Dirac Operator

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Miniseries of five lectures:

1. Dirac operator
2. Atiyah-Singer revisited
3. What is K-homology?
4. Beyond ellipticity
5. The Riemann-Roch theorem

Based on joint work with Ron Douglas.
Happy Birthday, Ron!
DIRAC OPERATOR

The Dirac operator of $\mathbb{R}^n$ will be defined. This is a first order elliptic differential operator with constant coefficients. Next, the class of differentiable manifolds which come equipped with an order one differential operator which is locally (modulo lower order terms) isomorphic to the Dirac operator of $\mathbb{R}^n$ will be considered. These are the Spin$^c$ manifolds. Spin$^c$ is slightly stronger than oriented, so Spin$^c$ can be viewed as “oriented plus epsilon”. Most of the oriented manifolds that occur in practice are Spin$^c$. The Dirac operator of a closed Spin$^c$ manifold is the basic example for the Hirzebruch- Riemann-Roch theorem and the Atiyah-Singer index theorem.
What is the Dirac operator of $\mathbb{R}^n$?

To answer this, shall construct matrices $E_1, E_2, \ldots, E_n$ with the following properties:
Properties of $E_1, E_2, \ldots, E_n$

- Each $E_j$ is a $2^r \times 2^r$ matrix of complex numbers, where $r$ is the largest integer $\leq n/2$.
- Each $E_j$ is skew adjoint, i.e. $E_j^* = -E_j$
- $E_j^2 = -I$, $j = 0, 1, \ldots, n$ (I is the $2^r \times 2^r$ identity matrix.)
- $E_j E_k + E_k E_j = 0$ whenever $j \neq k$.
- For $n$ odd, $i^{r+1} E_1 E_2 \cdots E_n = I$  
  
- For $n$ even, each $E_j$ is of the form $E_j = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$ and $i^r E_1 E_2 \cdots E_n = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$
These matrices are constructed by a simple inductive procedure.

\( n = 1, \quad E_1 = [-i] \)

\( n \mapsto n + 1 \) with \( n \) odd \quad \( (r \mapsto r + 1) \)

The new matrices \( \tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_{n+1} \) are

\[
\tilde{E}_j = \begin{bmatrix} 0 & E_j \\ E_j & 0 \end{bmatrix} \quad \text{for } j = 1, \ldots, n \quad \text{and} \quad \tilde{E}_{n+1} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}
\]

where \( E_1, E_2, \ldots, E_n \) are the old matrices.

\( n \mapsto n + 1 \) with \( n \) even \quad \( (r \text{ does not change}) \)

The new matrices \( \tilde{E}_1, \tilde{E}_2, \ldots, \tilde{E}_{n+1} \) are

\[
\tilde{E}_j = E_j \quad \text{for } j = 1, \ldots, n \quad \text{and} \quad \tilde{E}_{n+1} = \begin{bmatrix} -iI & 0 \\ 0 & iI \end{bmatrix}
\]

where \( E_1, E_2, \ldots, E_n \) are the old matrices.
Example

\[ n = 1: \quad E_1 = [-i] \]

\[ n = 2: \quad E_1 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

\[ n = 3: \quad E_1 = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \]
Example

\[ n = 4: \quad E_1 = \begin{bmatrix}
0 & 0 & 0 & -i \\
0 & 0 & -i & 0 \\
0 & -i & 0 & 0 \\
-i & 0 & 0 & 0
\end{bmatrix} \quad E_2 = \begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}\]

\[ E_3 = \begin{bmatrix}
0 & 0 & -i & 0 \\
0 & 0 & 0 & i \\
-i & 0 & 0 & 0 \\
0 & i & 0 & 0
\end{bmatrix} \quad E_4 = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix}\]
$D = \text{Dirac operator of } \mathbb{R}^n$

\[
\begin{cases}
n = 2r & \text{n even} \\
n = 2r + 1 & \text{n odd}
\end{cases}
\]

\[
D = \sum_{j=1}^{n} E_j \frac{\partial}{\partial x_j}
\]

$D$ is an unbounded symmetric operator on the Hilbert space
$L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n)$ (2r times)

To begin, the domain of $D$ is
$C^\infty_c(\mathbb{R}^n) \oplus C^\infty_c(\mathbb{R}^n) \oplus \ldots \oplus C^\infty_c(\mathbb{R}^n)$ (2r times)

$D$ is essentially self-adjoint (i.e. $D$ has a unique self-adjoint extension)
so it is natural to view $D$ as an unbounded self-adjoint operator on the Hilbert space
$L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n)$ (2r times)
QUESTION : Let $M$ be a $\mathcal{C}^\infty$ manifold of dimension $n$. Does $M$ admit a differential operator which (modulo lower order terms) is locally isomorphic to the Dirac operator of $\mathbb{R}^n$?

To answer this question, will define Spin$^c$ vector bundle.
What is a Spin$^c$ vector bundle?

Let $X$ be a paracompact Hausdorff topological space. On $X$ let $E$ be an $\mathbb{R}$ vector bundle which has been oriented, i.e. the structure group of $E$ has been reduced from $GL(n, \mathbb{R})$ to $GL^+(n, \mathbb{R})$

$$GL^+(n, \mathbb{R}) = \{ [a_{ij}] \in GL(n, \mathbb{R}) \mid \det[a_{ij}] > 0 \}$$

$n$ = fiber dimension ($E$)

Assume $n \geq 3$ and recall that for $n \geq 3$

$$H^2(GL^+(n, \mathbb{R}); \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

Denote by $\mathcal{F}^+(E)$ the principal $GL^+(n, \mathbb{R})$ bundle on $X$ consisting of all positively oriented frames.
A point of $\mathcal{F}^+(E)$ is a pair $(x, (v_1, v_2, \ldots, v_n))$ where $x \in X$ and $(v_1, v_2, \ldots, v_n)$ is a positively oriented basis of $E_x$. The projection $\mathcal{F}^+(E) \to X$ is

$$\left(x, (v_1, v_2, \ldots, v_n)\right) \mapsto x$$

For $x \in X$, denote by

$$\iota_x : \mathcal{F}_x^+(E) \hookrightarrow \mathcal{F}^+(E)$$

the inclusion of the fiber at $x$ into $\mathcal{F}^+(E)$.

Note that (with $n \geq 3$)

$$H^2(\mathcal{F}_x^+(E); \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$
A Spin\(^c\) vector bundle on \(X\) is an \(\mathbb{R}\) vector bundle \(E\) on \(X\) (fiber dimension \(E \geq 3\)) with

1. \(E\) has been oriented.
2. \(\alpha \in H^2(\mathcal{F}^+(E); \mathbb{Z})\) has been chosen such that \(\forall x \in X\)

\[ i_x^*(\alpha) \in H^2(\mathcal{F}_x^+(E); \mathbb{Z}) \] is non-zero.
Remarks

1. For $n = 1, 2$ “$E$ is a Spin$^c$ vector bundle” = “$E$ has been oriented and an element $\alpha \in H^2(X; \mathbb{Z})$” has been chosen. ($\alpha$ can be zero.)

2. For all values of $n = \text{fiber dimension}(E)$, $E$ is a Spin$^c$ vector bundle iff the structure group of $E$ has been changed from $GL(n, \mathbb{R})$ to Spin$^c(n)$. i.e. Such a change of structure group is equivalent to the above definition of Spin$^c$ vector bundle.
By forgetting some structure a complex vector bundle or a Spin vector bundle canonically becomes a Spin\(^c\) vector bundle.

\[
\text{complex} \downarrow \Rightarrow \text{Spin} \Rightarrow \text{Spin}^c \downarrow \Rightarrow \text{oriented}
\]

A Spin\(^c\) structure for an \(\mathbb{R}\) vector bundle \(E\) can be thought of as an orientation for \(E\) plus a slight extra bit of structure. Spin\(^c\) structures behave very much like orientations. For example, an orientation on two out of three \(\mathbb{R}\) vector bundles in a short exact sequence determines an orientation on the third vector bundle. An analogous assertion is true for Spin\(^c\) structures.
Let

\[ 0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0 \]

be a short exact sequence of $\mathbb{R}$-vector bundles on $X$. If two out of three are Spin$^c$ vector bundles, then so is the third.
Definition

Let $M$ be a $C^\infty$ manifold (with or without boundary). $M$ is a Spin$^c$ manifold iff the tangent bundle $TM$ of $M$ is a Spin$^c$ vector bundle on $M$.

The Two Out Of Three Lemma implies that the boundary $\partial M$ of a Spin$^c$ manifold $M$ with boundary is again a Spin$^c$ manifold.
Various well-known structures on a manifold $M$ make $M$ into a Spin\(^c\) manifold.
A Spin$^c$ manifold can be thought of as an oriented manifold with a slight extra bit of structure. Most of the oriented manifolds which occur in practice are Spin$^c$ manifolds.

A Spin$^c$ manifold comes equipped with a first-order elliptic differential operator known as its Dirac operator. This operator is locally isomorphic (modulo lower order terms) to the Dirac operator of $\mathbb{R}^n$. 
EXAMPLE. Let $M$ be a compact complex-analytic manifold. Set $\Omega^{p,q} = C^\infty(M, \Lambda^{p,q}T^*M)$, $\Omega^{p,q}$ is the $\mathbb{C}$ vector space of all $C^\infty$ differential forms of type $(p, q)$.

Dolbeault complex

$$0 \longrightarrow \Omega^{0,0} \longrightarrow \Omega^{0,1} \longrightarrow \Omega^{0,2} \longrightarrow \cdots \longrightarrow \Omega^{0,n} \longrightarrow 0$$

The Dirac operator (of the underlying Spin$^c$ manifold) is the assembled Dolbeault complex

$$\bar{\partial} + \bar{\partial}^* : \bigoplus_j \Omega^{0,2j} \longrightarrow \bigoplus_j \Omega^{0,2j+1}$$

The index of this operator is the arithmetic genus of $M$ — i.e. is the Euler number of the Dolbeault complex.
TWO POINTS OF VIEW ON SPIN\(^c\) MANIFOLDS

1. Spin\(^c\) is a slight strengthening of oriented. Most of the oriented manifolds that occur in practice are Spin\(^c\).

2. Spin\(^c\) is much weaker than complex-analytic. BUT the assembled Dolbeault complex survives (as the Dirac operator). AND the Todd class survives.

\[ M \text{ Spin}^c \implies \exists Td(M) \in H^*(M; \mathbb{Q}) \]
If $M$ is a Spin$^c$ manifold, then $Td(M)$ is

$$Td(M) := \exp^{c_1(M)/2} \hat{A}(M) \quad Td(M) \in H^*(M; \mathbb{Q})$$

If $M$ is a complex-analytic manifold, then $M$ has Chern classes $c_1, c_2, \ldots, c_n$ and

$$\exp^{c_1(M)/2} \hat{A}(M) = P_{Todd}(c_1, c_2, \ldots, c_n)$$
SPECIAL CASE OF ATIYAH-SINGER

Let $M$ be a compact even-dimensional Spin$^c$ manifold without boundary. Let $E$ be a $\mathbb{C}$ vector bundle on $M$.

$D_E$ denotes the Dirac operator of $M$ tensored with $E$.

$$D_E: C^\infty(M, S^+ \otimes E) \longrightarrow C^\infty(M, S^- \otimes E)$$

$S^+, S^-$ are the positive (negative) spinor bundles on $M$.

**THEOREM** $\text{Index}(D_E) = (\text{ch}(E) \cup \text{Td}(M))[M]$. 
SPECIAL CASE OF ATIYAH-SINGER

Let $M$ be a compact even-dimensional Spin$^c$ manifold without boundary. Let $E$ be a $\mathbb{C}$ vector bundle on $M$. $D_E$ denotes the Dirac operator of $M$ tensored with $E$.

**THEOREM** \( \text{Index}(D_E) = (ch(E) \cup Td(M))[M] \).

This theorem will be proved in the next lecture as a corollary of Bott periodicity.

In particular (since the assembled Dolbeault complex of a complex analytic manifold is the Dirac operator of the underlying Spin$^c$ manifold) this will prove the Hirzebruch-Riemann-Roch theorem.