after Corollary 3.31 for more details.

Let $X \times [1, \infty)$ be equipped with the product metric, where the metric on $[1, \infty)$ is the standard Euclidean metric. By using the standard subdivision of Section 4.2, there exists a triangulation $\text{Tri}_{X \times [1, \infty)}$ of $X \times [1, \infty)$ such that

1. $\text{Tri}_{X \times [1, \infty)}$ has bounded geometry in the sense of Definition 4.12;

2. the sizes\(^6\) of simplices in $\text{Tri}_{X \times [1, \infty)}$ uniformly go to zero, as we approach infinity along the cylindrical direction.

Recall that every locally finite simplicial complex carries a natural path metric, whose restriction to each $n$-simplex is the Riemannian metric obtained by identifying the $n$-simplex with the standard $n$-simplex in the Euclidean space $\mathbb{R}^n$. Such a metric is called a simplicial metric.

\(^6\)Here the size of a simplex is measured with respect to the product metric on $X \times [1, \infty)$. 
**Definition 4.32.** Let $X \times [1, \infty)$ be equipped with the triangulation $\text{Tri}_{X \times [1, \infty)}$ from above. We define the simplicial metric cone of $X$, denoted by $CX$, to be the space $X \times [1, \infty)$ equipped with the simplicial metric determined by $\text{Tri}_{X \times [1, \infty)}$.

**Remark 4.33.** From now on, the notation $X \times [1, \infty)$ will stand for the space $X \times [1, \infty)$ with the product metric. The reader should not confuse $CX$ with $X \times [1, \infty)$.

Recall that the space of $M$ attached with a cylinder is defined to be $CM = M \cup_{\partial M} (\partial M \times [1, \infty))$.

Let us fix a triangulation of $CM$ as follows. On $M$, it is the original triangulation of $M$. The triangulation on $\partial M \times [1, \infty)$ is the pullback triangulation of $\text{Tri}_{X \times [1, \infty)}$ under the map $\varphi \times \text{Id}: \partial M \times [1, \infty) \to CX$. More precisely, for every simplex $\Delta^k \subset CX$, the inverse image $(\varphi \times \text{Id})^{-1}(\Delta^k)$ is a product $K \times \Delta^k$, where $K$ is some triangulated submanifold of $\partial M$.

**Remark 4.34.** In fact, since it is easier to deal with products of cubical complexes than products of simplicial complexes, one could use cubical complexes instead of simplicial complexes for our discussion in this subsection.

Let $\Gamma = \pi_1 X$. We denote by $\widetilde{CM}$ (resp. $\widetilde{CN}$) the corresponding $\Gamma$-cover of $CM$ (resp. $CN$) induced by $\Phi: CM \to X \times [1, \infty)$ (resp. $\Psi: CN \to X \times [1, \infty)$). Here we have borrowed the same notation from Definition 3.3.

Note that the simplicial decomposition of $CM$ (resp. $CN$) naturally lifts to a $\Gamma$-equivariant simplicial decomposition of $\widetilde{CM}$ (resp. $\widetilde{CN}$). Consider the $\Gamma$-equivariant geometrically controlled Poincaré complex

$$E_0(\widetilde{CM}) \xleftarrow{b_1} E_1(\widetilde{CM}) \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n(\widetilde{CM})$$

associated to the above $\Gamma$-equivariant simplicial decomposition of $\widetilde{CM}$, where

1. $E_i(\widetilde{CM})$ is a geometrically controlled $(\widetilde{CX}, \Gamma)$-module,

2. $b_i$ is a geometrically controlled morphism,

3. and the Poincaré duality map $T$ is given by the usual cap product with the $\Gamma$-equivariant fundamental class of $\widetilde{CM}$.

The $\ell^2$-completion of this $\Gamma$-equivariant geometrically controlled Poincaré complex gives rises to a $\Gamma$-equivariant analytically controlled Poincaré complex, still denoted by $(E(\widetilde{CM}), b, T)$. We summarize this in the following lemma.

**Lemma 4.35.** $(E(\widetilde{CM}), b, T)$ is a $\Gamma$-equivariant analytically controlled Poincaré complex.

Similarly, we have the $\Gamma$-equivariant analytically controlled Poincaré complex $(E(\widetilde{CN}), b', T')$ associated to $\widetilde{CN}$:

$$E_0(\widetilde{CN}) \xleftarrow{b'_1} E_1(\widetilde{CN}) \xleftarrow{b'_2} \cdots \xleftarrow{b'_n} E_n(\widetilde{CN}).$$
Now let us proceed to define the higher rho invariant for elements in $S_n(X)$. We will only give the details for the odd dimensional case, that is, the case where $n$ is odd. The even dimensional case is completely similar.

In the following, all controls are measured with respect to the control maps

$$\Phi: CM \to X \times [1, \infty) \text{ and } \Psi: CN \to X \times [1, \infty).$$

For notational simplicity, we shall drop the term “$\Gamma$-equivariant” in the construction below, with the understanding that all steps below are done $\Gamma$-equivariantly. Also, we write $E = E(\tilde{CM})$ and $E' = E(\tilde{CN})$.

Let us consider the $\Gamma$-equivariant analytically controlled Poincaré complex $(E, b, T) = (E \oplus E', b \oplus b', T \oplus -T')$

Let $B = B \oplus B'$ and $S = S \oplus -S'$ (cf. Definition 4.10). The signature index of $(E, b, T)$ is defined to be the class of $(B + S)(B - S)^{-1}$ in $K_1(C^*(CX))$. Clearly, the map

$$\tau: CX \to X \times [1, \infty) \quad (7)$$

by $\tau(x, t) = (x, t)$ is a proper continuous map that induces a $C^*$-algebra homomorphism

$$\tau_*: C^*(CX) \to C^*_c(X \times [1, \infty)).$$

Similarly, we have

$$\text{and } \tau_*: C^*_{L,0}(CX) \to C^*_{L,0,c}(X \times [1, \infty)).$$

There are also obvious $\Gamma$-equivariant versions. In the following, unless otherwise specified, all elements below are to be thought of as their corresponding images under the map $\tau_*$.

Following Higson and Roe [21, Section 4], we shall first build an explicit path of invertible elements connecting

$$(B + S)(B - S)^{-1}$$

to the identity element, within the $C^*$-algebra $C^*_c(\tilde{X} \times [1, \infty))$.\Gamma.

Let $F: E' \to E$ and $G: E \to E'$ be the chain maps induced by $F: CN \to CM$ and $G: CM \to CN$.

**Lemma 4.36.** With the same notation above, $F: E' \to E$ and $G: E \to E'$ satisfy the following conditions:

1. the chain maps $F: (E', b') \to (E, b)$ and $G: (E, b) \to (E', b')$ are analytically controlled;

2. $GF$ and $FG$ are analytically controlled homotopic to the identity;

3. $GTG^*$ is analytically controlled homotopic to $T'$.

Moreover, for any $\varepsilon > 0$, there exists a positive number $k$ such that the chain maps $F$, $G$ and the various homotopies have propagation $< \varepsilon$ away from $N \cup (\partial N \times [1, k])$ and $M \cup (\partial M \times [1, k])$. 

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In particular, we see that the operator

\[
\begin{pmatrix}
T & 0 \\
0 & (s - 1)T' - sGT^*
\end{pmatrix}
\]

implements a Poincaré duality operator for the complex \((E, b)\), for each \(s \in [0, 1]\). This path connects the duality operator \(T \oplus -T'\) to \(T \oplus -GT^*\).

Now consider the operator

\[
\widehat{T}(s) = \begin{pmatrix}
\cos(s)T & \sin(s)TG^* \\
\sin(s)GT & -\cos(s)GT^*
\end{pmatrix}.
\]

**Lemma 4.37.** The operator \(\widehat{T}(s)\) implements a Poincaré duality operator for the complex \((E, b)\), for each \(s \in [0, \pi/2]\).

**Proof.** An analytically controlled homotopy inverse of \(\widehat{T}(s)\) is given by

\[
\begin{pmatrix}
\cos(s)\alpha & \sin(s)\alpha F \\
\sin(s)F^*\alpha & -\cos(s)F^*\alpha F
\end{pmatrix}
\]

where \(\alpha\) is an analytically controlled homotopy inverse of \(T\). \(\square\)

Now concatenate the two paths above and denote the resulting path by

\((E, b, \mathcal{J}_t)\)

with \(t \in [0, 1]\). Let \((B + S_t)(B - S_t)^{-1}\) be the invertible operator representing the corresponding signature of \((E, b, \mathcal{J}_t)\) (cf. Definition 4.11).

Note that the last duality operator

\[
\mathcal{J}_1 = \begin{pmatrix}
0 & TG^* \\
GT & 0
\end{pmatrix}
\]

is analytically controlled homotopic to its additive inverse along the path

\[
\begin{pmatrix}
0 & \exp(is)TG^* \\
\exp(-is)GT & 0
\end{pmatrix},
\]

with \(s \in [0, \pi]\). By Lemma 4.30, we see that \((B + S_1)(B - S_1)^{-1}\) is connected to the identity operator through a path of analytically controlled invertible elements.

To summarize, we have constructed a path of analytically controlled invertible elements in \(C^*_c(\tilde{X} \times [1, \infty))^\Gamma\) connecting \((B + S)(B - S)^{-1}\) to the identity element. Let us re-parametrize the time variable, and denote this path by

\[
V_s = (B + S_s)(B - S_s)^{-1}
\]

with \(V_0 = I\) and \(V_1 = (B + S)(B - S)^{-1}\).

Now we shall extend this path to obtain an element in \(C^*_c(L, 0, c)(\tilde{X} \times [1, \infty))^\Gamma\). In other words, we will construct an element \(W \in C^*_c(L, 0, c)(\tilde{X} \times [1, \infty))^\Gamma\) so that \(W_s = V_s \oplus I\) for all \(s \in [0, 1]\), where \(I\) is the identity operator.
In fact, the construction of $W_s$, starting at $s \geq 1$, coincides with the construction of the $K$-homology class of the signature operator on $CM \coprod -CN$ with control with respect to $CX$. To be more precise, there are in fact two equivalent ways of constructing the path $W_s$.

(i) One directly works with the geometrically controlled Poincaré complex and its refinements associated to the space $CM \coprod -CN$. In particular, everything is controlled over $CX$. This is what we have chosen to do for the construction of $V_s$ above. Note that, although $CM \coprod -CN$ is not a closed PL manifold, it is a complete manifold without boundary. It is easy to see that the construction in Appendix A.1 (not Appendix A.2) applies verbatim to the space $CM \coprod -CN$ controlled over $CX$. As a result, we obtain a $K$-theory class $(W_s)_{0 \leq s < \infty}$ in $K_n(C^*_\Gamma(X \times [1, \infty]))$.

(ii) Alternatively, we consider the Poincaré space $M \cup_f (-N)$. Although $M \cup_f (-N)$ is not a manifold, it is still a space equipped with a Poincaré duality. In fact, since $f: \partial N \to \partial M$ is a PL infinitesimally controlled homotopy equivalence, we can still make sense of $K$-homology class of its “signature operator”, as in Appendix A.2. Let us denote this $K$-homology class by a path of invertible elements $(U_s)_{0 \leq s < \infty}$. Moreover, in the current situation, we also have that $f: N \to M$ is a homotopy equivalence. Similar to the discussion following Lemma 4.36, the homotopy equivalence $f$ can be used to connect $U_0$ to the identity operator through a path of invertible elements. Re-parametrize the resulting new path, and define it to be the higher rho invariant of $\theta$ in $K_n(C^*_\Gamma(X))$. It is not difficult to see that these two constructions define the same $K$-theory class in $K_n(C^*_{\Gamma,0,c}(\widetilde{X} \times [1, \infty]))$.

To summarize, we have constructed a path of invertible elements $W(\theta)$ for each $\theta = (M, \partial M, \varphi, N, \partial N, \psi, f) \in S_n(X)$.

**Proposition 4.38.** For $\theta = (M, \partial M, \varphi, N, \partial N, \psi, f) \in S_n(X)$, we have $W(\theta) \in C^*_{\Gamma,0,c}(\widetilde{X} \times [1, \infty])$.

**Proof.** Note that the simplicial chain complex associated to the triangulation on $\widetilde{CM}$ (resp. $\widetilde{CN}$) is a $\Gamma$-equivariant geometrically controlled module over $\widetilde{CX}$. Since the map $f: \partial M \to \partial N$ is PL infinitesimally controlled over $X$, it follows that all maps $F$, $G$, $H$ and $H'$ as in Definition 3.3 are $\Gamma$-equivariantly geometrically controlled over $CX$. Therefore, our construction produces an element in $C^*_{\Gamma,0}(\widetilde{CX})$, whose image under the map $\tau$ from line (7) is precisely the element

$$W(\theta) \in C^*_{\Gamma,0,c}(\widetilde{X} \times [1, \infty])$$

$\Box$

**Definition 4.39.** For each element $\theta = (M, \partial M, \varphi, N, \partial N, \psi, f) \in S_n(X)$ when $n$ is odd, we define the higher rho invariant of $\theta$ to be

$$\rho(\theta) = [W(\theta)] \in K_n(C^*_{\Gamma,0,c}(\widetilde{X} \times [1, \infty])) \cong K_n(C^*_{\Gamma,0}(\widetilde{X}))$$.
Note that the definition of the higher rho invariant for the even dimensional case (i.e. for $S_n(X)$ when $n$ even) is completely similar. We omit the details.

Remark 4.40. We point out that, in the odd dimensional case, the higher rho invariant for signature operators in the literature (cf. [23, Section 3] [37, Remark 4.6] [67]) is twice of the higher rho invariant of this paper, cf. Remark 6.7 and Theorem 6.9 below.

We have the following main theorem of our paper.

**Theorem 4.41.** For $n \geq 5$, the map
$$\rho: S_n(X) \to K_n(C^*_L,0(\tilde{X})^\Gamma)$$
is a group homomorphism.

**Proof.** The well-definedness of the map $\rho: S_n(X) \to K_n(C^*_L,0(\tilde{X})^\Gamma)$ will be proved in Theorem 5.8. Now the group structure on $S_n(X)$ is given by disjoint union, and $\rho$ is obviously additive on disjoint unions. This finishes the proof. \hfill $\square$

**Remark 4.42.** Although we have chosen to work with the reduced version of various $C^*$-algebras, we point out that the exact same proofs work equally well for the maximal version of these $C^*$-algebras. In particular, we also have a well-defined group homomorphism
$$\rho: S_n(X) \to K_n(C^*_L,0(\tilde{X})_{\text{max}}^\Gamma).$$

## 5 Invariance properties of higher rho invariant

In this section, we prove that the map $\rho: S_n(X) \to K_n(C^*_L,0(\tilde{X})^\Gamma)$ is well-defined. Our method is modeled upon Higson and Roe’s proof for the bordism invariance of higher signature index [21, Section 7].

The following definitions are geometrically controlled analogues of the corresponding definitions in [21, Section 7]. We refer the reader to [21, Section 7] for more details.

**Definition 5.1.** A complemented subcomplex of the geometrically controlled complex $(E,b)$ is a family of complemented geometrically controlled submodules $E_p \subset E$, such that $b$ maps $E_p$ to $E_p - 1$, for all $p$.

For each complemented subcomplex $(PE,Pb)$ of $(E,b)$, there is a corresponding geometrically controlled complement complex $(P^\perp E, P^\perp b)$. The inclusion $PE \subset E$ is a chain map from $(PE,Pb)$ into $(E,b)$, whereas the orthogonal projection $E \to P^\perp E$ gives a chain map from $(E,b)$ onto $(P^\perp E, P^\perp b)$. Note that $P^\perp b = P^\perp b P^\perp$.

**Definition 5.2.** An $(n+1)$-dimensional geometrically controlled Poincaré pair is a geometrically controlled complex
$$E_0 \leftarrow b_1 \ E_1 \leftarrow b_2 \cdots \leftarrow b_n \ E_n$$
together with a family of geometrically controlled operators $T: E_p \to E_{n+1-p}$ and a family of geometrically controlled orthogonal projections $P: E_p \to E_p$ such that
(1) the orthogonal projections \( P \) determines a subcomplex of \((E, b)\); that is, \( PbP = bP \);

(2) the range of the operator \( Tb^* + (-1)^p bT : E_p \to E_{n-p} \) is contained within the range of \( P : E_{n-p} \to E_{n-p} \);

(3) \( T^* = (-1)^{(n+1-p)p} T : E_p \to E_{n+1-p} \);

(4) \( P^\perp T \) is a geometrically controlled chain homotopy equivalence from the dual complex \((E, b^*)\) to \((P^\perp E, P^\perp b)\).

**Example 5.3.** A typical example of geometrically controlled Poincaré complexes comes from a triangulation of a smooth manifold with boundary [22, Section 4.2].

The following lemma is geometrically controlled analogue of [21, Lemma 7.4].

**Lemma 5.4 ([21, Lemma 7.4]).** Let \((E, b, T, P)\) be an \((n+1)\)-dimensional geometrically controlled Poincaré pair. The operators

\[ T_0 = Tb^* + (-1)^p bT : E_p \to E_{n-p} \]

satisfy the following relations:

1. \( T_0^* = (-1)^{(n-p)p} T_0 : E_p \to E_{n-p} \);
2. \( T_0 = PT_0 = T_0 P \);
3. \( T_0 b^* + (-1)^p bT_0 = 0 : PE_p \to PE_p \);
4. \( T_0 = Tb^* + (-1)^p bT \) induces a geometrically controlled chain homotopy from \((PE, Pb^*)\) to \((PE, Pb)\).

**Proof.** The proof is a combination of the proof of [21, Lemma 7.4] together with [22, Lemma 4.2]. We leave out the details. \( \square \)

The above lemma asserts \((PE, Pb, T_0)\) is an \( n \)-dimensional geometrically controlled Poincaré complex.

**Definition 5.5.** The geometrically controlled Poincaré complex \((PE, Pb, T_0)\) is called the boundary of the geometrically controlled Poincaré pair \((E, b, T, P)\).

Note that there is an obvious analogue theory in the analytically controlled category. Moreover, there are obvious equivariant theories for both geometrically controlled Poincaré pairs and analytically controlled Poincaré pairs respectively.

The following theorem is a rephrasing of a theorem of Higson and Roe [22, Theorem 3.18].

**Theorem 5.6 ([22, Theorem 3.18]).** Every geometrically controlled Poincaré pair naturally defines an analytically controlled Poincaré pair, by \( \ell^2 \)-completion.
Before we prove that the higher rho invariant map is well-defined, let us give a proof of the bordism invariance of the $K$-homology class of signature operator (compare \cite[Theorem 2]{47}). Our proof below is modeled upon Higson and Roe’s proof for the bordism invariance of higher signature index \cite[Theorem 7.6]{21}. Note that, in the theorem below, we do not invert $2$.

**Theorem 5.7** (Bordism invariance of $K$-homology signature). Let $V$ be an $(n+1)$-dimensional oriented PL manifold with boundary $\partial V$, equipped with a continuous map $\psi: V \to X$, where $X$ is a proper metric space. Then

$$\text{Ind}_L(\partial V) = 0 \in K_n(C^*_L(\tilde{X})^\Gamma),$$

where $\tilde{X}$ is the universal cover of $X$ with $\Gamma = \pi_1 X$.

**Proof.** Fix a triangulation of $V$, together with a sequence of successive refinements $\text{Sub}^n(V)$ as in Section 4.2. Note that, for the geometrically controlled Poincaré pair associated to the triangulation $\text{Sub}^n(V)$, all maps appearing in Definition 5.2 are geometrically controlled, with their propagations go to zero as $n \to \infty$.

Let us denote the geometrically controlled Poincaré pair associated to the triangulation $\text{Sub}^n(V)$ by $(E^n, b^n, T^n, P^n)$. Since our construction below works for these refinements simultaneously, we shall omit the superscript $(n)$ from now on. Equivalently, one consider the direct sum $$(E, b, T, P) = \bigoplus_{n=1}^\infty (E^n, b^n, T^n, P^n).$$

In particular, by the construction in Appendix A, the geometrically controlled Poincaré complex $(PE, Pb, T_0)$ produces a specific representative of the local index $\text{Ind}_L(\partial V, \psi) \in K_n(C^*_L(X))$ of the signature operator of $\partial V$ (cf. Definition A.3).

Let $\lambda$ be a real number and define a complex $(\tilde{E}, \tilde{b}_\lambda)$ by

$$\tilde{E}_p = E_p \oplus P^\perp E_{p+1} \quad \text{and} \quad \tilde{b}_\lambda = \begin{pmatrix} b & 0 \\ \lambda P^\perp & -P^\perp b \end{pmatrix}.$$ 

This is the mapping cone complex for the chain map $\lambda P^\perp: (E, b) \to (P^\perp E, P^\perp b)$. Together with the operators

$$\tilde{T} = \begin{pmatrix} 0 & TP^\perp \\ (-1)^p P^\perp T & 0 \end{pmatrix} : \tilde{E}_p \to \tilde{E}_{n-p},$$

the triple $(\tilde{E}, \tilde{b}_\lambda, \tilde{T})$ is an $n$-dimensional geometrically controlled Poincaré complex for any $\lambda$ (including $\lambda = 0$). Of course, we need to check that $\tilde{T}$ is indeed a geometrically controlled homotopy equivalence. This can be verified by applying \cite[Lemma 4.2]{22} to the following commutative diagram:\footnote{One needs to take into account of the sign convention when verifying various identities. For example, the map $(-1)^p P^\perp T$ carries the sign $(-1)^p$ when it maps from $E_p$ to $P^\perp E_{n-p+1}$.}

$$\begin{array}{cccccc}
0 & \longrightarrow & (E, b^*) & \longrightarrow & (\tilde{E}, \tilde{b}_\lambda^*) & \longrightarrow & (P^\perp E, -b^* P^\perp) & \longrightarrow & 0 \\
\downarrow (-1)^p P^\perp T & & \downarrow \hat{\tau} & & \downarrow TP^\perp & \\
0 & \longrightarrow & (P^\perp E, -P^\perp b) & \longrightarrow & (\tilde{E}, \tilde{b}_\lambda) & \longrightarrow & (E, b) & \longrightarrow & 0.
\end{array}$$
Note that, when \( \lambda = -1 \), the map \( A(v) = v \oplus 0 \in E_p \oplus P^\perp E_{p+1} \) defines a geometrically controlled chain homotopy equivalence of geometrically controlled Poincaré complexes

\[
A: (PE, Pb, T_0) \to (\tilde{E}, \tilde{b}_{-1}, \tilde{T}).
\]

Indeed, we apply \([22, \text{Lemma 4.2}]\) to the following commutative diagram:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & (PE, Pb) & \xrightarrow{=} & (PE, Pb) & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow A & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (PE, Pb) & \xrightarrow{= \ A} & (\tilde{E}, \tilde{b}_{-1}) & \xrightarrow{Q} & (E', b') & \longrightarrow & 0 \\
\end{array}
\]

where \( E'_p = P^\perp E_p \oplus P^\perp E_{p+1} \) with \( b' = \left( \begin{array}{cc} p^\perp & 0 \\ -1 & -p^\perp \end{array} \right) \) and \( Q \) is the obvious orthogonal projection. It is easy to see that \((E', b')\) is geometrically controlled chain homotopy equivalent to the trivial chain 0. Moreover, we have\(^8\)

\[
AT_0 A^* - \tilde{T} = h_{p+1} \circ \tilde{b}_{-1}^* + (-1)^p \tilde{b}_{-1} \circ h_p: \tilde{E}_p \to \tilde{E}_{n-p},
\]

where \( h_p = \left( \begin{array}{cc} T & 0 \\ 0 & 0 \end{array} \right): E_p \oplus P^\perp E_{p+1} \to E_{n-p+1} \oplus P^\perp E_{n-p+2} \). This shows that \( AT_0 A^* \) and \( \tilde{T} \) are geometrically controlled homotopic to each other.

Let us abuse our notation, and denote by \((B+S)(B-S)^{-1} \in (C^*_L(\tilde{X})^\Gamma)^+\) the explicit representative for the local index \( \text{Ind}_L(\partial V, \psi) \) constructed out of \((PE, Pb, T_0)\) as in Appendix \( A \) (cf. Definition \( A.3 \)). Then the same argument from Section 4.6 produces a continuous path of invertible elements in \((C^*_L(\tilde{X})^\Gamma)^+\) connecting \((B+S)(B-S)^{-1}\) to

\[
(B_{-1} + S_{-1})(B_{-1} - S_{-1})^{-1} \in (C^*_L(\tilde{X})^\Gamma)^+
\]

where \((B_{-1} + S_{-1})(B_{-1} - S_{-1})^{-1}\) stands for the representative of the local index constructed out of \((\tilde{E}, \tilde{b}_{-1}, \tilde{T})\) (cf. Definition \( A.3 \)). To be precise, we in fact need to stabilize \((B+S)(B-S)^{-1}\) by the identity operator, and consider \((B+S)(B-S)^{-1} \oplus I\) instead. For notational simplicity, we will omit these stabilizing steps.

On the other hand, there is a continuous path of invertible elements

\[
(B_t + S_t)(B_t - S_t)^{-1} \in (C^*_L(\tilde{X})^\Gamma)^+
\]

representing the local index class constructed out of \((\tilde{E}, \tilde{b}_t, \tilde{T})\) for \( t \in [-1, 0] \) (cf. Definition \( A.3 \)). Of course, it is important to appropriately control the propagations of various terms. This can be achieved by Proposition \( B.3 \) in Appendix \( B \). Note that, for \((\tilde{E}, \tilde{b}_0, \tilde{T})\), the duality operator \( \tilde{T} \) is operator homotopic to its additive inverse along the path

\[
\tilde{T} = \left( \begin{array}{cc} 0 & \exp(is)TP^\perp \\ (-1)^p \exp(is)P^\perp T & 0 \end{array} \right)
\]

with \( s \in [0, \pi] \). Now the same argument from Section 4.6 again shows that \((B_0 + S_0)(B_0 - S_0)^{-1}\) is connected to the identity by a path of invertible elements in \((C^*_L(\tilde{X})^\Gamma)^+\). This finishes the proof. \( \square \)

---

\(^8\)Note that the appearance of \((-1)^p\) is due to our sign convention.
Theorem 5.8. The map $\rho: S_n(X) \to K_n(C^*_L(\tilde{X})^\Gamma)$ is well-defined.

Proof. Let $\theta = (M, \partial M, \varphi, N, \partial N, \psi, f)$ be an element in $S_n(X)$. Suppose $\theta$ is cobordant to zero in $S_n(X)$. Then we have $(W, \partial W, \Phi, V, \partial V, \Psi, F)$ as in Definition 3.6. Note that $F|_{\partial V}: \partial V \to \partial W$ is an infinitesimally controlled homotopy equivalence over $X$, thus $\partial V$ and $\partial W$ will not contribute to the higher rho invariant of $F|_{\partial V}: \partial V \to \partial W$.

More precisely, $F|_{\partial V}: \partial V \to \partial W$ induces an infinitesimally controlled chain homotopy equivalence between the Poincaré pair associated to $\partial V$ and the Poincaré pair associated to $\partial W$. It follows that the geometrically controlled Poincaré complex associated to $M \cup_f (-N)$ and its refinements are geometrically controlled equivalent to the geometrically controlled Poincaré complex associated to $\partial V \coprod (-\partial W)$ and its refinements. See Appendix A.2 for a related discussion. To summarize, we have

$$\rho(\theta) = \rho(F|_{\partial V}: \partial V \to \partial W).$$

Therefore, it suffices to show that

$$\rho(F|_{\partial V}: \partial V \to \partial W) = 0.$$

In the following, we use the reversed orientation of $W$. Since no confusion will arise, let us still write $W$ to denote $-W$ for the rest of the proof. Fix a triangulation of $V$ and of $W$, together with a sequence of successive refinements $\text{Sub}^n(V)$ and $\text{Sub}^n(W)$ as before. Note that, for the geometrically controlled Poincaré pair associated to the triangulation $\text{Sub}^n(V)$ (resp. $\text{Sub}^n(W)$), all maps appearing in Definition 5.2 are geometrically controlled, with their propagations go to zero as $n \to \infty$.

Now the theorem follows from a combination of the proof of Theorem 5.7 above with the construction of the higher rho invariant in Section 4.6. Indeed, let

$$(B_{\partial V} + S_{\partial V})(B_{\partial V} - S_{\partial V})^{-1} \text{ and } (B_{\partial W} + S_{\partial W})(B_{\partial W} - S_{\partial W})^{-1} \in (C^*_L(\tilde{X})^\Gamma)^+$$

be the representatives of the local indices for the signature operators of $\partial V$ and $\partial W$ respectively. By the proof of Theorem 5.7, we have an explicit continuous path of invertible elements $\{V_s\}_{0 \leq s \leq 1}$ in $(C^*_L(\tilde{X})^\Gamma)^+$ connecting

$$V_0 = (B_{\partial V} + S_{\partial V})(B_{\partial V} - S_{\partial V})^{-1}$$

to the identity operator $V_1 = I$. Similarly, there is an explicit continuous path of invertible elements $\{W_s\}_{0 \leq s \leq 1}$ in $(C^*_L(\tilde{X})^\Gamma)^+$ connecting

$$W_0 = (B_{\partial W} + S_{\partial W})(B_{\partial W} - S_{\partial W})^{-1}$$

to the identity operator $W_1 = I$.

Consider the following elements at time $t = 0$:

$$V_s(0) \text{ and } W_s(0) \in C^*(\tilde{X})^\Gamma.$$

Let $(E, b, T)_{V, \partial V}$ and $(E', b', T')_{W, \partial W}$ be the geometrically controlled Poincaré pairs associated to the triangulations of $V$ and $W$ respectively. Note that we are not taking
subdivisions at the moment. Then the homotopy equivalence $F: V \to W$ induces a geometrically controlled chain homotopy equivalence between the geometrically controlled Poincaré pairs $(E, b, T)_{V, \partial V}$ and $(E', b', T')_{W, \partial W}$. In fact, the homotopy equivalence $F: V \to W$ also induces corresponding geometrically controlled chain homotopy equivalences between various Poincaré complexes, such as $(\widetilde{E}, \widetilde{b}, \widetilde{T})_{V, \partial V}$ and $(\widetilde{E}', \widetilde{b}', \widetilde{T}')_{W, \partial W}$, that appear in the proof of Theorem 5.7. Consequently, the construction in Section 4.6 simultaneously produces continuous paths $\{U_s(t)\}_{-1 \leq t \leq 0}$ of invertible elements connecting

$$V_s(0) \oplus W_s(0)$$

to the identity operator, for $s \in [0, 1]$.

For each $s \in [0, 1]$, we concatenate the path $\{U_s(t)\}_{-1 \leq t \leq 0}$ with the path $\{V_s(t) \oplus W_s(t)\}_{0 \leq t < \infty}$. This produces an element, denoted by $\rho_s$, in $(C^*_{L,0}(\tilde{X})^\Gamma)^+$ for each $s \in [0, 1]$. Since $\{\rho_s\}_{0 \leq s \leq 1}$ is a norm continuous path of invertible elements in $(C^*_{L,0}(\tilde{X})^\Gamma)^+$, it is clear that

$$[\rho_0] = [\rho_1] \in K_1(C^*_{L,0}(\tilde{X})^\Gamma).$$

On the other hand, $\rho_0$ is precisely the definition of the higher rho invariant of $F_{\partial}: \partial V \to \partial W$, while $\rho_1 \equiv I$ is the constant identity operator. Therefore, $\rho(\theta) = [\rho_0] = 0$. This finishes the proof.

$$\square$$

## 6 Mapping surgery to analysis

In this section, for each closed oriented topological manifold $X$ of dimension $\geq 5$, we prove the commutativity of the following diagram of abelian groups:

$$
\begin{array}{cccc}
N_{n+1}(X) & \xrightarrow{i_*} & L_{n+1}(\Gamma) & \xrightarrow{j_*} & S_n(X) & \xrightarrow{k_{n, \rho}} & N_n(X) \\
\text{Ind}_L & & \text{Ind} & & k_{n, \text{Ind}_L} & & \\
K_{n+1}(C^*_L(\tilde{X})^\Gamma) & \xrightarrow{h_*} & K_{n+1}(C^*_r(\Gamma)) & \xrightarrow{k_n} & K_n(C^*_{L,0}(\tilde{X})^\Gamma) & \xrightarrow{k_n} & K_n(C^*_L(\tilde{X})^\Gamma)
\end{array}
$$

(9)

where $\Gamma = \pi_1 X$ and

$$k_n = \begin{cases} 
1 & \text{if } n \text{ is even}, \\
2 & \text{if } n \text{ is odd}.
\end{cases}$$

In the case of smooth manifolds, a similar commutative diagram was proved by Higson and Roe [22]. Since the structure set of a smooth manifold does not carry a group structure, the commutative diagram of Higson and Roe is a commutative diagram of sets in an appropriate sense [22, Section 5]. Piazza and Schick gave a different proof of Higson and Roe’s commutative diagram for smooth manifolds [37]. Zenobi proved a similar commutative diagram for topological manifolds, but only treating $S_n(X)$ as a set [67].

The local index map

$$\text{Ind}_L: N_n(X) \to K_n(C^*_L(\tilde{X})^\Gamma)$$
is defined by assigning each element in $\mathcal{N}_n(X)$ the $K$-homology class of its signature operator. See Appendix A for more details. The well-definedness of the map $\text{Ind}_L$ follows from the bordism invariance of $K$-homology class of signature operator. See Theorem 5.7 above.

**Remark 6.1.** Note that the well-definedness of the map $\text{Ind}_L : \mathcal{N}_n(X) \to K_n(C^*_r(\tilde{X}))$ implies Novikov’s theorem on the topological invariance of the rational Pontrjagin classes [35]. Also see [36].

The index map

$$\text{Ind} : L_{n+1}(\Gamma) \to K_{n+1}(C^*_r(\Gamma))$$

is defined as follows. Suppose we have an element $\theta = (M, \partial M, \varphi, N, \partial N, \psi, f) \in L_{n+1}(\Gamma)$ satisfying the conditions in Definition 3.7. Consider the space $M \cup_f (-N)$ obtained by gluing $-N$ with $M$ along the boundary by the map $f$, where $-N$ is the manifold $N$ with reversed orientation. Although $M \cup_f (-N)$ is not a manifold in general, it is still a space equipped with Poincaré duality. In particular, the higher signature index of $M \cup_f (-N)$ makes sense.

**Definition 6.2.** For each element $\theta = (M, \partial M, \varphi, N, \partial N, \psi, f) \in L_{n+1}(\pi_1 X, X)$, we define $\text{Ind}(\theta)$ to be the higher signature index of $M \cup_f (-N)$.

**Proposition 6.3.** The map $\text{Ind} : L_{n+1}(\Gamma) \to K_{n+1}(C^*_r(\Gamma))$ is a well-defined group homomorphism.

**Proof.** The well-definedness follows immediately from the bordism invariance of higher signature index. Moreover, the higher signature index is clearly additive on disjoint unions, hence the index map is a group homomorphism. This finishes the proof. □

We need some preparation before we prove the commutativity of diagram (9) above.

Recall that, by Proposition 3.16, there is a natural isomorphism

$$S_n(X) \cong L_{n+1}(\pi_1 X, X).$$

In the following, first we shall give another definition, denoted by $\hat{\rho}$, of higher rho invariant by using the description of $L_{n+1}(\pi_1 X, X)$. Then we will prove that $\hat{\rho} = k_n \rho$, where $k_n = 1$ if $n$ is even and 2 if $n$ is odd.

Recall from Definition 3.12 that, for each

$$\theta = (M, \partial_\pm M, \varphi, N, \partial N_\pm, \psi, f) \in L_{n+1}(\pi_1 X, X),$$

the map $f|_{\partial_\pm N} : \partial_+ N \to \partial_+ M$ is a homotopy equivalence, and $f|_{\partial_+ N}$ restricts to a PL infinitesimally controlled homotopy equivalence

$$f|_{\partial(\partial_\pm N)} : \partial(\partial_\pm N) \to \partial(\partial_\pm M).$$

Let $Z = M \cup_{f_+} (-N)$ be the space obtained by gluing $M$ and $N$ along the boundary $\partial_+ N$ to $\partial_+ M$ through the homotopy equivalence $f|_{\partial_+ N}$. Though $Z$ is not a manifold, it is a space equipped with Poincaré duality. Note that the “boundary” of $Z$ is the space $\partial_- M \cup_{f_+} \partial_- (-N)$, where the latter is obtained by gluing $\partial_- M$ and $\partial_- N$ along
the boundary $\partial(\partial_- N)$ to $\partial(\partial_- M)$ through the infinitesimally controlled homotopy equivalence $f|_{\partial(\partial_- N)}$. Let us write $\partial Z := \partial_- M \cup_{\partial f_+} \partial_- (-N)$.

Let us fix a triangulation of $CZ$ as follows. On $Z$, it is the original triangulation of $Z$. The triangulation on $\partial Z \times [1, \infty)$ is the pullback triangulation of $\text{Tri}_{X \times [1, \infty)}$ under the map $\Phi_{\theta} \times \text{Id}: \partial Z \times [1, \infty) \to CX$, where $\Phi_{\theta}$ is the restriction of $\Phi = \varphi \cup_{f_+} \psi$ on $\partial Z$. That is, for every simplex $\Delta^k \subset CX$, the inverse image $(\Phi_{\theta} \times \text{Id})^{-1}(\Delta^k)$ is a product $K \times \Delta^k$, where $K$ is some triangulated submanifold of $\partial Z$.

**Remark 6.4.** To be precise, we should be using a sequence of spaces $\{Z_i\}_{i \geq 1}$, where each $Z_i = M \cup_{(f_i)_+} (-N)$ be the space obtained by gluing $M$ and $N$ along the boundary $\partial_+ N$ to $\partial_+ M$ through the homotopy equivalence $(f_i)|_{\partial_+ N}$. Here $f_i$ is the map $f$ with the additional condition that the homotopy equivalence $f|_{\partial(\partial_+ N)}: \partial(\partial_+ N) \to \partial(\partial_+ M)$ has control $\leq \frac{1}{i}$. Since no confusion is likely to arise, we shall abuse our notation and continue as if we are working with a single space.

Now the geometrically controlled Poincaré complex associated to $CZ$ canonically defines a higher signature index in $K_{n+1}(C^*_c(\tilde{X} \times [1, \infty]))^\Gamma$.

**Definition 6.5.** With the same notation as above, for $\theta = (M, \partial_\pm M, \varphi, N, \partial N_\pm, \psi, f) \in L_{n+1}(\pi_1 X, X)$, we define $\hat{\rho}(\theta) \in K_{n+1}(C^*_c(\tilde{X} \times [1, \infty]))^\Gamma$ to be the higher signature index of $CZ$.

The following lemma is an immediate consequence of bordism invariance of higher signature index.

**Lemma 6.6.** $\hat{\rho}$ defines a group homomorphism

$$\hat{\rho} : L_{n+1}(\pi_1 X, X) \to K_{n+1}(C^*_c(\tilde{X} \times [1, \infty]))^\Gamma.$$
Recall that we have the following natural isomorphism

\[ c_* : \mathcal{S}_n(X) \to L_{n+1}(\pi_1 X, X) \]

by taking the product with the unit interval (see Section 3.3):

\[ \theta = \{M, \partial M, \varphi, N, \partial N, \psi, f\} \mapsto \theta \times I. \]

Moreover, by Proposition 4.4 and Corollary 4.7, we have

\[ K_{n+1}(C^*_\epsilon(\widetilde{X} \times [1, \infty]))^\Gamma \cong K_n(C^*_{L,0,\epsilon}(\widetilde{X} \times [1, \infty]))^\Gamma \cong K_n(C^*_{L,0}(\widetilde{X})^\Gamma). \]

It follows that \( \hat{\rho} \) also induces a group homomorphism

\[ \hat{\rho} : \mathcal{S}_n(X) \to K_n(C^*_{L,0}(\widetilde{X})^\Gamma). \]

Remark 6.7. In the case of smooth structure sets, it is easy to see that the definition \( \hat{\rho} \) above agrees with the structure invariant of Higson and Roe [23, Section 3].

In Theorem 6.9 below, we will prove that \( \hat{\rho} \) is equal to \( k_n \cdot \rho \), where \( \rho \) is the higher rho invariant from Definition 4.39 and \( k_n = 1 \) if \( n \) is even and \( 2 \) if \( n \) is odd. Before doing this, let us first prove a product formula for the higher rho invariant \( \rho \), which will be useful for the proof of Theorem 6.9.

Given \( \theta = (M, \partial M, \varphi, N, \partial N, \psi, f) \in \mathcal{S}_n(X) \), let \( \theta \times \mathbb{R} \in \mathcal{S}_{n+1}(X \times \mathbb{R}) \) be the product of \( \theta \) and \( \mathbb{R} \), which defines an element in \( \mathcal{S}_{n+1}(X \times \mathbb{R}) \). Here various undefined terms take the obvious meanings (see Section 3.3 for the definition of \( \theta \times I \) for example).

Note that the construction in Section 4.6 also applies to \( \theta \times \mathbb{R} \) and defines its higher rho invariant \( \rho(\theta \times \mathbb{R}) \in K_{n+1}(C^*_{L,0}(\widetilde{X} \times \mathbb{R})^\Gamma) \). Also there is a natural homomorphism \( \alpha : C^*_{L,0}(\widetilde{X})^\Gamma \otimes C^*_{L}(\mathbb{R}) \to C^*_{L,0}(\widetilde{X} \times \mathbb{R})^\Gamma \), which induces isomorphisms on \( K \)-theory groups.

**Theorem 6.8.** With the same notation as above, we have

\[ k_n \cdot \alpha_*(\rho(\theta) \otimes \text{Ind}_L(\mathbb{R})) = \rho(\theta \times \mathbb{R}) \]

in \( K_{n+1}(C^*_{L,0}(\widetilde{X} \times \mathbb{R})^\Gamma) \), where \( \text{Ind}_L(\mathbb{R}) \) is the \( K \)-homology class of the signature operator on \( \mathbb{R} \), and \( k_n = 1 \) if \( n \) is even and \( 2 \) if \( n \) is odd.

**Proof.** The proof is elementary, and will be given in Appendix D. \( \square \)

To prepare for the proof of Theorem 6.9, let us introduce some notation. Consider the \( C^* \)-algebra \( \mathcal{A} = C^*_{L,0}(\widetilde{X} \times \mathbb{R})^\Gamma \). Using the notation from Definition 2.2, we define

\[ \mathcal{A}_- = \bigcup_{n \in \mathbb{N}} C^*_{L,0}(\widetilde{X} \times (-\infty, n]; \widetilde{X} \times \mathbb{R})^\Gamma. \]

Similarly, we define \( \mathcal{A}_+ = \bigcup_{n \in \mathbb{N}} C^*_{L,0}(\widetilde{X} \times [-n, \infty); \widetilde{X} \times \mathbb{R})^\Gamma \), and

\[ \mathcal{A}_\cap = \bigcup_{n \in \mathbb{N}} C^*_{L,0}(\widetilde{X} \times [-n, n]; \widetilde{X} \times \mathbb{R})^\Gamma \]

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It is clear that \( \mathscr{A}_\pm \) and \( \mathscr{A}_\cap \) are closed two-sided ideals of \( \mathscr{A} \). Moreover, we have \( \mathscr{A}_+ + \mathscr{A}_- = \mathscr{A} \) and \( \mathscr{A}_+ \cap \mathscr{A}_- = \mathscr{A}_\cap \), which gives rise to the following Mayer-Vietoris sequence in \( K \)-theory:

\[
\begin{array}{ccc}
K_0(\mathscr{A}_\cap) & \longrightarrow & K_0(\mathscr{A}_+) \oplus K_0(\mathscr{A}_-)
\end{array}
\]

\[
\begin{array}{ccc}
\partial_{MV} & \longrightarrow & K_0(\mathscr{A})
\end{array}
\]

Similarly, consider the \( C^* \)-algebra \( \mathcal{B} = C_L^*(\mathbb{R}) \) and define \( \mathcal{B}_+ = \bigcup_{n \in \mathbb{N}} C_L^*([-n, \infty); \mathbb{R}) \) and \( \mathcal{B}_\cap = \bigcup_{n \in \mathbb{N}} C_L^*([-n, n]; \mathbb{R}) \).

These \( C^* \)-algebras again give rise to the following Mayer-Vietoris sequence in \( K \)-theory:

\[
\begin{array}{ccc}
K_0(\mathcal{B}_\cap) & \longrightarrow & K_0(\mathcal{B}_+) \oplus K_0(\mathcal{B}_-)
\end{array}
\]

\[
\begin{array}{ccc}
\partial_{MV} & \longrightarrow & K_0(\mathcal{B})
\end{array}
\]

\[
\begin{array}{ccc}
K_1(\mathcal{B}_+) \oplus K_1(\mathcal{B}_-) & \longleftarrow & K_1(\mathcal{B}_\cap)
\end{array}
\]

Note that there is a natural homomorphism \( \alpha: C_{L,0}^*(\widetilde{X})^G \otimes \mathcal{B} \to \mathscr{A} \), which restricts to homomorphisms

\[
\alpha: C_{L,0}^*(\widetilde{X})^G \otimes \mathcal{B}_\pm \to \mathscr{A}_\pm \quad \text{and} \quad \alpha: C_{L,0}^*(\widetilde{X})^G \otimes \mathcal{B}_\cap \to \mathscr{A}_\cap,
\]

such that the following diagram commutes:

\[
\begin{array}{c}
K_n(C_{L,0}^*(\widetilde{X})^G) \otimes K_0(\mathcal{B}) \xrightarrow{\partial_{MV}} K_{n+1}(C_{L,0}^*(\widetilde{X})^G \otimes \mathcal{B}) \xrightarrow{\alpha_\cap} K_{n+1}(C_{L,0}^*(\widetilde{X} \times \mathbb{R})^G)
\end{array}
\]

(10)

**Theorem 6.9.** The following diagram commutes:

\[
\begin{array}{c}
L_{n+1}(\pi_1X, X) \xrightarrow{\hat{\rho}} K_{n+1}(C_{c}^*(\widetilde{X} \times [1, \infty))^G) \xrightarrow{\partial_*} K_{n+1}(C_{c}^*(\widetilde{X} \times [1, \infty))^G)
\end{array}
\]

where \( \partial_* \) is the connecting map in the \( K \)-theory long exact sequence associated to

\[
0 \to C_{L,0}^*(\widetilde{X} \times [1, \infty))^G \to C_{L,c}^*(\widetilde{X} \times [1, \infty))^G \to C_{c}^*(\widetilde{X} \times [1, \infty))^G \to 0.
\]

Here \( k_n = 1 \) if \( n \) is even and \( 2 \) if \( n \) is odd.
Proof. Recall that a standard way to construct the connecting map \( \partial_{\ast} \) is by lifting a projection (resp. invertible) in \( C^{\ast}_{c}(\widetilde{X} \times [1, \infty])^{\Gamma} \) to an element in \( C^{\ast}_{L,c}(\widetilde{X} \times [1, \infty])^{\Gamma} \). For \( \theta = \{ M, \partial_{\pm} M, \varphi, N, \partial_{\pm} N, \psi, f \} \in L_{n+1}(\pi_{1}X, X) \), there exists an explicit lifting \( a_{\theta} \in C^{\ast}_{L,c}(\widetilde{X} \times [1, \infty])^{\Gamma} \) of \( \tilde{\rho}(\theta) \in C^{\ast}_{c}(\widetilde{X} \times [1, \infty])^{\Gamma} \) as follows. Let

\[
a_{\theta}(n) = \chi_{n}\tilde{\rho}(\theta)\chi_{n},
\]

where \( \chi_{n} \) is the characteristic function on \( \widetilde{X} \times [n, \infty) \). We define

\[
a_{\theta}(t) = (n + 1 - t)a_{\theta}(n) + (t - n)a_{\theta}(n + 1)
\]

for all \( n \leq t \leq n + 1 \). It is clear that \( a_{\theta} \) lies in \( C^{\ast}_{L,c}(\widetilde{X} \times [1, \infty])^{\Gamma} \) and is a lift of \( \tilde{\rho}(\theta) \).

On the other hand, it is not difficult to see that the same \( a_{\theta} \) above is also a lift of \( \rho(\theta \times \mathbb{R}) \), for the construction of the connecting map

\[
\partial_{MV} : K_{n+1}(C^{\ast}_{L,0}(\widetilde{X} \times \mathbb{R})^{\Gamma}) \to K_{n}(C^{\ast}_{L,0}(\widetilde{X})^{\Gamma})
\]

in diagram (10). In particular, we see that

\[
\partial_{\ast} \circ \tilde{\rho} \circ c_{\ast}(\theta) = \partial_{MV} [\rho(\theta \times \mathbb{R})].
\]

Now by Theorem 6.8 and the commutative diagram (10), it follows that

\[
\partial_{\ast} \circ \tilde{\rho} \circ c_{\ast}(\theta) = k_{n} \cdot \rho(\theta) \otimes \partial_{MV}[\text{Ind}_{L}(\mathbb{R})] = k_{n} \cdot \rho(\theta).
\]

This finishes the proof.

\[\square\]

**Theorem 6.10.** The diagram (9) is commutative.

**Proof.** The commutativity of the right square and the left square follows immediately from definition.

The commutativity of the middle square is an immediate consequence of Theorem 6.9 above. Indeed, by Theorem 6.9, we have the following commutative diagram

\[
\begin{array}{ccc}
L_{n}(\pi_{1}X) & \xrightarrow{j_{\ast}} & L_{n+1}(\pi_{1}X, X) \\
\ind & & \downarrow \rho_{k_{n}} \\
K_{n}(C^{\ast}(\widetilde{X})^{\Gamma}) & \xrightarrow{\partial_{\ast}} & K_{n+1}(C^{\ast}_{c}(\widetilde{X} \times [1, \infty])^{\Gamma}) \\
\end{array}
\]

Here the commutativity of the lower right square follows from the definition of the index map and the map \( \tilde{\rho} \). This finishes the proof.

\[\square\]

**Remark 6.11.** The exact same proof also applies to the maximal version, and we have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{N}_{n+1}(X) & \xrightarrow{i_{\ast}} & L_{n+1}(\Gamma) & \xrightarrow{j_{\ast}} & \mathcal{S}_{n}(X) & \xrightarrow{k_{n}} & \mathcal{N}_{n}(X) \\
\ind_{L} & & & & \ind & & \ind_{L} \\
K_{n+1}(C^{\ast}_{L}(\widetilde{X})_{\max}^{\Gamma}) & \xrightarrow{\partial_{\ast}} & K_{n+1}(C^{\ast}_{c}(\Gamma)) & \xrightarrow{k_{n}} & K_{n}(C^{\ast}_{L,0}(\widetilde{X})_{\max}^{\Gamma}) & \xrightarrow{k_{n}} & K_{n}(C^{\ast}_{L}(\widetilde{X})_{\max}^{\Gamma})
\end{array}
\]
7 Novikov rho invariant and strong Novikov conjecture

In this section, we define a homological version of the higher rho invariant from Section 4.6. This homological higher rho invariant will be called Novikov rho invariant for reasons that will be explained in later part of this section. One importance of the Novikov rho invariant is that it can be used to detect nontrivial elements in the structure group of a closed oriented topological manifold, even when the fundamental group of the manifold is torsion free. Throughout the section, we assume \( n \geq 5 \).

Let \( X \) be a proper metric space with \( \pi_1 X = \Gamma \). Suppose \( \tilde{X} \) is the universal cover of \( X \). We have the following commutative diagram:

\[
\begin{array}{cccccc}
K_{n+2}^\Gamma(EG, \tilde{X}) & \longrightarrow & K_{n+1}^\Gamma(\tilde{X}) & \longrightarrow & K_{n+1}^\Gamma(EG) & \longrightarrow & K_{n+1}^\Gamma(EG, \tilde{X}) \\
\Lambda & \downarrow \cong & \downarrow \mu & & \Lambda & \downarrow \mu \circ \rho \circ \Lambda & \\
K_{n+1}(C^*_L(\tilde{X})) & \longrightarrow & K_{n+1}(C^*_L(\tilde{X})^\Gamma) & \longrightarrow & K_{n+1}(C^*_r(\Gamma)) & \longrightarrow & K_{n+1}(C^*_r(\tilde{X})^\Gamma)
\end{array}
\]

(12)

where \( K_{n+1}^\Gamma(EG, \tilde{X}) \) is the \( \Gamma \)-equivariant relative \( K \)-homology group for the pair of spaces \((EG, \tilde{X})\). For example, we can define \( K_{n+1}^\Gamma(EG, \tilde{X}) \) to be the \( \Gamma \)-equivariant \( K \)-homology group of the mapping cone of \( \tilde{X} \rightarrow EG \). Also, \( K_{n+1}^\Gamma(EG, \tilde{X}) \) is naturally isomorphic to the \( K \)-theory group of the \( C^* \)-algebra mapping cone associated to the natural map \( C^*_L(\tilde{X}) \rightarrow C^*_L(EG)^\Gamma \). Furthermore, \( K \)-theory groups of \( C^*_L(\tilde{X})^\Gamma \) are naturally identified with the \( K \)-theory groups of the \( C^* \)-algebra mapping cone associated to the evaluation map \( C^*_r(\tilde{X})^\Gamma \rightarrow C^*_r(\Gamma) \), cf. [7]. In view of this mapping cone picture, the commutativity of the above diagram is clear.

We would like to see in what circumstances there exists a natural homomorphism \( \beta : K_{n+1}(C^*_L(\tilde{X})) \rightarrow K_{n+1}^\Gamma(EG, \tilde{X}) \) such that the following diagram remains commutative.

\[
\begin{array}{cccccc}
K_{n+2}^\Gamma(EG, \tilde{X}) & \longrightarrow & K_{n+1}^\Gamma(\tilde{X}) & \longrightarrow & K_{n+1}^\Gamma(EG) & \longrightarrow & K_{n+1}^\Gamma(EG, \tilde{X}) \\
\Lambda \uparrow \beta & & \cong & & \mu \circ \Lambda & \circ \mu \circ \Lambda & \\
K_{n+1}(C^*_L(\tilde{X})) & \longrightarrow & K_{n+1}(C^*_L(\tilde{X})^\Gamma) & \longrightarrow & K_{n+1}(C^*_r(\Gamma)) & \longrightarrow & K_{n+1}(C^*_r(\tilde{X})^\Gamma) \\
\rho & \downarrow & \Ind_L & \downarrow \Ind & \downarrow k_\rho & & \\
\mathcal{S}_{n+1}(X) & \longrightarrow & \mathcal{N}_{n+1}(X) & \longrightarrow & L_{n+1}(\Gamma) & \longrightarrow & \mathcal{S}_n(X)
\end{array}
\]

(13)

Now suppose that the strong Novikov conjecture holds for \( \Gamma \), that is, the Baum-Connes assembly map \( \mu_+ : K_{n+1}^\Gamma(EG) \rightarrow K_{n+1}(C^*_r(\Gamma)) \) is injective. In fact, let us assume a slightly stronger condition, that is,

\[
\mu_+ : K_{n+1}^\Gamma(EG) \rightarrow K_{n+1}(C^*_r(\Gamma))
\]

is a split injection. So far, in all known cases where the strong Novikov conjecture holds, the split injectivity of the Baum-Connes assembly map is known to be true as well, cf. [11, 17, 19, 20, 29, 30, 52, 64, 65].
In this case, let us denote the splitting map by $\alpha: K_{n+1}(C_r^*(\Gamma)) \to K_{n+1}^\Gamma(E\Gamma)$ which induces a direct sum decomposition:

$$K_{n+1}(C_r^*(\Gamma)) \cong K_{n+1}^\Gamma(E\Gamma) \oplus \mathcal{E}. $$

Then a routine diagram chasing shows that

1. the homomorphism $\Lambda: K_{n+1}^\Gamma(E\Gamma, \tilde{X}) \to K_{n}(C_{L,0}^*(\tilde{X})^\Gamma)$ is also an injection;
2. and $\partial(\mathcal{E}) \cap \partial(K_{n+1}^\Gamma(E\Gamma)) = 0.$

It follows that we have the following commutative diagram:

$$
\begin{array}{c}
\rightarrow K_{n+1}^\Gamma(\tilde{X}) \xrightarrow{=} K_{n+1}^\Gamma(E\Gamma) \xrightarrow{\mu} K_{n+1}^\Gamma(E\Gamma, \tilde{X}) \xrightarrow{\Lambda} K_{n}^\Gamma(\tilde{X}) \rightarrow \\
\xrightarrow{=} K_{n+1}(C_{L}^*(\tilde{X})^\Gamma) \xrightarrow{\partial} K_{n}(C_{L,0}^*(\tilde{X})^\Gamma) \xrightarrow{q} K_{n}(C_{L}^*(\tilde{X})^\Gamma) \rightarrow \\
\xrightarrow{=} K_{n+1}(C_{L}^*(\tilde{X})^\Gamma) \xrightarrow{\partial} K_{n}(C_{L,0}^*(\tilde{X})^\Gamma)/\partial(\mathcal{E}) \xrightarrow{\alpha} K_{n}(C_{L}^*(\tilde{X})^\Gamma) \rightarrow
\end{array}
$$

where $q$ is the quotient map $q: K_{n}(C_{L,0}^*(\tilde{X})^\Gamma) \to K_{n}(C_{L,0}^*(\tilde{X})^\Gamma)/\partial(\mathcal{E}).$

Note that the last row in diagram (14) is also a long exact sequence. By the five lemma, it follows that the composition

$$q \circ \Lambda: K_{n+1}^\Gamma(E\Gamma, \tilde{X}) \xrightarrow{\cong} K_{n}(C_{L,0}^*(\tilde{X})^\Gamma)/\partial(\mathcal{E})$$

is an isomorphism.

Now we define

$$\beta := (q \circ \Lambda)^{-1} \circ q: K_{n}(C_{L,0}^*(\tilde{X})^\Gamma) \to K_{n+1}^\Gamma(E\Gamma, \tilde{X}).$$

By definition, $\beta$ makes diagram (13) commute.

**Definition 7.1.** We define the Novikov rho invariant map $\rho^{Nov}$ to be the composition

$$\rho^{Nov} = \beta \circ (k_n \cdot \rho): S_n(X) \to K_{n+1}^\Gamma(E\Gamma, \tilde{X}),$$

where we have

$$k_n = \begin{cases} 
1 & \text{if } n \text{ is even}, \\
2 & \text{if } n \text{ is odd}.
\end{cases}$$

**Remark 7.2.** Note that our definition of the invariant $\rho^{Nov}$ only works when the strong Novikov conjecture holds. This is the reason that we name this homological higher rho invariant after Novikov. To be more precise, we have assumed a slightly stronger condition that the Baum-Connes assembly map is split injective. So far, in all known cases where the strong Novikov conjecture holds, the split injectivity of the Baum-Connes assembly map is known to true as well, cf. [11, 17, 19, 20, 29, 30, 52, 64, 65].
Remark 7.3. There is also a maximal version of the Novikov rho invariant defined above. In this case, we assume that the Baum-Connes assembly map is split injective for the maximal group $C^*$-algebra $C^*_{\text{max}}(\pi_1 X)$. The Novikov rho invariant is defined similarly. This split injectivity assumption for maximal group $C^*$-algebras is weaker than the same assumption for reduced group $C^*$-algebras.

We invert 2 for the rest of this section. With some minor modifications, all discussions in this paper work equally well for the real case. Roughly speaking, whenever the imaginary number $i = \sqrt{-1}$ appears in a formula, we replace it by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. More precisely, for a geometrically controlled Poincaré complex $(E, b, T)$, we consider the direct sum $(E, b, T) \oplus (E, b, T)$.

For the operator $S$ in Definition 4.10, we define

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{p(p-1)+T}.$$

The same remark applies to various other formulas, such as the formula in line (8), where complex numbers are used. In the case of dimension $n = 0$ or 1 mod 4, this gives rise to a signature operator that is twice of the actual signature operator with real coefficients. Now taking product with $\mathbb{R}^2$ takes care of the case of dimension $n = 2$ or 3 mod 4. The analogue of the diagram (9) for the real case involves extra factors of 2 in front of various maps. We will leave out the details. In any case, it is harmless to introduce these extra factors of 2, since we have already inverted 2.

Recall that, after inverting 2, the maps $\tau_1$ and $\tau_2$ in the following commutative diagram are split injective:

$$\begin{array}{ccc}
KO_\Gamma^\Gamma(E\Gamma)[\frac{1}{2}] & \xrightarrow{\mu_\mathbb{R}} & K_i(C^*_\Gamma(\Gamma, \mathbb{R}))[\frac{1}{2}] \\
\downarrow \tau_1 & & \downarrow \tau_2 \\
K_\Gamma^\Gamma(E\Gamma)[\frac{1}{2}] & \xrightarrow{\mu} & K_i(C^*_\Gamma(\Gamma))[\frac{1}{2}] 
\end{array}$$

where the notation $[\frac{1}{2}]$ means $\otimes \mathbb{Z}[\frac{1}{2}]$, $\mu_\mathbb{R}$ and $\mu$ are Baum-Connes assembly maps, and $\tau_1$ and $\tau_2$ are induced by changing the scalars from $\mathbb{R}$ to $\mathbb{C}$. In particular, if we assume the split injectivity of the Baum-Connes assembly map, then we also have the Novikov rho invariant map in the real case:

$$\rho_{\text{Nov}}: S_n(X) \rightarrow KO_{n+1}^\Gamma(E\Gamma, \tilde{X})[\frac{1}{2}].$$

Moreover, we shall see that $\rho_{\text{Nov}}$ is surjective in this case.

Recall that, after inverting 2, the split injectivity of Baum-Connes assembly map implies that the split injectivity of Farrell-Jones assembly map (cf. [33, Proposition 95, page 758]), that is,

$$A: H^\Gamma_1(\mathbb{E}\Gamma; \mathbb{L}_\bullet)[\frac{1}{2}] \rightarrow L_i(\Gamma)[\frac{1}{2}] = L_i(Z\Gamma)[\frac{1}{2}],$$

is a split injection. Denote the splitting map by

$$\gamma: L_i(\Gamma)[\frac{1}{2}] \rightarrow H^\Gamma_1(\mathbb{E}\Gamma; \mathbb{L}_\bullet)[\frac{1}{2}].$$
Recall that, for any $Y$, we have

$$H_i(Y; \mathbb{L}_\bullet)[\frac{1}{2}] \cong KO_i(Y)[\frac{1}{2}],$$

It follows that we have the following commutative diagram:

$$
\begin{array}{cccc}
KO^\Gamma_{n+2}(E\Gamma, \tilde{X})[\frac{1}{2}] & \longrightarrow & KO^\Gamma_{n+1}(\tilde{X})[\frac{1}{2}] & \longrightarrow & KO^\Gamma_{n+1}(E\Gamma)[\frac{1}{2}] & \longrightarrow & KO^\Gamma_{n+1}(E\Gamma, \tilde{X})[\frac{1}{2}] \\
\rho_{Nov}^\circ & \cong & \gamma & \cong & \rho_{Nov} \\
S_{n+1}(X)[\frac{1}{2}] & \longrightarrow & H^\Gamma_{n+1}(\tilde{X}; \mathbb{L}_\bullet)[\frac{1}{2}] & \longrightarrow & L_{n+1}(\Gamma)[\frac{1}{2}] & \longrightarrow & S_{n}(X)[\frac{1}{2}] \\
\end{array}
$$

(15)

By the splitting map $\gamma: L_{n+1}(\Gamma)[\frac{1}{2}] \to H^\Gamma_{n+1}(E\Gamma; \mathbb{L}_\bullet)[\frac{1}{2}]$, we have the following direct sum decomposition

$$L_{n+1}(\Gamma)[\frac{1}{2}] \cong KO^\Gamma_{n+1}(E\Gamma)[\frac{1}{2}] \oplus \mathcal{F}$$

such that $\partial(KO^\Gamma_{n+1}(E\Gamma)[\frac{1}{2}]) \cap \partial(\mathcal{F}) = 0$. It follows that commutative diagram (15) above descends to the following diagram:

$$
\begin{array}{cccc}
\longrightarrow & KO^\Gamma_{n+2}(E\Gamma, \tilde{X})[\frac{1}{2}] & \longrightarrow & KO^\Gamma_{n+1}(\tilde{X})[\frac{1}{2}] & \longrightarrow & KO^\Gamma_{n+1}(E\Gamma)[\frac{1}{2}] & \longrightarrow & KO^\Gamma_{n+1}(E\Gamma, \tilde{X})[\frac{1}{2}] \\
\rho_{Nov}^\circ & \cong & \gamma & \cong & \rho_{Nov}^\circ \\
\longrightarrow & S_{n+1}(X)[\frac{1}{2}]/\partial(\mathcal{F}) & \longrightarrow & H^\Gamma_{n+1}(\tilde{X}; \mathbb{L}_\bullet)[\frac{1}{2}] & \longrightarrow & L_{n+1}(\Gamma)[\frac{1}{2}]/\mathcal{F} & \longrightarrow & S_{n}(X)[\frac{1}{2}]/\partial(\mathcal{F}) \\
\end{array}
$$

(16)

By the five lemma, it follows that

$$\tilde{\rho}_{Nov}^\circ: S_{n}(X)[\frac{1}{2}]/\partial(\mathcal{F}) \cong KO^\Gamma_{n+1}(E\Gamma, \tilde{X})[\frac{1}{2}]$$

is an isomorphism. In particular, it implies that

$$\rho_{Nov}^\circ: S_{n}(X)[\frac{1}{2}] \to KO^\Gamma_{n+1}(E\Gamma, \tilde{X})[\frac{1}{2}]$$

is surjective.

This surjection can be used to construct many elements in $S_{n}(X)$. For example, if $KO^\Gamma_{n+1}(E\Gamma)[\frac{1}{2}]$ is infinitely generated as module over $\mathbb{Z}[\frac{1}{2}]$, then $KO^\Gamma_{n+1}(E\Gamma, \tilde{X})[\frac{1}{2}]$ is also infinitely generated, for any closed oriented manifold $X$. It follows that, in this case, $S_{n}(X)[\frac{1}{2}]$ is also infinitely generated.

8 Non-rigidity of topological manifolds

In this section, we apply our main theorem 4.41 to give a lower bound of the free rank of $\tilde{S}_{n}(X)$ of an $n$-dimensional closed oriented topological manifold $X$. The two versions of the reduced structure group, $\tilde{S}_{n}(X)$ and $\tilde{ST}_{TOP}(X)$, are different in general. The group $\tilde{S}_{n}(X)$ is more algebraically significant. It is functorial and fits well with the surgery long exact sequence. On the other hand, the group $\tilde{ST}_{TOP}(X)$ is geometrically more significant. It measures the size of the collection of closed manifolds homotopic equivalent but not homeomorphic to $X$. 

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Since we will be using the maximal version of various $C^*$-algebras throughout this section, we will omit the subscript “max” for notational simplicity.

Let $X$ be an $n$-dimensional oriented closed topological manifold. Denote the group of orientation-preserving self-homotopy equivalences of $X$ by $\text{Aut}_h(X)$. There are two different actions of $\text{Aut}_h(X)$ on $\mathcal{S}_n(X) = \mathcal{S}^{\text{TOP}}(X)$, which induce two different versions of reduced structure groups as follows, cf. [43] for the essentially same discussion in the context of algebraic surgery long sequence.

One one hand, $\text{Aut}_h(X)$ acts naturally on $\mathcal{S}_n(X)$ by

$$\alpha_g(\theta) = (M, \partial M, g \circ \varphi, N, \partial N, g \circ \psi, f)$$

for all $g \in \text{Aut}_h(X)$ and all $\theta = (M, \partial M, \varphi, N, \partial N, \psi, f) \in \mathcal{S}_n(X)$. Recall that, the natural identification $\mathcal{S}^{\text{TOP}}(X) = \mathcal{S}_n(X)$ identifies an element $\theta = (f, M) \in \mathcal{S}^{\text{TOP}}(X)$ with

$$\begin{array}{ccc}
M & \xrightarrow{f} & X \\
\downarrow f & & \downarrow \text{Id} \\
X & \quad & \in \mathcal{S}_n(X).
\end{array}$$

In this case, the action $\alpha_g$ on $\mathcal{S}_n(X)$ maps

$$\begin{array}{ccc}
M & \xrightarrow{f} & X \\
\downarrow f & & \downarrow \text{Id} \\
X & \quad & \in \mathcal{S}_n(X).
\end{array}$$

Clearly, $\alpha_g$ is a group homomorphism from $\mathcal{S}_n(X)$ to $\mathcal{S}_n(X)$. Note that this action $\alpha$ is compatible the actions of $\text{Aut}_h(X)$ on other terms in the surgery long exact sequence.

On the other hand, $\text{Aut}_h(X)$ also naturally acts on $\mathcal{S}^{\text{TOP}}(X) = \mathcal{S}_n(X)$ by compositions of homotopy equivalences, that is,

$$g_*(\theta) = (g \circ f, M)$$

for all $g \in \text{Aut}_h(X)$ and all $\theta = (f, M) \in \mathcal{S}^{\text{TOP}}(X)$. In comparison with the action $\alpha_g$ above, the action $g_*$ on $\mathcal{S}^{\text{TOP}}(X) = \mathcal{S}_n(X)$ maps

$$\begin{array}{ccc}
M & \xrightarrow{f} & X \\
\downarrow f & & \downarrow \text{Id} \\
X & \quad & \in \mathcal{S}_n(X).
\end{array}$$

Clearly, $g_*$ is a bijective map of sets, and is not a group homomorphism in general.

**Definition 8.1.** With the same notation as above,

1. we define $\mathcal{S}_n(X) / \langle \alpha_g(\theta) \rangle$ to be the quotient group of $\mathcal{S}_n(X)$ by the subgroup generated by elements of the form $\theta - \alpha_g(\theta)$ for all $\theta \in \mathcal{S}_n(X)$ and all $g \in \text{Aut}_h(X)$;
we define $\widetilde{S}^{TOP}(X)$ to be the quotient group of $S_n(X)$ by the subgroup generated by elements of the form $\theta - g_*(\theta)$ for all $\theta \in S_n(X)$ and all $g \in \text{Aut}_h(X)$.

Recall the following definitions and theorems from [58] and [61]. Let $G$ be a countable group. An element $g \in G$ is said to have order $d$ if $d$ is the smallest positive integer such that $g^d = e$, where $e$ is the identity element of $G$. If no such positive integer exists, we say that the order of $g$ is $\infty$.

**Definition 8.2.** A countable discrete group $\Gamma$ is said to be finitely embeddable into Hilbert space $H$ if for any finite subset $F \subseteq \Gamma$, there exist a group $\Gamma'$ that is coarsely embeddable into $H$ and a map $\phi : F \to \Gamma'$ such that

1. if $\gamma, \beta$ and $\gamma \beta$ are all in $F$, then $\phi(\gamma \beta) = \phi(\gamma) \phi(\beta)$;
2. if $\gamma$ is a finite order element in $F$, then $\text{ord}(\phi(\gamma)) = \text{ord}(\gamma)$. Here $\text{ord}(\gamma)$ is the order of $\gamma$.

The class of groups with finite embeddability into Hilbert space is quite large, including all residually finite groups, amenable groups, Gromov’s monster groups, virtually torsion free groups (e.g. $\text{Out}(F_n)$), and any group of analytic diffeomorphisms of a connected analytic manifold fixing a given point [58].

Let $G$ be a countable group. If $g \in G$ has finite order $d$, then we can define an idempotent in the group algebra $\mathbb{Q}G$ by:

$$p_g = \frac{1}{d} \left( \sum_{k=1}^{d} g^k \right).$$

For the rest of this paper, we denote the maximal group $C^*$-algebra of $G$ by $C^*(G)$.

**Definition 8.3.** We define $K_0^\text{fin}(C^*(G))$, called the finite part of $K_0(C^*(G))$, to be the abelian subgroup of $K_0(C^*(G))$ generated by $\lfloor p_g \rfloor$ for all elements $g \neq e$ in $G$ with finite order.

We remark that rationally all representations of finite groups are induced from finite cyclic groups [51]. This explains that the finite part of K-theory, rationally, contains all K-theory elements which can be constructed using finite subgroups, despite being constructed using only cyclic subgroups.

**Theorem 8.4 ([58, Theorem 1.4]).** Suppose $\Gamma$ is finitely embeddable into Hilbert space. If $\{g_1, \cdots, g_m\}$ is a collection of elements in $\Gamma$ with distinct finite orders such that $g_i \neq e$ for all $1 \leq i \leq m$, then the following holds:

1. $\{[p_{g_1}], \cdots, [p_{g_m}]\}$ generates an abelian subgroup of $K_0^\text{fin}(C^*(\Gamma))$ with rank $m$;
2. any nonzero element in the abelian subgroup of $K_0^\text{fin}(C^*(\Gamma))$ generated by the elements $\{[p_{g_1}], \cdots, [p_{g_m}]\}$ is not in the image of the assembly map $\mu_* : K_0(B\Gamma) \cong K_0^\Gamma(EG) \to K_0(C^*(\Gamma))$, where $EG$ is the universal space for proper and free $\Gamma$-actions.
Before we go into the main result of this section, let us recall the following key step of constructing elements in the structure group by the finite part of $K$-theory [58, Theorem 3.4].

**Example 8.5.** Let $M$ be an $n$-dimensional closed oriented topological manifold with $\pi_1 M = \Gamma$. Suppose $\{g_1, \cdots, g_m\}$ is a collection of elements in $\Gamma$ with distinct finite orders such that $g_i \neq e$ for all $1 \leq i \leq m$. Recall the surgery exact sequence:

$$\cdots \to H_{4k}(M, L\bullet) \to L_{4k}(Z\Gamma) \xrightarrow{\partial} S_{4k-1}(M) \to H_{4k-1}(M, L\bullet) \to \cdots.$$ 

For each finite subgroup $H$ of $\Gamma$, we have the following commutative diagram:

$$\begin{array}{ccc}
H^H_{4k}(EH, L\bullet) & \longrightarrow & L_{4k}(ZH) \\
\downarrow & & \downarrow \\
H^G_{4k}(EG, L) & \longrightarrow & L_{4k}(Z\Gamma),
\end{array}$$

where the vertical maps are induced by the inclusion homomorphism from $H$ to $\Gamma$. For each element $g$ in $H$ with finite order $d$, the idempotent $p_g = \frac{1}{d}(\sum_{k=1}^{d} g^k)$ produces a class in $L_0(QH)$. Let $[q_g]$ be the corresponding element in $L_{4k}(QH)$ given by periodicity. Recall that

$$L_{4k}(ZH) \otimes Q \simeq L_{4k}(QH) \otimes Q.$$ 

For each element $g$ in $H$ with finite order, we use the same notation $[q_g]$ to denote the element in $L_{4k}(ZH) \otimes Q$ corresponding to $[q_g] \in L_{4k}(QH)$ under the above isomorphism.

We also have the following commutative diagram:

$$\begin{array}{ccc}
H^\Gamma_{4k}(EG, L\bullet) \otimes Q & \longrightarrow & L_{4k}(Z\Gamma) \otimes Q \\
\downarrow & & \downarrow \\
K^\Gamma_0(EG) \otimes Q & \longrightarrow & K_0(C^* (\Gamma)) \otimes Q,
\end{array}$$

where the left vertical map is induced by a map at the spectra level and the right vertical map is induced by the inclusion map:

$$L_{4k}(ZH) \to L_{4k}(C^*(\Gamma)) \cong K_0(C^*(\Gamma))$$

(see [45] for the last identification).

Now if $\Gamma$ is finitely embeddable into Hilbert, then the abelian subgroup of $K_0(C^*(\Gamma))$ generated by $\{[p_{g_1}], \cdots, [p_{g_m}]\}$ is not in the image of of the map

$$\mu_* : K^\Gamma_0(EG) \to K_0(C^*(\Gamma)).$$

It follows that

1. any nonzero element in the abelian subgroup of $L_{4k}(Z\Gamma) \otimes Q$ generated by the elements $\{[q_{g_1}], \cdots, [q_{g_m}]\}$ is not in the image of the rational assembly map

$$A : H^\Gamma_{4k}(EG, L\bullet) \otimes Q \to L_{4k}(Z\Gamma) \otimes Q;$$
the abelian subgroup of $L_{4k}(\mathbb{Z}\Gamma) \otimes \mathbb{Q}$ generated by $\{[q_{g_1}], \cdots, [q_{g_n}]\}$ has rank $m$.

By exactness of the surgery sequence, we know that the map

$$\mathcal{S} : L_{4k}(\mathbb{Z}\Gamma) \otimes \mathbb{Q} \to S_n(M) \otimes \mathbb{Q},$$

is injective on the abelian subgroup of $L_{4k}(\mathbb{Z}\Gamma) \otimes \mathbb{Q}$ generated by $\{[q_{g_1}], \cdots, [q_{g_n}]\}$.

In fact, to prove the main result (Theorem 8.7) of this section, we need to apply the above argument not only to $\Gamma$, but also to certain semi-direct products of $\Gamma$ with free groups of finite generators. Let us introduce some more terminology.

Let $\Gamma$ be a countable discrete group. Note that any set of $n$ automorphisms of $\Gamma$, say, $\psi_1, \cdots, \psi_n \in \text{Aut}(\Gamma)$, induces a natural action of $F_n$ the free group of $n$ generators on $\Gamma$. More precisely, if we denote the set of generators of $F_n$ by $\{s_1, \cdots, s_n\}$, then we have a homomorphism $F_n \to \text{Aut}(\Gamma)$ by $s_i \mapsto \psi_i$. This homomorphism induces an action of $F_n$ on $\Gamma$. We denote by $\Gamma \rtimes \{\psi_1, \cdots, \psi_n\} F_n$ the semi-direct product of $\Gamma$ and $F_n$ with respect to this action. If no confusion arises, we shall write $\Gamma \rtimes F_n$ instead of $\Gamma \rtimes \{\psi_1, \cdots, \psi_n\} F_n$.

**Definition 8.6.** A countable discrete group $\Gamma$ is said to be strongly finitely embeddable into Hilbert space $H$, if $\Gamma \rtimes \{\psi_1, \cdots, \psi_n\} F_n$ is finitely embeddable into Hilbert space $H$ for all $n \in \mathbb{N}$ and all $\psi_1, \cdots, \psi_n \in \text{Aut}(\Gamma)$.

We remark that all coarsely embeddable groups are strongly finitely embeddable. Indeed, if a group $\Gamma$ is coarsely embeddable into Hilbert space, then $\Gamma \rtimes \{\psi_1, \cdots, \psi_n\} F_n$ is coarsely embeddable (hence finitely embeddable) into Hilbert space for all $n \in \mathbb{N}$ and all $\psi_1, \cdots, \psi_n \in \text{Aut}(\Gamma)$. Moreover, if a group $\Gamma$ has a torsion free normal subgroup $\Gamma'$ such that $\Gamma/\Gamma'$ is residually finite, then $\Gamma$ is strongly finitely embeddable into Hilbert space, cf. [61, Section 4]. In particular, all residually finite groups are strongly finitely embeddable into Hilbert space.

We denote by $N_{\text{fin}}(\Gamma)$ the the cardinality of the following collection of positive integers:

$$\{d \in \mathbb{N}_+ \mid \exists \gamma \in \Gamma \text{ such that } \gamma \neq e \text{ and } \text{ord}(\gamma) = d\}.$$ 

Then we have the following main theorems of this section. At the moment, we are only able to prove the theorem for $\tilde{S}_n(M)$. We will give a brief discussion to indicate the difficulties in proving the version for $\tilde{S}^{\text{TOP}}(M)$ after the theorem.

**Theorem 8.7.** Let $M$ be a closed oriented topological manifold with dimension $n = 4k - 1$ ($k > 1$) and $\pi_1 M = \Gamma$. If $\Gamma$ is strongly finitely embeddable into Hilbert space, then the free rank of the abelian group $\tilde{S}_n(M)$ is $\geq N_{\text{fin}}(\Gamma)$.

**Proof.** The method of proof is a combination of methods from [58] and [61]. Let us fix a metric on $M$ once and for all.

Consider the following short exact sequence

$$0 \to C^*_L(\tilde{M})^\Gamma \to C^*_L(\tilde{M})^\Gamma \to C^*(\tilde{M})^\Gamma \to 0$$

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where $\tilde{M}$ is the universal cover of $M$. It induces the following six-term long exact sequence:

$$
\begin{array}{ccc}
K_0(C^*_L(M) \Gamma) & \longrightarrow & K_0(C^*_L(\tilde{M}) \Gamma) \\
\downarrow & & \downarrow \\
K_1(C^*(\tilde{M}) \Gamma) & \xleftarrow{\partial} & K_1(C^*_L(\tilde{M}) \Gamma)
\end{array}
$$

Recall that we have $K_0(C^*_L(\tilde{M}) \Gamma) \cong K^\Gamma_0(\tilde{M})$ and $K_0(C^*(\tilde{M}) \Gamma) \cong K_0(C^*(\Gamma))$.

Now a key point of the proof is to construct a certain group homomorphism on $\tilde{S}_n(M)$ which can be used to distinguish elements in $\tilde{S}_n(M)$. First, let us consider the higher rho invariant homomorphism from Theorem 4.41:

$$\rho : S_n(M) \rightarrow K_1(C^*_L,0(\tilde{M}) \Gamma).$$

Note that every self-homotopy equivalence $\psi \in Aut_h(M)$ induces a homomorphism\(^9\)

$$\psi_* : K_1(C^*_L,0(\tilde{M}) \Gamma) \rightarrow K_1(C^*_L,0(\tilde{M}) \Gamma).$$

Let $I_1(C^*_L,0(\tilde{M}) \Gamma)$ be the subgroup of $K_1(C^*_L,0(\tilde{M}) \Gamma)$ generated by elements of the form $[x] - \psi_*[x]$ for all $[x] \in K_1(C^*_L,0(\tilde{M}) \Gamma)$ and all $\psi \in Aut_h(M)$. Note that, by the definition of the higher rho invariant, we have

$$\rho(\alpha_\psi(\theta)) = \psi_* (\rho(\theta)) \in K_1(C^*_L,0(\tilde{M}) \Gamma)$$

for all $\theta \in S_n(M)$ and $\psi \in Aut_h(M)$. It follows that $\rho$ descends to a group homomorphism

$$\tilde{\rho} : \tilde{S}_n(M) \rightarrow K_1(C^*_L,0(\tilde{M}) \Gamma) / I_1(C^*_L,0(\tilde{M}) \Gamma).$$

Now for a collection of elements $\{\gamma_1, \ldots, \gamma_\ell\}$ with distinct finite orders, we consider the elements $\mathcal{J}(p_{\gamma_1}), \ldots, \mathcal{J}(p_{\gamma_\ell}) \in S_n(M)$ as in line (17). To be precise, the elements $\mathcal{J}(p_{\gamma_1}), \ldots, \mathcal{J}(p_{\gamma_\ell})$ actually lie in $S_n(M) \otimes \mathbb{Q}$. Consequently, all abelian groups in the following need to be tensored by the rationals $\mathbb{Q}$. For notational simplicity, we shall omit $\otimes \mathbb{Q}$ from our notation, with the understanding that various abelian groups in the following are to be viewed as tensored with $\mathbb{Q}$.

For notational simplicity, let us write

$$\rho(\gamma_i) := \rho(\mathcal{J}(p_{\gamma_i})) \in K_1(C^*_L,0(\tilde{M}) \Gamma).$$

\(^9\)There is one subtlety when defining the homomorphism $\psi_* : K_1(C^*_L,0(\tilde{M}) \Gamma) \rightarrow K_1(C^*_L,0(\tilde{M}) \Gamma)$. Indeed, the map $\tilde{\psi} : M \rightarrow \tilde{M}$ lifts to a map $\tilde{\psi} : M \rightarrow \tilde{M}$. However, to view $\tilde{\psi}$ as a $\Gamma$-equivariant map, we need to use two different actions of $\Gamma$ on $M$. Denote by $\tau$ the action of $\Gamma$ on $M$ through deck transformations. Then we define a new action $\tau'$ of $\Gamma$ on $\tilde{M}$ by $\tau'_g = \tau_{\psi^{-1}(g)}$, where $\psi : \Gamma \rightarrow \Gamma$ is the automorphism induced by $\psi$. It is easy to see that $\tilde{\psi} : M \rightarrow \tilde{M}$ is $\Gamma$-equivariant, when $\Gamma$ acts on the first copy of $\tilde{M}$ by $\tau'$ and the second copy of $\tilde{M}$ by $\tau$. Let us denote the corresponding $C^*$-algebras by $C^*_L,0,\tilde{M}^\tau_r$ and $C^*_L,0,\tilde{M}^\tau_l$. Observe that, despite of the two different actions of $\Gamma$ on $M$, the two $C^*$-algebras $C^*_L,0,\tilde{M}^\tau_r$ and $C^*_L,0,\tilde{M}^\tau_l$, are canonically identical, since an operator is invariant under the action $\tau$ if and only if it is invariant under the action $\tau'$.
To prove the theorem, it suffices to show that for any collection of elements \( \{ \gamma_1, \cdots, \gamma_\ell \} \) with distinct finite orders, the elements

\[ \tilde{\rho}(\gamma_1), \cdots, \tilde{\rho}(\gamma_\ell) \]

are linearly independent in \( K_1(C^*_L(\widetilde{M})^\Gamma)/\mathcal{I}_1(C^*_L(\widetilde{M})^\Gamma) \).

Let us assume the contrary, that is, there exist \( [x_1], \cdots, [x_m] \in K_1(C^*_L(\widetilde{M})^\Gamma) \) and \( \psi_1, \cdots, \psi_m \in \text{Aut}_h(M) \) such that

\[ \sum_{i=1}^\ell c_i \rho(\gamma_i) = \sum_{j=1}^m ([x_j] - (\psi_j)_*[x_j]), \tag{18} \]

where \( c_1, \cdots, c_\ell \in \mathbb{Z} \) with at least one \( c_i \neq 0 \).

We denote by \( W \) the wedge sum of \( m \) circles. The fundamental group \( \pi_1(W) \) is the free group \( F_m \) of \( m \) generators \( \{ s_1, \cdots, s_m \} \). We denote the universal cover of \( W \) by \( \widetilde{W} \). Clearly, \( \widetilde{W} \) is the Cayley graph of \( F_m \) with respect to the generating set \( \{ s_1, \cdots, s_m, s_1^{-1}, \cdots, s_m^{-1} \} \). Let \( F_m \) act on \( M \) through the self-homotopy equivalences \( \psi_1, \cdots, \psi_m \). In other words, we have a homomorphism \( F_m \to \text{Aut}_h(M) \) by \( s_i \mapsto \psi_i \). We define

\[ X = M \times_{F_m} \widetilde{W}. \]

Notice that \( \pi_1(X) = \Gamma \rtimes \langle \psi_1, \cdots, \psi_m \rangle F_m \). Let us write \( \Gamma \rtimes F_m \) for \( \Gamma \rtimes \langle \psi_1, \cdots, \psi_m \rangle F_m \).

Let \( \widetilde{X} \) be the universal cover of \( X \). We have the following short exact sequence:

\[ 0 \to C^*_L(\widetilde{X})^{\Gamma \rtimes F_m} \to C^*_L(\widetilde{X})^{\Gamma \rtimes F_m} \to C^*(\widetilde{X})^{\Gamma \rtimes F_m} \to 0. \]

Recall that \( K_0(C^*_L(\widetilde{X})^{\Gamma \rtimes F_m}) \cong K_0^{\Gamma \rtimes F_m}(\widetilde{X}) \) and \( K_0(C^*(\widetilde{X})^{\Gamma \rtimes F_m}) \cong K_0(C^*(\Gamma \rtimes F_m)) \). So we have the following six-term long exact sequence:

\[ \begin{array}{c}
K_0(C^*_L(\widetilde{X})^{\Gamma \rtimes F_m}) \longrightarrow K_0^{\Gamma \rtimes F_m}(\widetilde{X}) \longrightarrow K_0(C^*(\Gamma \rtimes F_m)) \\
\uparrow \\
K_1(C^*(\Gamma \rtimes F_m)) \longleftarrow K_1^{\Gamma \rtimes F_m}(\widetilde{X}) \longleftarrow K_1(C^*_L(\widetilde{X})^{\Gamma \rtimes F_m})
\end{array} \tag{19} \]

Now recall the following Pimsner-Voiculescu exact sequence [38]:

\[ \begin{array}{c}
\bigoplus_{j=1}^m K_0(C^*(\Gamma)) \longrightarrow \sum_{j=1}^m 1 - (\psi_j)_* \longrightarrow K_0(C^*(\Gamma)) \longrightarrow i_* \longrightarrow K_0(C^*(\Gamma \rtimes F_m)) \\
\uparrow \\
K_1(C^*(\Gamma \rtimes F_m)) \longleftarrow K_1(C^*(\Gamma)) \longleftarrow \sum_{j=1}^m 1 - (\psi_j)_* \longrightarrow \bigoplus_{j=1}^m K_1(C^*(\Gamma))
\end{array} \]

where \( (\psi_j)_* \) is induced by \( \psi_j \) and \( i_* \) is induced by the inclusion map of \( \Gamma \) into \( \Gamma \rtimes F_m \). Similarly, we also have the following two Pimsner-Voiculescu type exact sequences for
\( K \)-homology and the \( K \)-theory groups of \( C^{*}_{L,0} \)-algebras in the diagram (19) above.

\[
\bigoplus_{i=1}^{m} K_0^F(M) \xrightarrow{\sum_{j=1}^{m} 1 - (\psi_j)_*} K_0^F(M) \xrightarrow{i_*} K_0^{F \times F_m}(\tilde{X})
\]

\[
\bigoplus_{i=1}^{m} K_0^F(M) \xrightarrow{\sum_{j=1}^{m} 1 - (\psi_j)_*} K_0^F(M) \xrightarrow{i_*} K_0^{F \times F_m}(\tilde{X})
\]

where again \((\psi_j)_*\) and \(i_*\) are defined in the obvious way.

Combining these Pimsner-Voiculescu exact sequences together, we have the following commutative diagram:

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\bigoplus_{j=1}^{m} K_0^F(M) & \bigoplus_{j=1}^{m} K_0^F(M) & \sigma & K_0^F(M) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
K_0(L) & K_0(L) & i_* & K_0^{F \times F_m}(\tilde{X}) \\
\end{array}
\]

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\bigoplus_{j=1}^{m} K_0^F(M) & \bigoplus_{j=1}^{m} K_0^F(M) & \sigma & K_0^F(M) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
K_0(L) & K_0(L) & i_* & K_0^{F \times F_m}(\tilde{X}) \\
\end{array}
\]

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\bigoplus_{j=1}^{m} K_0^F(M) & \bigoplus_{j=1}^{m} K_0^F(M) & \sigma & K_0^F(M) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
K_0(L) & K_0(L) & i_* & K_0^{F \times F_m}(\tilde{X}) \\
\end{array}
\]

where \(\sigma = \sum_{j=1}^{m} 1 - (\psi_j)_*\). Notice that all rows and columns are exact.

Now on one hand, if we pass Equation (18) to \(K_1(C^{*}_{L,0}(\tilde{X})^{F \times F_m})\) under the map \(i_*\), then it follows immediately that

\[
\sum_{k=1}^{\ell} c_k \cdot i_*[\rho(\gamma_k)] = 0 \text{ in } K_1(C^{*}_{L,0}(\tilde{X})^{F \times F_m}),
\]

where at least one \(c_k \neq 0\). On the other hand, by assumption, \(\Gamma\) is strongly finitely embeddable into Hilbert space. Hence \(\Gamma \times F_m\) is finitely embeddable into Hilbert space. By Theorem 8.4, we have the following.

(i) \(\{[p_{\gamma_1}], \cdots, [p_{\gamma_n}]\}\) generates a rank \(n\) abelian subgroup of \(K_0^{\text{fin}}(C^*(\Gamma \times F_m))\), since \(\gamma_1, \cdots, \gamma_n\) have distinct finite orders. In other words,

\[
\sum_{k=1}^{n} c_k[p_{\gamma_k}] \neq 0 \in K_0^{\text{fin}}(C^*(\Gamma \times F_m))
\]
if at least one $c_k \neq 0$.

(ii) Every nonzero element in $K_0^\text{fin}(C^*(\Gamma \rtimes F_m))$ is not in the image of the assembly map

$$\mu : K_0^\Gamma \times F_m(E(\Gamma \rtimes F_m)) \to K_0(C^*(\Gamma \rtimes F_m)),$$

where $E(\Gamma \rtimes F_m)$ is the universal space for proper and free $\Gamma \rtimes F_m$-action. In particular, every nonzero element in $K_0^\text{fin}(C^*(\Gamma \rtimes F_m))$ is not in the image of the map

$$\mu : K_0^\Gamma \times F_m(\tilde{X}) \to K_0(C^*(\Gamma \rtimes F_m))$$

in diagram (20). It follows that the map

$$\partial_{\Gamma \rtimes F_m} : K_0^\text{fin}(C^*(\Gamma \rtimes F_m)) \to K_1(C_{L,0}^*(\tilde{X})^{\Gamma \times F_m})$$

is injective. In other words, $\partial_{\Gamma \rtimes F_m}$ maps a nonzero element in $K_0^\text{fin}(C^*(\Gamma \rtimes F_m))$ to a nonzero element in $K_1(C_{L,0}^*(\tilde{X})^{\Gamma \times F_m})$.

To summarize, we have

(a) $\sum_{k=1}^\ell c_k[p_{\gamma_k}] \neq 0$ in $K_0^\text{fin}(C^*(\Gamma \rtimes F_m))$,

(b) $\sum_{k=1}^\ell c_k \cdot i_\ast[\rho(\gamma_k)] = 0$ in $K_1(C_{L,0}^*(\tilde{X})^{\Gamma \times F_m})$,

(c) the map $\partial_{\Gamma \rtimes F_m} : K_0^\text{fin}(C^*(\Gamma \rtimes F_m)) \to K_1(C_{L,0}^*(\tilde{X})^{\Gamma \times F_m})$ is injective,

(d) and by commutative diagram (9), we have

$$\partial_{\Gamma \rtimes F_m}\left(\sum_{k=1}^\ell c_k[p_{\gamma_k}]\right) = 2 \cdot \left(\sum_{k=1}^\ell c_k \cdot i_\ast[\rho(\gamma_k)]\right).$$

Therefore, we arrive at a contradiction. This finishes the proof. \qed

It is tempting to use a similar argument to prove an analogue of Theorem 8.7 above for $\tilde{S}_{TOP}(M)$. However, there are some subtleties. Note that, under the natural identification $S_n(M) \cong S_{TOP}(M)$ from Section 3.3, it is clear (cf. [43]) that

$$\alpha_\varphi(\theta) + [\varphi] = \varphi_*(\theta)$$

for all $\theta = (f,N) \in S_{TOP}(M)$ and all $\varphi \in \text{Aut}_h(M)$, where $[\varphi] = (\varphi,M)$ is the element given by $\varphi : M \to M$ in $S_{TOP}(M)$. It follows that

$$\rho(\varphi_*(\theta)) = \rho(\alpha_\varphi(\theta)) + \rho([\varphi]) = \varphi_*(\rho(\theta)) + \rho([\varphi]).$$

In other words, in general, $\rho(\varphi_*(\theta)) \neq \varphi_*(\rho(\theta))$, and consequently the homomorphism

$$\rho : S_{TOP}(M) \to K_1(C_{L,0}^*(\tilde{M})^\Gamma)$$

does not descend to a homomorphism from $\tilde{S}_{TOP}(M)$ to $K_1(C_{L,0}^*(\tilde{M})^\Gamma)/L_1(C_{L,0}^*(\tilde{M})^\Gamma)$. New ingredients are needed to take care of this issue. On the other hand, there is strong
evidence which suggests an analogue of Theorem 8.7 for $\widetilde{S}^{TOP}(M)$. For example, this has been verified by Weinberger and Yu for residually finite groups [58, Theorem 3.9]. Also, Chang and Weinberger gave a different lower bound of $\widetilde{S}^{TOP}(M)$ that works for all non-torsion-free groups, although the lower bound is weaker [6, Theorem 1]. In any case, we hope to deal with this question in a separate paper.

We close this section by proving the following theorem, which is an analogue of a theorem by Chang and Weinberger [6, Theorem 1].

**Theorem 8.8.** Let $X$ be a closed oriented topological manifold with dimension $n = 4k - 1$ ($k > 1$) and $\pi_1 X = \Gamma$. If $\Gamma$ is not torsion free, then the free rank of the abelian group $\tilde{S}_n(X)$ is $\geq 1$.

**Proof.** Recall that for any non-torsion-free countable discrete group $G$, if $\gamma \neq e$ is a finite order element of $G$, then $[p_{\gamma}]$ generates a subgroup of rank one in $K_0(C^*(G))$ and any nonzero multiple of $[p_{\gamma}]$ is not in the image of the assembly map $\mu: K^T_0(EG) \to K_0(C^*(G))$ [58, Theorem 2.3]. Now we apply this fact to $\Gamma \rtimes \mathbb{Z}$, and the theorem is proved the same way as Theorem 8.7.

## 9 Signatures and Lipschitz structures

In this section, we show how our approach can be adapted to deal with signature operators arising from Lipschitz structures on topological manifolds. Throughout this section, all manifolds are assumed to have dimension $\geq 5$.

As we have seen, for our main theorem (Theorem 4.41) and our main application 8.7, it suffices to work with the smooth or PL representatives, that is, the groups $N_n^\infty(X;w)$, $L_n^\infty(\pi_1 X;w)$ and $S_n^\infty(X,w)$, or $N_n^{PL}(X;w)$, $L_n^{PL}(\pi_1 X;w)$ and $S_n^{PL}(X,w)$, cf. Section 3.4. On the other hand, our approach to the higher rho invariant given in Section 4.6 applies essentially verbatim to signature operators associated to Lipschitz structures on topological manifolds. In particular, with some minor modifications given below, we can directly deal with signature operators arising from Lipschitz structures as well.

There are two modifications that are needed for the Lipschitz case.

(i) We use the unbounded theory (cf. [21, Section 5]) instead of the bounded theory that is used in Section 4.6. For various properties of the signature operator associated to a Lipschitz structure, we refer the reader to [25, 26] [27, Section 3]. Now the higher rho invariant for the Lipschitz case is defined by the same formula as in Section 4.6, and the proofs are literally identical.

(ii) To prove the well-definedness of the higher rho invariant map (for the Lipschitz case), the techniques in Section 5 do not quite apply to the unbounded theory. Recall that, for an even dimensional manifold $Y$ with boundary $\partial Y$, the restriction of the signature operator $D_Y$ of $Y$ is 2 times the signature operator $D_{\partial Y}$ on $\partial Y$. In order to take care this factor of 2, we use the results of Stern on topological vector fields ([53, Corollary 1.5]) and techniques developed by Pedersen,
Roe and Weinberger in [36, Section 4]. In particular, these results allow us to use a vector field to split the signature operator $D_Y$ into two halves. The rest of the proof is similar to the proof for the commutativity of the middle square in Theorem 6.10.

Since every element in $\mathcal{S}_n(X)$ is cobordant to a smooth representative, it follows that this higher rho invariant defined using Lipschitz structures above coincides with our definition of higher rho invariant in Definition 4.39.

Remark 9.1. We point out that, if one is only interested in the well-definedness of the higher rho invariant map after inverting 2, that is, if one only wants to prove the map

$$\rho: \mathcal{S}_n(X) \to K_n(C_{L,0}^*(\tilde{X})^\Gamma) \otimes \mathbb{Z}[1/2]$$

is well-defined, then there is a simpler argument than the one outlined in item (ii) above. Indeed, in this case, an argument similar to the proof for the commutativity of the middle square in Theorem 6.10 suffices.

Appendix A  K-homology class of signature operator

In this section of the appendix, we give a detailed construction of the $K$-homology classes of signature operators on PL manifolds. The material of this section is taken from [24]. We will only give the details for the odd case. The even case is similar.

A.1 Special case: closed PL manifolds

In this subsection, we construct the $K$-homology classes of signature operators on closed PL manifolds. The construction for the more general case of elements in $\mathcal{N}_m(X)$ will be considered in the next subsection.

Let $M$ be a closed PL manifold. Assume that $M$ is equipped with a triangulation that has bounded geometry. Suppose there is a control map $\varphi: M \to X$.

Let $\text{Sub}(M)$ be the subdivision from Section 4.2. We define

$$H_1 = \bigoplus_k C_k(\text{Sub}(M)) \otimes \mathbb{C}.$$ 

Note that $H_1$ carries a natural $X$-module structure. Similarly, we define

$$H_j = \bigoplus_k C_k(\text{Sub}^j(M)) \otimes \mathbb{C}.$$ 

Define $\mathcal{H}$ to be the $\ell^2$-completion of $\bigoplus_{j=0}^{\infty} H_j$. Then $\mathcal{H}$ is an analytically controlled $X$-module.

\footnote{Technically speaking, we may need to punch out a disc in $Y$, replace it with an infinite cylinder, and control this cylinder appropriately over the reference control space.}
Each $H_k$, together with the maps $b, b^*$ and $T$, gives rise to a geometrically controlled Poincaré complex over $X$. Let $B_k = (b + b^*)_k$ and $S_k$ be the operators on $H_k$ as defined in Section 4.4. By construction, $B_k$ and $S_k$ have finite propagation. Moreover, $B_k \pm S_k$ are invertible for each $k \geq 0$. The following lemma shows that in fact $B_k \pm S_k$ are uniformly bounded below for all $k \geq 0$.

**Lemma A.1.** There exist constants $\varepsilon > 0$ and $C > 0$ such that

$$\varepsilon < \|B_k \pm S_k\|_{H_k} < C$$

for all $k$.

*Proof.* Consider the disjoint union of countably many copies of $M$, denoted by $\bigsqcup_j M_j$, where $M_j = M$ endowed with the triangulation $\text{Sub}^j(M)$. Then $\bigsqcup_j H_j$ is a geometrically controlled Poincaré complex over $\bigsqcup_j X$. Let $\mathcal{H}$ be the $\ell^2$ completion of $\bigsqcup_j H_j$. Then by the discussion in Section 4.4 and 4.5, the operators

$$\bigoplus_j (B_j + S_j) \text{ and } \bigoplus_j (B_j - S_j)$$

are bounded and invertible [21, Lemma 3.5]. In particular, there exist $\varepsilon > 0, C > 0$ such that

$$\varepsilon < \|\bigoplus_j (B_j + S_j)\|_{\mathcal{H}} < C \text{ and } \varepsilon < \|\bigoplus_j (B_j - S_j)\|_{\mathcal{H}} < C.$$  

It follows that $\varepsilon < \|B_j \pm S_j\|_{H_j} < C$ for all $j \geq 0$. \qed

Let $p(x)$ be a polynomial on $[\varepsilon, C] \cup [-C, -\varepsilon]$ such that

$$\sup_{x \in [\varepsilon, C]} |p(x) - x^{-1}| < \frac{1}{C}.$$  

Then $\|p(B_j - S_j) - (B_j - S_j)^{-1}\| < \frac{1}{\|B_j - S_j\|}$, which implies that $p(B_j - S_j)$ is invertible. Moreover, the element

$$(B_j + S_j) \cdot p(B_j - S_j)$$

has finite propagation. Since the propagation of $B_j - S_j$ goes to zero as $j$ goes to $\infty$, we have that the propagation of $(B_j + S_j) \cdot p(B_j - S_j)$ goes to zero, as $j$ goes to infinity.

The standard subdivision (cf. Section 4.2) induces a geometrically controlled chain homotopy equivalence

$$A_j : (H_j, b) \to (H_{j+1}, b).$$

Observe that the propagation of $A_j$ goes to zero, as $j \to \infty$. Moreover, $A_j S_j A_j^*$ is controlled chain homotopic to $S_{j+1}$. We shall use these controlled chain homotopy

\[\text{[11 More precisely, the metrics on various copies of } X \text{ need to be rescaled appropriately. See a similar discussion for the cone } CX \text{ at the beginning of Section } 4.6. \text{ Also, note that the space } \bigsqcup_j X \text{ is only used as a reference to obtain norm estimates. In the end, it will not enter into our construction of local index, i.e., } K\text{-homology class.}\]
equivalences to construct a norm-bounded and uniformly continuous path that connects all \((B_j + S_j) \cdot p(B_j - S_j)\). The resulting path represents a class in \(K_1(C^*_L(X))\), which is precisely the image of the \(K\)-homology class of the signature operator on \(M\) in \(K_1(C^*_L(X))\).

Consider the duality operator \((-S_j) \oplus S_{j+1}\) on the chain complex \(H_j \oplus H_{j+1}\). We shall construct a continuous path of invertible elements (with well-behaved propagations) connecting

\[
\left[ \begin{pmatrix} B_j & B_{j+1} \\ B_{j+1} & B_{j+1} \end{pmatrix} + \begin{pmatrix} -S_j \\ S_{j+1} \end{pmatrix} \right] \cdot p \left[ \begin{pmatrix} B_j & B_{j+1} \\ B_{j+1} & B_{j+1} \end{pmatrix} - \begin{pmatrix} -S_j \\ S_{j+1} \end{pmatrix} \right]
\]

to the identity operator \((1 0 \ 0 1)\). The construction is adapted from [21, Section 4]. First consider the path of duality operators

\[
\begin{pmatrix} -S_j & 0 \\ 0 & (1 - t)S_{j+1} + tA_j S_j A^*_j \end{pmatrix}
\]

which connects the duality operator \((-S_j) \oplus S_{j+1}\) on the chain complex \(H_j \oplus H_{j+1}\) to the operator \((-S_j) \oplus A_j S_j A^*_j\). Following this, the path of duality operators

\[
\begin{pmatrix} -\cos(t)S_j & \sin(t)S_j A^*_j \\ \sin(t)A_j S_j & \cos(t)A_j S_j A^*_j \end{pmatrix}, \ t \in [0, \pi/2],
\]

connecting \((-S_j) \oplus A_j S_j A^*_j\) to \(\begin{pmatrix} 0 & S_j A^*_j \\ A_j S_j & 0 \end{pmatrix}\).

By using these paths of duality operators, we see that

\[
\left[ \begin{pmatrix} B_j & B_{j+1} \\ B_{j+1} & B_{j+1} \end{pmatrix} + \begin{pmatrix} -S_j \\ S_{j+1} \end{pmatrix} \right] \cdot p \left[ \begin{pmatrix} B_j & B_{j+1} \\ B_{j+1} & B_{j+1} \end{pmatrix} - \begin{pmatrix} -S_j \\ S_{j+1} \end{pmatrix} \right]
\]

is connected to

\[
\left[ \begin{pmatrix} B_j & B_{j+1} \\ B_{j+1} & B_{j+1} \end{pmatrix} + \begin{pmatrix} 0 & S_j A^*_j \\ A_j S_j & 0 \end{pmatrix} \right] \cdot p \left[ \begin{pmatrix} B_j & B_{j+1} \\ B_{j+1} & B_{j+1} \end{pmatrix} - \begin{pmatrix} 0 & S_j A^*_j \\ A_j S_j & 0 \end{pmatrix} \right]
\]

by a norm-continuous path of invertibles.

Now observe that the duality operator \(\begin{pmatrix} 0 & S_j A^*_j \\ A_j S_j & 0 \end{pmatrix}\) is connected to its additive inverse \(\begin{pmatrix} 0 & -S_j A^*_j \\ -A_j S_j & 0 \end{pmatrix}\) by the path of duality operators

\[
\begin{pmatrix} 0 & \exp(it)S_j A^*_j \\ \exp(-it)A_j S_j & 0 \end{pmatrix}, \ t \in [0, \pi].
\]

To proceed, we need the following lemma.

**Lemma A.2.** The elements

\[
E^\pm_j(t) = \begin{pmatrix} B_j \\ B_{j+1} \end{pmatrix} \pm \begin{pmatrix} -S_j \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ (1 - t)S_{j+1} + tA_j S_j A^*_j \end{pmatrix},
\]

\[
F^\pm_j(t) = \begin{pmatrix} B_j \\ B_{j+1} \end{pmatrix} \pm \begin{pmatrix} -\cos(t)S_j \\ \sin(t)A_j S_j \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \cos(t)A_j S_j A^*_j \end{pmatrix}
\]

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and
\[
G_j^\pm(t) = \begin{pmatrix} B_j & 0 \\ B_{j+1} & \exp(it)A_jA_j^* \end{pmatrix} \pm \begin{pmatrix} 0 & \exp(-it)A_jS_j \\ \exp(it)A_jS_j & 0 \end{pmatrix}
\]
are invertible. Moreover, there exist $\varepsilon > 0, C > 0$ such that
\[
\varepsilon \leq \|\mathcal{E}_j^\pm(t)\|, \|\mathcal{F}_j^\pm(t)\|, \|G_j^\pm(t)\| \leq C
\]
for all $j$ and $t$.

**Proof.** The proof uses the same idea from Lemma A.1. It also follows from applying Proposition B.3 to mapping cone chain complexes.

Consider the map
\[
A = \bigoplus_{j \geq 0} A_j : \bigoplus_{j \geq 0} H_j \to \bigoplus_{j \geq 1} H_j
\]
Similarly, we define
\[
S = \bigoplus_{j \geq 0} S_j \quad \text{and} \quad S' = \bigoplus_{j \geq 1} S_j;
\]
\[
B = \bigoplus_{j \geq 0} B_j \quad \text{and} \quad B' = \bigoplus_{j \geq 1} B_j;
\]
Note that $ASA^*$ is controlled chain homotopic to $S'$. Now define the paths of operators
\[
\mathcal{E}^+(t) = \begin{pmatrix} B & B' \\ -S & 0 \end{pmatrix} + \begin{pmatrix} 0 & (1-t)S' + t ASA^* \end{pmatrix}.
\]
By the discussion in Section 4.4 and Section 4.5, the operators $\mathcal{E}^+(t)$ are bounded and invertible [21, Lemma 3.5]. Therefore there exists a constant $\varepsilon > 0$ and $C > 0$ such that
\[
\varepsilon \leq \|\mathcal{E}^+(t)\| \leq C.
\]
The same argument applies to the other terms. This finishes the proof. \(\Box\)

Without loss of generality, we assume that we have chosen $\varepsilon$ and $C$ as in the above lemma. It follows that the element
\[
v_0 = \left[ \begin{pmatrix} B_j & 0 \\ B_{j+1} & A_jS_j \end{pmatrix} + \begin{pmatrix} 0 & S_jA_j^* \\ A_jS_j & 0 \end{pmatrix} \right] \cdot p \left[ \begin{pmatrix} B_j & 0 \\ B_{j+1} & A_jS_j \end{pmatrix} + \begin{pmatrix} 0 & S_jA_j^* \\ A_jS_j & 0 \end{pmatrix} \right]
\]
is connected to
\[
v_1 = \left[ \begin{pmatrix} B_j & 0 \\ B_{j+1} & A_jS_j \end{pmatrix} + \begin{pmatrix} 0 & S_jA_j^* \\ A_jS_j & 0 \end{pmatrix} \right] \cdot p \left[ \begin{pmatrix} B_j & 0 \\ B_{j+1} & A_jS_j \end{pmatrix} + \begin{pmatrix} 0 & S_jA_j^* \\ A_jS_j & 0 \end{pmatrix} \right]
\]
by the path
\[
v_t = \left[ \begin{pmatrix} B_j & 0 \\ B_{j+1} & A_jS_j \end{pmatrix} + \begin{pmatrix} 0 & S_jA_j^* \\ A_jS_j & 0 \end{pmatrix} \right] \cdot p \left[ \begin{pmatrix} B_j & 0 \\ B_{j+1} & A_jS_j \end{pmatrix} + \begin{pmatrix} 0 & S_jA_j^* \\ A_jS_j & 0 \end{pmatrix} \right].
\]
Notice that, since $p(x)$ is approximating the function $f(x) = x^{-1}$, the element $v_1$ in this path is very close to the identity operator $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$. More precisely, the linear path between $v_1$ and $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$ is a path of invertible elements connecting $v_1$ and $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$. 71
To summarize, we have obtained a norm-continuous path of invertible elements that connects
\[
\left[ \begin{pmatrix} B_j & B_{j+1} \\ B_{j+1} & B_j \end{pmatrix} + \begin{pmatrix} -S_j & S_{j+1} \\ S_{j+1} & -S_j \end{pmatrix} \right] \cdot p \left[ \begin{pmatrix} B_j & B_{j+1} \\ B_{j+1} & B_j \end{pmatrix} - \begin{pmatrix} -S_j & S_{j+1} \\ S_{j+1} & -S_j \end{pmatrix} \right]
= \left( \frac{(B_j - S_j) \cdot p(B_j + S_j) \quad 0}{0 \quad (B_{j+1} + S_{j+1}) \cdot p(B_{j+1} - S_{j+1})} \right)
\]
to the identity operator \((\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})\). In particular, by multiplying this path by the element
\[
\left( \begin{pmatrix} (B_j + S_j) \cdot p(B_j - S_j) & 0 \\ 0 & 1 \end{pmatrix} \right),
\]
we have a path of invertible elements connecting \((\begin{pmatrix} (B_j + S_j) \cdot p(B_j - S_j) & 0 \\ 0 & 1 \end{pmatrix})\) to
\[
\left( \begin{pmatrix} (B_j + S_j) \cdot p(B_j - S_j) \quad 0 \\ 0 \quad (B_{j+1} + S_{j+1}) \cdot p(B_{j+1} - S_{j+1}) \end{pmatrix} \right).
\]
Observe that again the entry
\[
(B_j + S_j) \cdot p(B_j - S_j)(B_j - S_j) \cdot p(B_j + S_j)
\]
in the last element is connected to the identity operator by a linear path of invertible elements. Therefore, combining these paths together, we obtain a path of invertible elements, denoted by \(U_t, t \in \{j,j+1\}\), connecting
\[
U_j = \left( \begin{pmatrix} (B_j + S_j) \cdot p(B_j - S_j) & 0 \\ 0 & 1 \end{pmatrix} \right)
\]
to
\[
U_{j+1} = \left( \begin{pmatrix} 1 & 0 \\ 0 & (B_{j+1} + S_{j+1}) \cdot p(B_{j+1} - S_{j+1}) \end{pmatrix} \right).
\]
Let \(c_j\) be the maximum of the propagations of \(B_j\), \(S_j\) and \(A_j\). By construction, the propagation of \(U_t\) is uniformly bounded by \(\deg(p) \cdot c_j\) for all \(t \in \{j,j+1\}\). Here \(\deg(p)\) is the degree of the polynomial \(p(x)\).

Now we stabilize \(U_t\), that is, for \(t \in \{j,j+1\}\), we view \(U_t\) as an invertible operator on \(H\) by making it act as the identity operator on \(H_i\) for \(i \neq j,j+1\). By concatenating all these paths \(\{U_t\}_{j \leq t \leq j+1}\), we obtain a norm-bounded and uniformly continuous path of invertible elements
\[
U : [0, \infty) \to C^*(X)^+
\]
such that the propagation of \(U_t\) goes to zero, as \(t \to \infty\). Here \(C^*(X)^+\) is the unitization of \(C^*(X)\).

**Definition A.3.** The local index \(\text{Ind}_L(M, \varphi)\) of the signature operator of \(M\) under the control \(\varphi : M \to X\) is defined to be the \(K\)-theory class of the path \(U\) in \(K_1(C^*_L(X))\).

The even dimensional case is similar. We leave the details to the reader.
A.2 General case: elements in $\mathcal{N}_m(X)$

In this subsection, we construct the $K$-homology classes of signature operators for elements in $\mathcal{N}_m(X)$. See Definition 3.10 for a description of $\mathcal{N}_m(X)$.

Let $\xi = (M, \partial M, \varphi, N, \partial, N, f) \in \mathcal{N}_m(X)$ (cf. Definition 3.10). In this case, we consider the space $M \cup_f (-N)$ obtained by gluing $M$ and $-N$ along the boundary by the map $f: \partial N \to \partial M$. Although $M \cup_f (-N)$ is not a manifold, it is still a space equipped with a Poincaré duality. In fact, since $f: \partial N \to \partial M$ is a PL infinitesimally controlled homotopy equivalence, we can still make sense of the $K$-homology class of its “signature operator”.

More precisely, let us denote the geometrically controlled Poincaré pair associated to the triangulation $\text{Sub}^n(M)$ and $\text{Sub}^n(N)$ by $(E_M^n, b_M^n, T_M^n, P_M^n)$ and $(E_N^n, b_N^n, T_N^n, P_N^n)$ respectively. Consider the mapping cone complex $(E^{(n)}, b^{(n)})$ of

$$(P_N^n, E_N^n, P_M^n, b_M^n) \xrightarrow{f^{(n)} P_N^n \oplus P_M^n} (E_M^n, b_M^n) \oplus (E_N^n, b_N^n)$$

where $f^{(n)}$ is a chain homotopy equivalence

$$(f^{(n)}): (P_N^n E_N^n, P_M^n b_M^n, T_0^{(n)}) \to (P_M^n E_M^n, P_M^n b_M^n, T_0^{(n)})$$

induced by the PL infinitesimally controlled homotopy equivalence $f: \partial N \to \partial M$. Here $T_0^{(n)}$ and $T_0^{(n)}$ are the Poincaré duality operators on the boundary as defined in Lemma 5.4. Note that

(1) the natural inclusion $\iota_n: (E_M^n, b_M^n, T_M^n, P_M^n) \to (E_M^{(n+1)}, b_M^{(n+1)}, T_M^{(n+1)}, P_M^{(n+1)})$ is geometrically controlled homotopy equivalence of Poincaré pairs;

(2) since $f: \partial N \to \partial M$ is a PL infinitesimally controlled homotopy equivalence, $f^{(n)}$ can be chosen so that the propagation of $f^{(n)}$ goes to 0, as $n$ approaches infinity;

(3) under the above inclusion $\iota_n$, the map $f^{(n)}$ is geometrically controlled chain homotopic to $f^{(n+1)}$; moreover, the propagation of the homotopy goes to 0, as $n$ goes to infinity;

In particular, it follows that the maps $T_M^{(n)}$ and $T_N^{(n)}$ induce a natural Poincaré duality operator $T^{(n)}$ on $(E^{(n)}, b^{(n)})$. To summarize, $(E^{(n)}, b^{(n)}, T^{(n)})$ is a geometrically controlled Poincaré complex such that the propagations of all relevant maps go to 0, as $n$ goes to infinity. Let us write

$$(E, b, T) = \bigoplus_{n=1}^{\infty} (E^{(n)}, b^{(n)}, T^{(n)}).$$

Now the exact same construction from Subsection A.1 above can be applied to $(E, b, T)$. Consequently, for each $\xi \in \mathcal{N}_m(X)$, we obtain a $K$-theory class in $K_m(C(X))$. We denote this class by $\text{Ind}_M(\xi)$, and call it the $K$-homology class of signature operator associated to $\xi$, or simply the local index of $\xi$. 

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Appendix B  Uniform control and uniform invertibility

In this part of the appendix, we show that a uniform family of geometrically controlled Poincaré complexes gives rise to a uniform family of analytically controlled Poincaré complexes.

First, let us introduce the notion of uniform families of geometrically controlled Poincaré complexes.

Definition B.1. A uniform family of geometrically controlled Poincaré complexes over $X$ is a family of geometrically controlled Poincaré complexes over $X$

$$\{(E_\lambda, b_\lambda, T_\lambda)\}_{\lambda \in \Lambda}$$

such that the following conditions are satisfied:

1. the propagations of $b_\lambda$ and $T_\lambda$ are uniformly bounded;
2. the propagations of the chain homotopy inverses $T'_\lambda$ of $T_\lambda$ are uniformly bounded;
3. the propagations of the chain homotopies between $T'_\lambda \circ T_\lambda$ and 1 are uniformly bounded, and the propagations of the chain homotopies between $T_\lambda \circ T'_\lambda$ and 1 are uniformly bounded;
4. the matrix coefficients of all maps above (including the chain homotopies) are uniformly bounded.

There is a natural counterpart of the above notion of uniform families in the analytically controlled category. Recall from Definition 4.10 that, for an analytically controlled Poincaré complex $(E, b, T)$, we have

$$B = b + b^*$$

and

$$S = i^{p(p-1)+l}T.$$

Definition B.2. A uniform family of analytically controlled Poincaré complexes over $X$ is a family of analytically controlled Poincaré complexes over $X$

$$\{(E_\lambda, b_\lambda, T_\lambda)\}_{\lambda \in \Lambda}$$

such that the following conditions are satisfied:

1. the norms of $b_\lambda$ and $T_\lambda$ are uniformly bounded;
2. the norms of the chain homotopy inverses $T'_\lambda$ of $T_\lambda$ are uniformly bounded;
3. there exist $\varepsilon > 0, C > 0$ such that

$$\varepsilon < \|B_\lambda \pm S_\lambda\| < C.$$
Proposition B.3. Suppose \( \{ (E_\lambda, b_\lambda, T_\lambda) \}_{\lambda \in \Lambda} \) is a uniform family of geometrically controlled Poincaré complexes over \( X \). Then their \( \ell^2 \)-completions give rise to a uniform family of analytically controlled Poincaré complexes over \( X \). In particular, there exists \( \varepsilon > 0, C > 0 \) such that

\[
\varepsilon < \| B_\lambda \pm S_\lambda \| < C
\]

for all \( \lambda \in \Lambda \).

Proof. Note that \( \bigoplus_\lambda E_\lambda \) is a geometrically controlled Poincaré complex over \( \bigsqcup_\lambda X \). Let \( \mathcal{E} \) be the \( \ell^2 \) completion of \( \bigoplus_\lambda E_\lambda \). Then by the discussion in Section 4.4 and 4.5, the operators

\[
\bigoplus_\lambda (B_\lambda + S_\lambda) \quad \text{and} \quad \bigoplus_\lambda (B_\lambda - S_\lambda)
\]

are bounded and invertible [21, Lemma 3.5]. In particular, there exist \( \varepsilon > 0, C > 0 \) such that

\[
\varepsilon < \| \bigoplus_{\lambda \in \Lambda} (B_\lambda + S_\lambda) \| \varepsilon < C \quad \text{and} \quad \varepsilon < \| \bigoplus_{\lambda \in \Lambda} (B_\lambda - S_\lambda) \| \varepsilon < C.
\]

It follows that \( \varepsilon < \| B_\lambda \pm S_\lambda \| \varepsilon_\lambda < C \) for all \( \lambda \in \Lambda \), where \( \mathcal{E}_j \) is the \( \ell^2 \)-completion of \( E_\lambda \). \( \square \)

Appendix C  Tensor products of Poincaré complexes

In this section, we briefly review tensor products of Poincaré complexes. The discussion below works simultaneously for geometrically controlled or analytically controlled Poincaré complexes. We will not specify which category, and simply call them Poincaré complexes.

Let \((E, d, T)\) and \((F, b, R)\) be two Poincaré complexes of dimension \( n \) and \( m \) respectively. Recall that the tensor product of two chain complexes is naturally a double complex. The total complex \((E \otimes F, \partial)\) of this double complex can be described as follows:

1. the \( k \)-th term of the total chain complex is

\[
(E \otimes F)_k = \bigoplus_{k=p+q} E_p \otimes F_q;
\]

2. the differential is defined as

\[
\partial(x \otimes y) = dx \otimes y + (-1)^{|x|} x \otimes by \in (E_{p-1} \otimes F_q) \oplus (E_p \otimes F_{q-1})
\]

for \( x \otimes y \in E_p \otimes F_q \), where \( |x| = p \) for \( x \in E_p \).

Roughly speaking, \( \partial = d \bar{\otimes} 1 + 1 \bar{\otimes} b \), where \( \bar{\otimes} \) stands for graded tensor product. And the sign convention is that a sign \( (-1)^{|\alpha|+|\beta|} \) is introduced when an symbol \( \alpha \) (a chain
element or a map) passes over another symbol $\beta$ (a chain element or a map). Now it is easy to verify that

$$\partial^*(x \otimes y) = d^*x \otimes y + (-1)^{|x|}x \otimes b^*y \in (E_{p+1} \otimes F_q) \oplus (E_p \otimes F_{q+1})$$

for $x \otimes y \in E_p \otimes F_q$.

The Poincaré duality operator $T$ and $R$ also naturally induce a Poincaré duality operator $T \hat{\otimes} R$ on $(E \otimes F, \partial)$ as follows.

**Definition C.1.** We define

$$(T \hat{\otimes} R)(x \otimes y) = (-1)^{(n-|x|)-|y|} T x \otimes R y.$$  

The following lemma shows that $T \hat{\otimes} R$ satisfied the conditions in Definition 4.9, hence implements a Poincaré duality operator for $(E \otimes F, \partial)$.

**Lemma C.2.** We have

1. $(T \hat{\otimes} R)\partial^*v + (-1)^k \partial(T \hat{\otimes} R)v = 0$ for all $v \in (E \otimes F)_k$;
2. $(T \hat{\otimes} R)^*v = (-1)^{(n+m-k)}(T \hat{\otimes} R)v$ for all $v \in (E \otimes F)_k$.

**Proof.** Let $x \otimes y \in E_p \otimes F_q$ with $p + q = k$. Then we have

$$\partial(T \hat{\otimes} R)(x \otimes y) = (-1)^{(n-|x|)-|y|} \partial(T x \otimes R y)$$

$$= (-1)^{(n-|x|)-|y|} (dT x \otimes R y + (-1)^{|T x|} T x \otimes b R y)$$

and

$$(T \hat{\otimes} R)\partial^*(x \otimes y) = (T \hat{\otimes} R)(d^*x \otimes y + (-1)^{|x|} x \otimes b^*y)$$

$$= (-1)^{(n-|x|-1)-|y|} T d^*x \otimes R y + (-1)^{|x|} (-1)^{(n-|x|)-(|y|+1)} T x \otimes R b^* y$$

$$= (-1)^{(n-|x|-1)-|y|} T d^*x \otimes R y + (-1)^n (-1)^{(n-|x|)-|y|} T x \otimes R b^* y$$

It follows that

$$(T \hat{\otimes} R)\partial^*(x \otimes y) + (-1)^k \partial(T \hat{\otimes} R)(x \otimes y) = 0,$$

since $Td^*x + (-1)^{|x|}dT x = 0$ and $Rb^* y + (-1)^{|y|}bR y = 0$. This prove part (1).

The calculation for Part (2) is similar. We leave out the details.

\[ \square \]

**Appendix D**

**Product formula for higher rho invariant**

In the section, we prove the product formula for higher rho invariant (Theorem 6.8).

Given $\theta = (M, \partial M, \varphi, N, \partial N, \psi, f) \in S_n(X)$, let $\theta \times \mathbb{R} \in S_{n+1}(X \times \mathbb{R})$ be the product of $\theta$ and $\mathbb{R}$, which defines an element in $S_{n+1}(X \times \mathbb{R})$. Here various undefined terms take the obvious meanings (see Section 3.3 for the definition of $\theta \times I$ for example).

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Note that the construction in Section 4.6 also applies to $\theta \times \mathbb{R}$ and defines its higher rho invariant $\rho(\theta \times \mathbb{R}) \in K_{n+1}(\mathcal{C}_L^*(\mathcal{X} \times \mathbb{R})^\Gamma)$. Also there is a natural homomorphism $\alpha: \mathcal{C}_L^*(\mathcal{X})^\Gamma \otimes \mathcal{C}_L^*(\mathbb{R}) \to \mathcal{C}_L^*(\mathcal{X} \times \mathbb{R})^\Gamma$, which induces isomorphisms on $K$-theory groups.

**Theorem D.1.** With the same notation as above, we have

$$k_n \cdot \alpha_*(\rho(\theta) \otimes \text{Ind}_L(\mathbb{R})) = \rho(\theta \times \mathbb{R})$$

in $K_{n+1}(\mathcal{C}_L^*(\mathcal{X} \times \mathbb{R})^\Gamma)$, where $\text{Ind}_L(\mathbb{R})$ is the $K$-homology class of the signature operator on $\mathbb{R}$, and $k_n = 1$ if $n$ is even and $2$ if $n$ is odd.

**Proof.** We will prove the even case and the odd case separately.

**Even case.** Let us first consider the case where $n$ is even. We use the unbounded theory to define the $K$-homology class of the signature operator on $\mathbb{R}$. The Hilbert-Poincaré complex $(F, d, R)$ associated to the signature operator on $\mathbb{R}$ is

$$\Omega_{L^2}^0(\mathbb{R}) \xleftarrow{d} \Omega_{L^2}^1(\mathbb{R})$$

where $d$ is the adjoint of the de Rham differential map and the duality operator $R$ is the Hodge star operator. See [21, Section 5] and [22, Section 5].

Let $(E, b, T)$ be the analytically controlled Poincaré complex associated to the space $M \cup_f (-N)$ as in the definition of $\rho(\theta)$. See Section 4.6 and Appendix A. Then the tensor product $(E \otimes F, \partial, T \otimes R)$ gives rise to a specific representative of the Hilbert-Poincaré complex associated to $\theta \times \mathbb{R}$. See Appendix C for more details on tensor products of Poincaré complexes. It is straightforward to verify that the self-adjoint duality operator $S_{T \otimes R}$ (as in Definition 4.10) associated to $T \otimes R$ is precisely $S_T \otimes R$. For notational simplicity, let us write $S = S_T$. Note that we have

$$\partial = b \otimes 1 + 1 \otimes d: E_{\text{even}} \otimes F_1 \to E_{\text{odd}} \otimes F_1 \oplus E_{\text{even}} \otimes F_0.$$ 

$$\partial^* = b^* \otimes 1 - 1 \otimes d^*: E_{\text{odd}} \otimes F_0 \to E_{\text{even}} \otimes F_0 \oplus E_{\text{odd}} \otimes F_1.$$ 

Let us identify $F_1 = \Omega_{L^2}^1(\mathbb{R})$ with $F_0 = \Omega_{L^2}^0(\mathbb{R})$ by $hdt \mapsto h$. With this identification, we have $d^* = -d$ and $d$ is skew-adjoint. Moreover, we have the following:

(i) $(E \otimes F)_{\text{odd}} = (E_{\text{even}} \otimes F_1) \oplus (E_{\text{odd}} \otimes F_0) \cong (E_{\text{even}} \oplus E_{\text{odd}}) \otimes F_0 = E \otimes F_0$;

(ii) $(E \otimes F)_{\text{even}} = (E_{\text{even}} \otimes F_0) \oplus (E_{\text{odd}} \otimes F_1) \cong (E_{\text{even}} \oplus E_{\text{odd}}) \otimes F_0 = E \otimes F_0$;

(iii) $\partial + \partial^* \pm S \otimes R = B \otimes 1 - 1 \otimes iD \pm S \otimes 1: E \otimes F_0 \to E \otimes F_0$, where $B = b + b^*$ and $D = i \cdot d$ with $i = \sqrt{-1}$.

**Claim D.2.** $\rho(\theta \times \mathbb{R})$ is represented by a path of invertibles:

$$V_t = (B \otimes 1 - 1 \otimes iD_t + S \otimes 1)(B \otimes 1 - 1 \otimes iD_t - S \otimes 1)^{-1}: E \otimes F_0 \to E \otimes F_0, \quad (21)$$

where $D_t = (1 + t)^{-1}D$ is the operator $D$ rescaled by $(1 + t)^{-1}$, for $0 \leq t < \infty$. 

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Proof of Claim D.2. We point out that, when defining $V_t$, we should in fact use refinements $B_j$ of $B$ and $S_j$ of $S$ respectively such as in Appendix A. For notational simplicity, we will leave out the details, and continue writing $B$ instead. Note that there exists $\varepsilon_0 > 0$ such that $(B_j \pm S_j)^2 > \varepsilon_0$ for all $j$.

Note that

$$(B \otimes 1 - 1 \otimes iD_t + S \otimes 1)(B \otimes 1 - 1 \otimes iD_t - S \otimes 1)^{-1}$$

$$= ((B + S) \otimes 1 - 1 \otimes iD_t)((B - S) \otimes 1 + 1 \otimes iD_t)((B - S)^2 \otimes 1 + 1 \otimes D_t^2)^{-1}$$

$$= 1 + (2S(B - S) \otimes 1 + 2S \otimes iD_t)((B - S)^2 \otimes 1 + 1 \otimes D_t^2)^{-1}$$

Clearly, we have $(B - S)^2 \otimes 1 + 1 \otimes D_t^2 = ((B - S)^2 - \varepsilon_0) \otimes 1 + 1 \otimes (\varepsilon_0 + D_t^2)$, where both $(B - S)^2 - \varepsilon_0$ and $(\varepsilon_0 + D_t^2)$ are invertible. It follows that

$$((B - S)^2 \otimes 1 + 1 \otimes D_t^2)^{-1}$$

$$= (1 \otimes (\varepsilon_0 + D_t^2)^{-1}) \left[ ((B - S)^2 - \varepsilon_0) \otimes (\varepsilon_0 + D_t^2)^{-1} + 1 \otimes 1 \right]^{-1}.$$ 

Now one can use an argument similar to the discussion in [60, Page 841-843] to show that all terms in line (22) can be approximated arbitrarily well operator-norm-wise by elements with arbitrary small propagations. Alternatively, we can argue as follows. Recall the following functional calculus by using Fourier transform and wave operator:

$$f(D_t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\xi} \hat{f}(\xi) d\xi,$$

where $\hat{f}$ is the Fourier transform of $f$. Now by the propagation estimate of the wave operator $e^{i\xi D_t}$, there exists $G_t$ such that $||((\varepsilon_0 + D_t^2)^{-1} - G_t||$ is sufficiently small for all $t$ and the propagation of $G_t$ goes to 0, as $t$ goes to infinity. A similar statement holds for $D_t(\varepsilon_0 + D_t^2)^{-1}$.

Now approximate $\left[ ((B - S)^2 - \varepsilon_0) \otimes (\varepsilon_0 + D_t^2)^{-1} + 1 \otimes 1 \right]^{-1}$ by

$$\left[ ((B - S)^2 - \varepsilon_0) \otimes G_t + 1 \otimes 1 \right]^{-1}.$$ 

Note that there exists $\delta > 0, C > 0$ such that

$$\delta < \|(B - S)^2 - \varepsilon_0) \otimes G_t + 1 \otimes 1\| < C$$

uniformly for all $t$. Now use a polynomial to approximate $h(x) = x^{-1}$ on the interval $[\delta, C]$. We see that there exists $K_t$ that approximates

$$\left[ ((B - S)^2 - \varepsilon_0) \otimes (\varepsilon_0 + D_t^2)^{-1} + 1 \otimes 1 \right]^{-1}$$

sufficiently well, and the propagation of $K_t$ goes to 0, as $t$ goes to infinity.

To summarize, for any $\varepsilon > 0$, there exists a path of invertible elements $(\check{V}_t^\varepsilon)_{0 \leq t < \infty}$ such that

1. $V_0^\varepsilon = 1$;
(2) \( \| V_t - \tilde{V}_t^\epsilon \| < \epsilon \) for all \( t \in [0, \infty) \),

(3) and the propagation of \( \tilde{V}_t^\epsilon \) goes to 0, as \( t \) goes to infinity.

It follows that, at the \( K \)-theory level, the path \((V_t)_{0 \leq t < \infty}\) is a representative of the \( K \)-theory class \( \rho(\theta \times \mathbb{R}) \in K_{n+1}(C^*_L(\tilde{X} \times \mathbb{R})^s) \). This finishes the proof.

Now we return to the proof of the theorem. Recall that \( B \pm S \) is a self-adjoint invertible operator. Therefore, \( B \pm S \) is homotopic to \( P_\pm - Q_\pm \) through a path of invertible elements, where \( P_\pm \) is the positive projection of \( B \pm S \) and \( Q_\pm \) is the negative projection of \( B \pm S \). Note that \( P_\pm + Q_\pm = 1 \). We see that the path

\[ B \otimes 1 - 1 \otimes iD_t \pm S \otimes 1 = (B \pm S) \otimes 1 - 1 \otimes iD_t \]

is homotopic to the path

\[ (P_\pm - Q_\pm) \otimes 1 - (P_\pm + Q_\pm) \otimes iD_t = P_\pm \otimes (1 - iD_t) + Q_\pm \otimes (-iD_t - 1). \]

To be precise, again we need to approximate \( P_\pm \) and \( Q_\pm \) by elements with appropriate propagations, and use these approximations instead of \( P_\pm \) and \( Q_\pm \). This is straightforward. In particular, the calculation below can be easily modified to work for these approximations of \( P_\pm \) and \( Q_\pm \) as well. For notational simplicity, we will leave out the details, and continue using \( P_\pm \) and \( Q_\pm \).

A routine calculation shows that

\[ (P_\pm \otimes (-iD_t + 1) + Q_\pm \otimes (-iD_t - 1))^{-1} = P_\pm \otimes (-iD_t + 1)^{-1} + Q_\pm \otimes (-iD_t - 1)^{-1}. \]

It follows that, at the \( K \)-theory level, the path \((V_t)_{0 \leq t < \infty}\) is equivalent to

\[
[(P_\pm \otimes (-iD_t + 1) + Q_\pm \otimes (-iD_t - 1))(P_\pm \otimes (-iD_t + 1)^{-1} + Q_\pm \otimes (-iD_t - 1)^{-1})] \\
= [P_\pm P_\pm \otimes 1 + P_\pm Q_\pm \otimes (iD_t - 1)(iD_t + 1)^{-1} + Q_\pm P_\pm \otimes (iD_t + 1)(iD_t + 1)^{-1} + Q_\pm Q_\pm \otimes 1] \\
= [(P_\pm \otimes (iD_t - 1)(iD_t + 1)^{-1} + (1 - P_\pm) \otimes 1)(P_\pm \otimes (iD_t + 1)(iD_t + 1)^{-1} + (1 - P_\pm) \otimes 1)] \\
= ([P_\pm] - [P_-]) \otimes ([D_t + i](D_t - i)^{-1})
\]

where the last term is precisely \( \rho(\theta) \otimes \text{Ind}_L(\mathbb{R}) \). To summarize, when \( n \) is even, we have proved that

\[ \alpha_* (\rho(\theta) \otimes \text{Ind}_L(\mathbb{R})) = \rho(\theta \times \mathbb{R}). \]

**Odd case.** Now we consider the case where \( n \) is odd. Note that we have the following commutative diagram:

\[
\begin{array}{ccc}
K_n(C^*_{L,0}(\tilde{X})^\Gamma) \otimes K_1(C^*_L(\mathbb{R})) & \xrightarrow{\alpha \otimes 1} & K_{n+1}(C^*_{L,0}(\tilde{X} \times \mathbb{R})^s) \otimes K_1(C^*_L(\mathbb{R})) \\
\cong & \cong & \\
K_n(C^*_{L,0}(\tilde{X})^\Gamma) \otimes K_0(C^*_L(\mathbb{R}^2)) & \xrightarrow{\tau} & K_n(C^*_{L,0}(\tilde{X} \times \mathbb{R}^2)^s). \\
\end{array}
\]

where various isomorphisms are induced by the natural homomorphisms

\[ \alpha: C^*_{L,0}(\tilde{X})^\Gamma \otimes C^*_L(\mathbb{R}) \to C^*_L(\tilde{X} \times \mathbb{R})^\Gamma, \quad \beta: C^*_L(\mathbb{R}) \otimes C^*_L(\mathbb{R}) \to C^*_L(\mathbb{R}^2), \]

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γ: $C^*_L(\tilde{X} \times \mathbb{R}) \otimes C^*_L(\mathbb{R}) \to C^*_L(\tilde{X} \times \mathbb{R}^2)^\Gamma$ and τ: $C^*_L(\tilde{X})\otimes C^*_L(\mathbb{R}) \to C^*_L(\tilde{X} \times \mathbb{R}^2)^\Gamma$.

In Proposition D.3 below, we will show that $\tau(\rho(\theta) \otimes \text{Ind}_L(\mathbb{R}^2)) = \rho(\theta \times \mathbb{R}^2)$, where $\text{Ind}_L(\mathbb{R}^2)$ is the $K$-homology class of the signature operator on $\mathbb{R}^2$. Assuming this for the moment, by the commutativity of diagram (23), it follows that

$$\gamma_* [\rho(\theta \times \mathbb{R}) \otimes \text{Ind}_L(\mathbb{R})] = \rho(\theta \times \mathbb{R}^2)$$

$$= 2 \cdot \tau_* [\rho(\theta) \otimes \beta_* (\text{Ind}_L(\mathbb{R}) \otimes \text{Ind}_L(\mathbb{R}))]$$

$$= 2 \cdot \gamma_* [\alpha_* (\rho(\theta) \otimes \text{Ind}_L(\mathbb{R})) \otimes \text{Ind}_L(\mathbb{R})]$$

where the first equality is a consequence of the even case. Here we have used the fact that

$$\text{Ind}_L(\mathbb{R}^2) = 2 \cdot \beta_* (\text{Ind}_L(\mathbb{R}) \otimes \text{Ind}_L(\mathbb{R})).$$

Therefore, we have

$$\rho(\theta \times \mathbb{R}) \otimes \text{Ind}_L(\mathbb{R}) = 2 \cdot \alpha_* (\rho(\theta) \otimes \text{Ind}_L(\mathbb{R})) \otimes \text{Ind}_L(\mathbb{R}),$$

which implies that $\rho(\theta \times \mathbb{R}) = 2 \cdot \alpha_* (\rho(\theta) \otimes \text{Ind}_L(\mathbb{R}))$. This finishes the proof. \hfill \Box

**Proposition D.3.** We have $\tau_* (\rho(\theta) \otimes \text{Ind}_L(\mathbb{R}^2)) = \rho(\theta \times \mathbb{R}^2)$, where $\text{Ind}_L(\mathbb{R}^2)$ is the $K$-homology class of the signature operator on $\mathbb{R}^2$.

**Proof.** The proof is similar to the even case above, but the details are more involved. Here again the precise details of the proof would involve a discussion about approximations by finite propagation elements. Since this is very similar to the even case above, we will leave out the details.

Let $(F, d, R)$ be the Hilbert-Poincaré complex associated to $\mathbb{R}^2$:

$$\Omega^0_{L^2}(\mathbb{R}^2) \xleftarrow{d} \Omega^1_{L^2}(\mathbb{R}^2) \xrightarrow{\cdot d} \Omega^2_{L^2}(\mathbb{R}^2)$$

where $d$ is the dual of the de Rham differential and $R$ is the Hodge star operator. Let us write $F_{\text{even}} = \Omega^0_{L^2}(\mathbb{R}^2) \oplus \Omega^2_{L^2}(\mathbb{R}^2)$ and $F_{\text{odd}} = \Omega^1_{L^2}(\mathbb{R}^2)$.

Let $(E, b, T)$ be the analytically controlled Poincaré complex associated to the space $M \cup_f (-N)$ as in the definition of $\rho(\theta)$. See Section 4.6 and Appendix A. Then the tensor product $(E \otimes F, \partial, T \otimes R)$ gives rise to a specific representative of the Hilbert-Poincaré complex associated to $\theta \times \mathbb{R}^2$. Let $S_{T \otimes R}$, $S_T$ and $S_R$ be the self-adjoint operators (as in Definition 4.10) associated to $T \otimes R$, $T$ and $R$ respectively. Now a straightforward calculation shows that

1. $S_{T \otimes R} = S_T \otimes S_R$ on $E \otimes F_{\text{even}}$, and $S_{T \otimes R} = -S_T \otimes S_R$ on $E \otimes F_{\text{odd}}$;

2. $\partial = b \otimes 1 + 1 \otimes d$ on $E_{\text{even}} \otimes F$ and $\partial = b \otimes 1 - 1 \otimes d$ on $E_{\text{odd}} \otimes F$.

It follows that

$$\partial + \partial^* \pm S_{T \otimes R} = B \otimes 1 - 1 \otimes D \pm S_T \otimes S_R \quad \text{on} \quad E_{\text{odd}} \otimes F_{\text{even}},$$

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and
\[ \partial + \partial^* \pm S_{T \otimes R} = B \otimes 1 + 1 \otimes D \mp S_T \otimes S_R \quad \text{on } E_{\text{even}} \otimes F_{\text{odd}}, \]

where \( B = b + b^* \) and \( D = d + d^* \). Let \( F^\pm \) be the eigenspace of \( S_R \) belonging to the eigenvalue \( \pm 1 \). We make the following identifications:

\[
\begin{align*}
E_{\text{odd}} \otimes F & = E_{\text{odd}} \otimes F_{\text{odd}} \oplus E_{\text{odd}} \otimes F_{\text{even}} \oplus E_{\text{odd}} \otimes F_{\text{odd}} \\
& \quad \rightarrow (E \otimes F)_{\text{odd}}, \\
E_{\text{even}} \otimes F & = E_{\text{even}} \otimes F_{\text{even}} \oplus E_{\text{even}} \otimes F_{\text{odd}} \oplus E_{\text{even}} \otimes F_{\text{odd}} \\
& \quad \rightarrow (E \otimes F)_{\text{even}}.
\end{align*}
\]

and

\[
\begin{align*}
(E \otimes F)_{\text{even}} & = E_{\text{even}} \otimes F_{\text{even}} \oplus E_{\text{even}} \otimes F_{\text{odd}} \oplus E_{\text{even}} \otimes F_{\text{odd}} \\
& \quad \rightarrow E_{\text{even}} \otimes F.
\end{align*}
\]

With these identifications, we have

\[
\begin{align*}
\partial + \partial^* + S_{T \otimes R} = & \begin{cases} 
(B + S_T) \otimes 1 + (B + S_T) \otimes D & \text{on } E_{\text{odd}} \otimes F_{\text{even}} \\
(B - S_T)^2(B + S_T) \otimes 1 + (B + S_T) \otimes D & \text{on } E_{\text{odd}} \otimes F_{\text{odd}^*} \\
-(B - S_T) \otimes 1 + (B - S_T) \otimes D & \text{on } E_{\text{odd}} \otimes F_{\text{odd}} \\
-(B + S_T)^2(B - S_T) \otimes 1 + (B - S_T) \otimes D & \text{on } E_{\text{odd}} \otimes F_{\text{even}}
\end{cases}
\end{align*}
\]

Note that \( (B \pm S_T)^2 : E_{\text{even}} \xrightarrow{B \pm S_T} E_{\text{odd}} \xrightarrow{B \pm S_T} E_{\text{even}} \) are positive invertible operators. It follows that the invertible element \( \partial + \partial^* + S_{T \otimes R} \) is homotopic to

\[
\begin{pmatrix} B + S_T \\ B - S_T \end{pmatrix} S_R + \begin{pmatrix} B - S_T \\ B + S_T \end{pmatrix} D : E_{\text{odd}} \otimes F \to E_{\text{even}} \otimes F \quad (24)
\]

Here the matrix form is written with respect to the decomposition \( F = F^+ \oplus F^- \). Note that \( D \) is off-diagonal in this case. Now the invertible element from line (24) in turn is homotopic to the invertible element

\[
V = \begin{pmatrix} B + S_T \\ B - S_T \end{pmatrix} S_R f(D) + \begin{pmatrix} B - S_T \\ B + S_T \end{pmatrix} g(D)
\]

where\[ g(x) = x(1 + x^2)^{-1/2} \] and \( f(x) = \sqrt{1 - g^2(x)} = (1 + x^2)^{-1/2} \). Similarly, \( \partial + \partial^* - S_{T \otimes R} \) is homotopic to

\[
U = \begin{pmatrix} B - S_T \\ B + S_T \end{pmatrix} S_R f(D) + \begin{pmatrix} B - S_T \\ B + S_T \end{pmatrix} g(D)
\]

\[ ^{12} \text{In fact, any normalizing function } g \text{ and } f(x) = \sqrt{1 - g^2(x)} \text{ will also work.} \]
Note that we have

\[
U^{-1} = (S_R f(D) + g(D)) \begin{pmatrix} (B - S_T)^{-1} \\ (B + S_T)^{-1} \end{pmatrix}
\]

since \((S_R f(D) + g(D))^2 = 1\).

Similarly, if we replace \(D\) by \(D_t = (1 + t)^{-1}D\), we have

\[
V_t = \begin{pmatrix} B + S_T & B - S_T \\ B - S_T & B + S_T \end{pmatrix} S_R f(D_t) + \begin{pmatrix} B - S_T & B + S_T \\ B + S_T & B - S_T \end{pmatrix} g(D_t)
\]

and

\[
U_t = \begin{pmatrix} B - S_T & B + S_T \\ B + S_T & B - S_T \end{pmatrix} S_R f(D_t) + \begin{pmatrix} B - S_T & B + S_T \\ B + S_T & B - S_T \end{pmatrix} g(D_t).
\]

It follows that the path of invertibles \((V_t U_t^{-1})_{0 \leq t < \infty}\) is a representative of the \(K\)-theory class \(\rho(\theta \times \mathbb{R}^2)\). Now note that

\[
V_t U_t^{-1} = [B \otimes 1 + S_T \otimes (S_R (f^2(D_t) - g^2(D_t)) + 2f(D_t)g(D_t))] \begin{pmatrix} (B - S_T)^{-1} \\ (B + S_T)^{-1} \end{pmatrix}.
\]

Following the notation of [22, Section 5.2.1], let us denote

\[
S_1 = S_R \quad \text{and} \quad (S_2)_t = g(D_t) + S_R f(D_t).
\]

We immediately see that

\[
S_R (f^2(D_t) - g^2(D_t)) + 2f(D_t)g(D_t) = (S_2)_t (S_1 (S_2)_t).
\]

Let us denote the latter by \(\mathcal{J}_t := (S_2)_t S_1 (S_2)_t\). Note that \(\mathcal{J}_t\) is a symmetry for each \(t\). We define a projection \(P_t := (\mathcal{J}_t + 1)/2\). To summarize, \(\rho(\theta \times \mathbb{R}^2)\) is represented by the path of invertibles

\[
\begin{align*}
&= (B \otimes 1 + S_T \otimes \mathcal{J}_t) \begin{pmatrix} (B - S_T)^{-1} & 0 \\ 0 & (B + S_T)^{-1} \end{pmatrix}^{-1} \\
&= \frac{1}{2} (B + S_T) \otimes P_t + (B - S_T) \otimes (1 - P_t) \begin{pmatrix} (B - S_T)^{-1} & 0 \\ 0 & (B + S_T)^{-1} \end{pmatrix}
\end{align*}
\]

which is precisely \(\tau_* (\rho(\theta) \otimes \text{Ind}_L(\mathbb{R}^2))\). This finishes the proof. \(\square\)

\[13\]There appears to be a switch of sign convention in [22, Section 5.2.1]. In any case, our sign convention is consistent with the usual sign convention in the literature.
References


