WHAT IS K-HOMOLOGY?

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April 2, 2014
Let $X$ be a compact $C^\infty$ manifold without boundary. $X$ is not required to be oriented. $X$ is not required to be even dimensional. On $X$ let

$$\delta : C^\infty(X, E_0) \longrightarrow C^\infty(X, E_1)$$

be an elliptic differential (or pseudo-differential) operator.

$$(S(TX \oplus 1_\mathbb{R}), E_\sigma) \in K_0(\cdot), \text{ and}$$

$$\text{Index}(D_{E_\sigma}) = \text{Index} (\delta).$$
\[(S(TX \oplus 1_R), E_\sigma)\]

\[
\text{Index}(\delta) = (\text{ch}(E_{\sigma}) \cup \text{Td}((S(TX \oplus 1_R))))[(S(TX \oplus 1_R)]
\]

and this is the general Atiyah-Singer formula.

\(S(TX \oplus 1_R)\) is the unit sphere bundle of \(TX \oplus 1_R\).

\(S(TX \oplus 1_R)\) is even dimensional and is — in a natural way — a Spin\(^c\) manifold.

\(E_\sigma\) is the \(\mathbb{C}\) vector bundle on \(S(TX \oplus 1_R)\) obtained by doing a clutching construction using the symbol \(\sigma\) of \(\delta\).
Minicourse of five lectures:

1. Dirac operator ✓
2. Atiyah-Singer revisited ✓
3. What is K-homology?
4. Beyond ellipticity
5. The Riemann-Roch theorem

The minicourse is based on joint work with Ron Douglas and is dedicated to Ron Douglas.
Let $X$ be a (possibly singular) projective algebraic variety over $\mathbb{C}$.

Grothendieck defined two abelian groups:

$K^0_{alg}(X) =$ Grothendieck group of algebraic vector bundles on $X$.

$K^0_{alg}(X) =$ Grothendieck group of coherent algebraic sheaves on $X$.

$K^0_{alg}(X) =$ the algebraic geometry $K$-theory of $X$ (contravariant).

$K^0_{alg}(X) =$ the algebraic geometry $K$-homology of $X$ (covariant).
Problem

How can $K$-homology be taken from algebraic geometry to topology?
$K$-homology is the dual theory to $K$-theory. There are three ways in which $K$-homology in topology has been defined:

**Homotopy Theory** $K$-theory is the cohomology theory and $K$-homology is the homology theory determined by the Bott (i.e. $K$-theory) spectrum.
This is the spectrum $\ldots, \mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U, \ldots$

**$K$-Cycles** $K$-homology is the group of $K$-cycles.

**$C^*$-algebras** $K$-homology is the Atiyah-BDF-Kasparov group $KK^*(A, \mathbb{C})$. 
Let $X$ be a finite CW complex. The three versions of $K$-homology are isomorphic.

\[ K^\text{homotopy}_j(X) \xrightarrow{\sim} K_j(X) \xrightarrow{} KK_j(C(X), \mathbb{C}) \]

homotopy theory  K-cycles  Atiyah-BDF-Kasparov

\[ j = 0, 1 \]
Problem
How can $K$-homology be taken from algebraic geometry to topology?

There are three ways in which this has been done:

Homotopy Theory $K$-homology is the homology theory determined by the Bott spectrum.

K-Cycles $K$-homology is the group of $K$-cycles.

$C^*$-algebras $K$-homology is the Kasparov group $KK^*(A,\mathbb{C})$. 
Let $X$ be a finite CW complex.

$C(X) = \{ \alpha : X \to \mathbb{C} \mid \alpha \text{ is continuous} \}$

$L(H) = \{ \text{bounded operators } T : H \to H \}$

Any element in the Atiyah-BDF-Kasparov K-homology group $KK^0(C(X), \mathbb{C})$

is given by a 5-tuple $(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T)$ such that:
- \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) are separable Hilbert spaces.
- \( \psi_0 : C(X) \to \mathcal{L}(\mathcal{H}_0) \) and \( \psi_1 : C(X) \to \mathcal{L}(\mathcal{H}_1) \) are unital *-homomorphisms.
- \( T : \mathcal{H}_0 \to \mathcal{H}_1 \) is a (bounded) Fredholm operator.
- For every \( \alpha \in C(X) \) the commutator \( T \circ \psi_0(\alpha) - \psi_1(\alpha) \circ T \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1) \) is compact.

\[
KK^0(C(X), \mathbb{C}) := \{(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T)\}/ \sim
\]
\[ KK^0(C(X), \mathbb{C}) := \{ (\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) \} / \sim \]

\[(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) + (\mathcal{H}_0', \psi_0', \mathcal{H}_1', \psi_1', T') =
(\mathcal{H}_0 \oplus \mathcal{H}_0', \psi_0 \oplus \psi_0', \mathcal{H}_1 \oplus \mathcal{H}_1', \psi_1 \oplus \psi_1', T \oplus T') \]

\[-(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) = (\mathcal{H}_1, \psi_1, \mathcal{H}_0, \psi_0, T^*) \]
Let $X$ be a finite CW complex.
Any element in the Atiyah-BDF-Kasparov K-homology

group $KK^1(C(X), \mathbb{C})$
is given by a 3-tuple $(\mathcal{H}, \psi, T)$ such that :

- $\mathcal{H}$ is a separable Hilbert space.
- $\psi: C(X) \to \mathcal{L}(\mathcal{H})$ is a unital $\ast$-homomorphism.
- $T: \mathcal{H} \to \mathcal{H}$ is a (bounded) self-adjoint Fredholm operator.
- For every $\alpha \in C(X)$ the commutator $T \circ \psi(\alpha) - \psi(\alpha) \circ T \in \mathcal{L}(\mathcal{H})$
is compact.
\[ KK^1(C(X), \mathbb{C}) := \{(\mathcal{H}, \psi, T)\}/ \sim \]

\[(\mathcal{H}, \psi, T) + (\mathcal{H}', \psi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', T \oplus T') \]

\[-(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi, -T) \]
Let $X, Y$ be CW complexes and let $f : X \to Y$ be a continuous map.

Denote by $f^\# : C(X) \leftarrow C(Y)$ the $\ast$-homomorphism

$$f^\#(\alpha) := \alpha \circ f \quad \alpha \in C(Y)$$

Then $f_* : KK_j(C(X), \mathbb{C}) \to KK_j(C(Y), \mathbb{C})$ is

$$f_*(\mathcal{H}, \psi, T) := (\mathcal{H}, \psi \circ f^\#, T) \quad j = 1$$

$$f_*(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) := (\mathcal{H}_0, \psi_0 \circ f^\#, \mathcal{H}_1, \psi_1 \circ f^\#, T) \quad j = 0$$
Cycles for $K$-homology

Let $X$ be a CW complex.

Definition

A $K$-cycle on $X$ is a triple $(M, E, \varphi)$ such that:

1. $M$ is a compact Spin$^c$ manifold without boundary.
2. $E$ is a $\mathbb{C}$ vector bundle on $M$.
3. $\varphi: M \to X$ is a continuous map from $M$ to $X$. 
Set $K_\ast(X) = \{(M, E, \varphi)\}/\sim$ where the equivalence relation $\sim$ is generated by the three elementary steps

- Bordism
- Direct sum - disjoint union
- Vector bundle modification
**Isomorphism** \( (M, E, \varphi) \) is isomorphic to \((M', E', \varphi')\) iff \( \exists \) a diffeomorphism

\[
\psi: M \to M'
\]

preserving the \( \text{Spin}^c \)-structures on \( M, M' \) and with

\[
\psi^*(E') \cong E
\]

and commutativity in the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\psi} & M' \\
\downarrow{\varphi} & & \downarrow{\varphi'} \\
X & & \\
\end{array}
\]
**Bordism** \((M_0, E_0, \varphi_0)\) is **bordant** to \((M_1, E_1, \varphi_1)\) iff \(\exists (\Omega, E, \varphi)\) such that:

1. \(\Omega\) is a compact Spin\(^c\) manifold with boundary.
2. \(E\) is \(\mathbb{C}\) vector bundle on \(\Omega\).
3. \((\partial \Omega, E|_{\partial \Omega}, \varphi|_{\partial \Omega}) \cong (M_0, E_0, \varphi_0) \sqcup (-M_1, E_1, \varphi_1)\)

\(-M_1\) is \(M_1\) with the Spin\(^c\) structure reversed.
\[ (M_0, E_0, \varphi_0) \quad \rightarrow \quad X \quad \rightarrow \quad (-M_1, E_1, \varphi_1) \]
Direct sum - disjoint union

Let $E, E'$ be two $\mathbb{C}$ vector bundles on $M$

$$(M, E, \varphi) \sqcup (M, E', \varphi) \sim (M, E \oplus E', \varphi)$$
Vector bundle modification

\[(M, E, \varphi)\]

Let \(F\) be a \(\text{Spin}^c\) vector bundle on \(M\).

Assume that

\[
\dim_{\mathbb{R}}(F_p) \equiv 0 \mod 2 \quad p \in M
\]

for every fiber \(F_p\) of \(F\).

\[
1_{\mathbb{R}} = M \times \mathbb{R}
\]

\[
S(F \oplus 1_{\mathbb{R}}) := \text{unit sphere bundle of } F \oplus 1_{\mathbb{R}}
\]

\[
(M, E, \varphi) \sim (S(F \oplus 1_{\mathbb{R}}), \beta \otimes \pi^*E, \varphi \circ \pi)
\]
This is a fibration with even-dimensional spheres as fibers.

$F \oplus 1_R$ is a Spin$^c$ vector bundle on $M$ with odd-dimensional fibers. Let $\Sigma$ be the spinor bundle for $F \oplus 1$

$$Cliff_{\mathbb{C}}(F_p \oplus \mathbb{R}) \otimes \Sigma_p \to \Sigma_p$$

$$\pi^* \Sigma = \beta^* \oplus \beta_-$$

$$(M, E, \varphi) \sim (S(F \oplus 1_R), \beta \otimes \pi^* E, \varphi \circ \pi)$$
\[ \{(M, E, \varphi)\}/\sim = K_0(X) \oplus K_1(X) \]

\[ K_j(X) = \text{subgroup of } \{(M, E, \varphi)\}/\sim \text{ consisting of all } (M, E, \varphi) \text{ such that every connected component of } M \text{ has dimension } \equiv j \mod 2, j = 0, 1 \]
Addition in $K_j(X)$ is disjoint union.

$$(M, E, \varphi) + (M', E', \varphi') = (M \sqcup M', E \sqcup E', \varphi \sqcup \varphi')$$

Additive inverse of $(M, E, \varphi)$ is obtained by reversing the Spin$^c$ structure of $M$.

$$-(M, E, \varphi) = (-M, E, \varphi)$$
**DEFINITION.** \((M, E, \varphi)\) *bounds* \iff \exists (\Omega, \tilde{E}, \tilde{\varphi})\) with:

1. \(\Omega\) is a compact \(\text{Spin}^c\) manifold with boundary.
2. \(\tilde{E}\) is a \(\mathbb{C}\) vector bundle on \(\Omega\).
3. \(\tilde{\varphi}: \Omega \to X\) is a continuous map.
4. \((\partial \Omega, \tilde{E}|_{\partial \Omega}, \tilde{\varphi}|_{\partial \Omega}) \cong (M, E, \varphi)\)

**REMARK.** \((M, E, \varphi) = 0\) in \(K_\ast(X)\) \iff \((M, E, \varphi) \sim (M', E', \varphi')\) where \((M', E', \varphi')\) bounds.
Let $X, Y$ be CW complexes and let $f: X \to Y$ be a continuous map.

Then $f_*: K_j(X) \to K_j(Y)$ is

$$f_*(M, E, \varphi) := (M, E, f \circ \varphi)$$
Chern character in $K$-theory

Let $X$ be a finite CW complex. The Chern character $ch$ is defined as

$$ch: K^j(X) \longrightarrow \bigoplus_{l} H^{j+2l}(X; \mathbb{Q})$$

for $j = 0, 1$. Moreover, the map

$$\mathbb{Q} \otimes_{\mathbb{Z}} K^j(X) \longrightarrow \bigoplus_{l} H^{j+2l}(X; \mathbb{Q})$$

is an isomorphism of $\mathbb{Q}$ vector spaces.
Chern character in $K$-homology

$X$ finite CW complex

$(M, E, \varphi) \mapsto \varphi_* (ch(E) \cup Td(M) \cap [M])$

$ch: K_j(X) \longrightarrow \bigoplus_{l} H_{j+2l}(X; \mathbb{Q})$

$j = 0, 1$

$\mathbb{Q} \otimes_{\mathbb{Z}} K_j(X) \longrightarrow \bigoplus_{l} H_{j+2l}(X; \mathbb{Q})$

is an isomorphism of $\mathbb{Q}$ vector spaces.
$X$ a finite CW complex.

$K^*(X)$ is a ring and $K_*(X)$ is a module over this ring. Chern character respects the ring and module structure.
Theorem (PB and R. Douglas and M. Taylor, PB and N. Higson and T. Schick)

Let $X$ be a finite CW complex.

Then for $j = 0, 1$ the natural map of abelian groups

$$K_j(X) \to KK^j(C(X), \mathbb{C})$$

is an isomorphism.
For $j = 0, 1$ the natural map of abelian groups

$$K_j(X) \to KK^j(C(X), \mathbb{C})$$

is $(M, E, \varphi) \mapsto \varphi_*[D_E]$

where

1. $D_E$ is the Dirac operator of $M$ tensored with $E$.
2. $[D_E] \in KK^j(C(M), \mathbb{C})$ is the element in the Atiyah-BDF-Kasparov $K$-homology of $M$ determined by $D_E$.
3. $\varphi_* : KK^j(C(M), \mathbb{C}) \to KK^j(C(X), \mathbb{C})$ is the homomorphism of abelian groups determined by $\varphi : M \to X$. 

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