BEYOND ELLIPTICITY

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Minicourse of five lectures:

1. Dirac operator✓
2. Atiyah-Singer revisited✓
3. What is K-homology?✓
4. Beyond ellipticity
5. The Riemann-Roch theorem

The minicourse is based on joint work with Ron Douglas and is dedicated to Ron Douglas.

Theorem (PB and R.Douglas and M.Taylor, PB and N. Higson and T. Schick)

Let X be a finite CW complex.

Then for $j = 0, 1$ the natural map of abelian groups

$$K_j(X) \to KK^j(C(X), \mathbb{C})$$

is an isomorphism.

For $j = 0, 1$ the natural map of abelian groups

$$K_j(X) \to KK^j(C(X), \mathbb{C})$$

is $(M, E, \varphi) \mapsto \varphi_*[D_E]$

where

1. D_E is the Dirac operator of M tensored with E.
2. $[D_E] \in KK^j(C(M), \mathbb{C})$ is the element in the Atiyah-BDF-Kasparov K-homology of M determined by D_E.
3. $\varphi_* \colon KK^j(C(M), \mathbb{C}) \to KK^j(C(X), \mathbb{C})$ is the homomorphism of abelian groups determined by $\varphi \colon M \to X$.

K-cycles and string theory

K-cycles are very closely connected to the D-branes of string theory. A D-brane is a K-cycle for the twisted K-homology of space-time.

In some models, the D-branes are allowed to evolve with time. This evolution is achieved by permitting the D-branes to change by the three elementary steps. Thus the underlying *charge* of a D-brane (i.e. the element in the twisted K-homology of space-time determined by the D-brane) remains unchanged as the D-brane evolves.

For more, see Jonathan Rosenberg's CBMS string theory lectures.
Also, see paper in the Journal of K-Theory
"K-cycles for twisted K-homology" by Baum-Carey-Wang.
Also, lectures (at Erwin Schrodinger Institute) by Bai-Ling Wang.

Comparison of $K_*(X)$ and $KK^*(C(X), \mathbb{C})$

Given some analytic data on X (i.e. an index problem) it is usually easy to construct an element in $KK^*(C(X), \mathbb{C})$. This does not solve the given index problem. $KK^*(C(X), \mathbb{C})$ does not have a simple explicitly defined chern character mapping it to $H_*(X; \mathbb{Q})$.

$K_*(X)$ does have a simple explicitly defined chern character mapping it to $H_*(X; \mathbb{Q})$.

$$ch\colon K_j(X) \longrightarrow \bigoplus_l H_{j+2l}(X; \mathbb{Q}) \qquad j = 0, 1$$

$$(M, E, \varphi) \mapsto \varphi_*(\mathrm{ch}(E) \cup Td(M) \cap [M])$$

With X a finite CW complex, suppose a datum (i.e. some analytical information) is given which then determines an element $\xi \in KK^j(C(X), \mathbb{C})$.

QUESTION : What does it mean to solve the index problem for ξ?

ANSWER : It means to explicitly construct the K-cycle (M, E, φ) such that

$$\mu(M, E, \varphi) = \xi$$

where $\mu\colon K_j(X) \to KK^j(C(X), \mathbb{C})$ is the natural map of abelian groups.

Suppose that $j = 0$ and that a K-cycle (M, E, φ) with

$$\mu(M, E, \varphi) = \xi$$

has been constructed. It then follows that for any $\mathbb{C}$ vector bundle F on X

$$\mathrm{Index}(F \otimes \xi) = \epsilon_*(ch(F) \cap ch(M, E, \varphi))$$

$\epsilon\colon X \longrightarrow \cdot$ $\qquad$ ϵ is the map of X to a point.

$$ch(M, E, \varphi) := \varphi_*(\mathrm{ch}(E) \cup Td(M) \cap [M])$$

EQUIVALENTLY Suppose that $j = 0$ and that a K-cycle (M, E, φ) with

$$\mu(M, E, \varphi) = \xi$$

has been constructed. It then follows that

$$\mathcal{I}(\xi) = \varphi_*(ch(E) \cup Td(M) \cap [M])$$

where $\mathcal{I}(\xi)$ is (by definition) the unique element of $H_{even}(X; \mathbb{Q}) = \bigoplus_l H_{2l}(X; \mathbb{Q})$ such that for any $\mathbb{C}$ vector bundle F on X

$$\mathrm{Index}(F \otimes \xi) = \epsilon_*(ch(F) \cap \mathcal{I}(\xi))$$

REMARK. If the construction of the K-cycle (M, E, φ) with

$$\mu(M, E, \varphi) = \xi$$

has been done correctly, then it will work in the equivariant case and in the case of families of operators.

Example

General case of the Atiyah-Singer index theorem

Let X be a compact C^∞ manifold without boundary.
X is not required to be oriented.
X is not required to be even dimensional.
On X let

$$\delta : C^\infty(X, E_0) \longrightarrow C^\infty(X, E_1)$$

be an elliptic differential (or pseudo-differential) operator.
Then δ determines an element

$$[\delta] \in KK^0(C(X), \mathbb{C})$$

The K-cycle on X – which solves the index problem for δ – is

$$(S(TX \oplus 1_{\mathbb{R}}), E_\sigma, \pi).$$

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$S(TX \oplus 1_{\mathbb{R}})$ is the unit sphere bundle of $TX \oplus 1_{\mathbb{R}}$.

$\pi\colon S(TX \oplus 1_{\mathbb{R}}) \longrightarrow X$ is the projection of $S(TX \oplus 1_{\mathbb{R}})$ onto X.

$S(TX \oplus 1_{\mathbb{R}})$ is even-dimensional and is a Spin^c manifold.

E_σ is the $\mathbb{C}$ vector bundle on $S(TX \oplus 1_{\mathbb{R}})$ obtained by doing a clutching construction using the symbol σ of δ.

$$\mu((S(TX \oplus 1_{\mathbb{R}}), E_\sigma, \pi)) = [\delta]$$

$$\Downarrow$$

$$\text{Index}(\delta) = (ch(E_\sigma) \cup Td(S(TX \oplus 1_{\mathbb{R}})))[(S(TX \oplus 1_{\mathbb{R}})]$$

which is the general Atiyah-Singer formula.

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K-homology is the dual theory to K-theory. The Baum-Douglas isomorphism of Atiyah-BDF-Kasparov K-homology and K-cycle K-homology provides a framework within which the Atiyah-Singer index theorem can be extended to certain non-elliptic operators. This talk will consider a class of non-elliptic differential operators on compact contact manifolds. These operators have been studied by a number of mathematicians. Working within the Baum-Douglas framework the index problem will be solved for these operators.

This is joint work with Erik van Erp.

P. Baum and E. van Erp "K homology and index theory on contact manifolds", to appear in Acta. Math.

FACT:
If M is a closed odd-dimensional C^∞ manifold and D is any elliptic differential operator on M, then Index$(D) = 0$.

EXAMPLE:
$M = S^3 = \{(a_1, a_2, a_3, a_4) \in \mathbb{R}^4 \mid a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1\}$
x_1, x_2, x_3, x_4 are the usual co-ordinate functions on $\mathbb{R}^4$.

$$x_j(a_1, a_2, a_3, a_4) = a_j \qquad j = 1, 2, 3, 4$$

$\partial/\partial x_j$ usual vector fields on $\mathbb{R}^4$ $\qquad j = 1, 2, 3, 4$

On S^3 consider the (tangent) vector fields V_1, V_2, V_3

$$V_1 = -x_2\partial/\partial x_1 + x_1\partial/\partial x_2 - x_4\partial/\partial x_3 + x_3\partial/\partial x_4$$

$$V_2 = -x_3\partial/\partial x_1 + x_4\partial/\partial x_2 + x_1\partial/\partial x_3 - x_2\partial/\partial x_4$$

$$V_3 = -x_4\partial/\partial x_1 - x_3\partial/\partial x_2 + x_2\partial/\partial x_3 + x_1\partial/\partial x_4$$

Let r be a positive integer and let $\gamma\colon S^3 \longrightarrow M(r, \mathbb{C})$ be a C^∞ map.
$M(r, \mathbb{C})$:= {r×r matrices of complex numbers}.
Form the operator $P_\gamma := 2i\gamma(V_1 \otimes I_r) - V_2^2 \otimes I_r - V_3^2 \otimes I_r$.
$I_r := r \times r$ identity matrix.

$$P_\gamma\colon C^\infty(S^3, S^3 \times \mathbb{C}^r) \longrightarrow C^\infty(S^3, S^3 \times \mathbb{C}^r)$$

$$P_\gamma := 2i\gamma(V_1 \otimes I_r) - V_2^2 \otimes I_r - V_3^2 \otimes I_r$$

$I_r := r \times r$ identity matrix. $\quad i = \sqrt{-1}.$

$$P_\gamma \colon C^\infty(S^3, S^3 \times \mathbb{C}^r) \longrightarrow C^\infty(S^3, S^3 \times \mathbb{C}^r)$$

LEMMA.

Assume that for all $p \in S^3, \gamma(p)$ does not have any odd integers among its eigenvalues i.e.

$$\forall p \in S^3,\ \forall \lambda \in \{\ldots -3, -1, 1, 3, \ldots\} \Longrightarrow \lambda I_r - \gamma(p) \in GL(r, \mathbb{C})$$

then $\dim_{\mathbb{C}}$ (Kernel $P_\gamma) < \infty$ and $\dim_{\mathbb{C}}$ (Cokernel $P_\gamma) < \infty$.

With γ as in the above lemma, for each odd integer n , let

$$\gamma_n \colon S^3 \longrightarrow GL(r, \mathbb{C}) \quad \text{be}$$

$$p \longmapsto nI_r - \gamma(p)$$

By Bott periodicity if $r \geq 2$, then $\pi_3 GL(r, \mathbb{C}) = \mathbb{Z}$.
Hence for each odd integer n have the Bott number $\beta(\gamma_n)$.

PROPOSITION. With γ as above and $r \geq 2$

$$\text{Index}(P_\gamma) = \sum_{n \text{ odd}} \beta(\gamma_n)$$

Contact Manifolds

A contact manifold is an odd dimensional C^∞ manifold X
dimension$(X) = 2n + 1$
with a given C^∞ 1-form θ such that

$$\theta(d\theta)^n \text{ is non zero at every } x \in X - i.e.\ \theta(d\theta)^n \text{ is a volume form for } X.$$

Let X be a compact connected contact manifold without boundary ($\partial X = \emptyset$).
Set dimension$(X) = 2n+1$.
Let r be a positive integer and let $\gamma\colon X \longrightarrow M(r,\mathbb{C})$ be a C^∞ map.
$M(r,\mathbb{C})$:= {r×r matrices of complex numbers}.

Assume: For each $x \in X$,
$\{$Eigenvalues of $\gamma(x)\} \cap \{\ldots, -n-4, -n-2, -n, n, n+2, n+4, \ldots\} = \emptyset$
i.e. $\forall x \in X$,
$\lambda \in \{\ldots -n-4, -n-2, -n, n, n+2, n+4, \ldots\} \Longrightarrow \lambda I_r - \gamma(x) \in GL(r,\mathbb{C})$

$\gamma\colon X \longrightarrow M(r, \mathbb{C})$
Are assuming : $\forall x \in X$,
$\lambda \in \{\ldots -n-4, -n-2, -n, n, n+2, n+4, \ldots\} \Longrightarrow \lambda I_r - \gamma(x) \in GL(r, \mathbb{C})$

Associated to γ is a differential operator P_γ which is hypoelliptic and Fredholm.

$$P_\gamma\colon C^\infty(X, X \times \mathbb{C}^r) \longrightarrow C^\infty(X, X \times \mathbb{C}^r)$$

P_γ is constructed as follows.

The sub-Laplacian Δ_H

Let H be the null-space of θ.

$$H = \{v \in TX \mid \theta(v) = 0\}$$

H is a C^∞ sub vector bundle of TX with

$$\text{For all } x \in X, \dim_{\mathbb{R}}(H_x) = 2n$$

The sub-Laplacian

$$\Delta_H \colon C^\infty(X) \to C^\infty(X)$$

is locally $-W_1^2 - W_2^2 - \cdots - W_{2n}^2$
where $W_1, W_2, \ldots, W_{2n}$ is a locally defined C^∞ orthonormal frame for H.
These locally defined operators are then patched together using a C^∞ partition of unity to give the sub-Laplacian Δ_H.

The Reeb vector field

The Reeb vector field is the unique C^∞ vector field W on X with :

$$\theta(W) = 1 \text{ and } \forall v \in TX,\ d\theta(W, v) = 0$$

Let

$$\gamma\colon X \longrightarrow M(r, \mathbb{C})$$

be as above, $P_\gamma\colon C^\infty(X, X \times \mathbb{C}^r) \to C^\infty(X, X \times \mathbb{C}^r)$ is defined:

$$P_\gamma = i\gamma(W \otimes I_r) + (\Delta_H) \otimes I_r \qquad I_r = r \times r \text{ identity matrix} \quad i = \sqrt{-1}$$

P_γ is a differential operator (of order 2) and is hypoelliptic but not elliptic.

These operators P_γ have been studied by :

- R.Beals and P.Greiner *Calculus on Heisenberg Manifolds* Annals of Math. Studies 119 (1988).
- C.Epstein and R.Melrose.
- E. van Erp *The Atiyah-Singer index formula for subelliptic operators on contact manifolds. Part 1 and Part 2* Annals of Math. 171(2010).

A class of operators with somewhat similar analytic and topological properties has been studied by A. Connes and H. Moscovici.
M. Hilsum and G. Skandalis.

Set $T_\gamma = P_\gamma(I + P_\gamma^* P_\gamma)^{-1/2}$.

Let $\psi\colon C(X) \to \mathcal{L}(L^2(X) \otimes_{\mathbb{C}} \mathbb{C}^r)$ be

$$\psi(\alpha)(u_1, u_2, \ldots, u_r) = (\alpha u_1, \alpha u_2, \ldots, \alpha u_r)$$

where for $x \in X$ and $u \in L^2(X), (\alpha u)(x) = \alpha(x)u(x)$

$$\alpha \in C(X) \qquad\qquad u \in L^2(X)$$

Then

$$(L^2(X) \otimes_{\mathbb{C}} \mathbb{C}^r,\ \psi,\ \ L^2(X) \otimes_{\mathbb{C}} \mathbb{C}^r,\ \psi,\ \ T_\gamma) \in KK^0(C(X), \mathbb{C})$$

Denote this element of $KK^0(C(X), \mathbb{C})$ by $[P_\gamma]$.

$$[P_\gamma] \in KK^0(C(X), \mathbb{C})$$

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QUESTION.What is the K-cycle that solves the index problem for $[P_\gamma]$?

ANSWER. To construct this K-cycle, first recall that the given 1-form θ which makes X a contact manifold also makes X a stably almost complex manifold :

$$(\text{contact}) \Longrightarrow (\text{stably almost complex})$$

(contact) $\Longrightarrow$ (stably almost complex)

Let θ, H, and W be as above. Then :

- $TX = H \oplus 1_{\mathbb{R}}$ where $1_{\mathbb{R}}$ is the (trivial) $\mathbb{R}$ line bundle spanned by W.
- A morphism of C^∞ $\mathbb{R}$ vector bundles $J : H \to H$ can be chosen with $J^2 = -I$ and $\forall x \in X$ and $u, v \in H_x$

$$d\theta(Ju, Jv) = d\theta(u, v) \qquad d\theta(Ju, u) \geq 0$$

- J is unique up to homotopy.

(contact) $\Longrightarrow$ (stably almost complex)

$J \colon H \to H$ is unique up to homotopy.
Once J has been chosen :

H is a C^∞ $\mathbb{C}$ vector bundle on X.

$$\Downarrow$$

$TX \oplus 1_{\mathbb{R}} = H \oplus 1_{\mathbb{R}} \oplus 1_{\mathbb{R}} = H \oplus 1_{\mathbb{C}}$ is a C^∞ $\mathbb{C}$ vector bundle on X.

$$\Downarrow$$

$X \times S^1$ is an almost complex manifold.

REMARK. An almost complex manifold is a $\mathbb{C}^\infty$ manifold Ω with a given morphism $\zeta\colon T\Omega \to T\Omega$ of C^∞ $\mathbb{R}$ vector bundles on Ω such that

$$\zeta \circ \zeta = -I$$

The conjugate almost complex manifold is Ω with ζ replaced by $-\zeta$.

NOTATION. As above $X \times S^1$ is an almost complex manifold, $\overline{X \times S^1}$ denotes the conjugate almost complex manifold.

Since (almost complex)$\Longrightarrow$ (Spin^c), the disjoint union $X \times S^1 \sqcup \overline{X \times S^1}$ can be viewed as a Spin^c manifold.

Let

$$\pi\colon X \times S^1 \sqcup \overline{X \times S^1} \longrightarrow X$$

be the evident projection of $X \times S^1 \sqcup \overline{X \times S^1}$ onto X.
i.e.

$$\pi(x, \lambda) = x \qquad (x, \lambda) \in X \times S^1 \sqcup \overline{X \times S^1}$$

The solution K-cycle for $[P_\gamma]$ is $(X \times S^1 \sqcup \overline{X \times S^1},\ E_\gamma,\ \pi)$

$$E_\gamma = \Big(\bigoplus_{j=0}^{j=N} L(\gamma, n{+}2j)\otimes\pi^*\mathrm{Sym}^j(H)\Big)\bigsqcup\Big(\bigoplus_{j=0}^{j=N} L(\gamma, -n{-}2j)\otimes\pi^*\mathrm{Sym}^j(H^*)\Big)$$

1. “Sym^j” is “ j-th symmetric power”.
2. H^* is the dual vector bundle of H.
3. N is any positive integer such that : $n + 2N > \sup\{||\gamma(x)||, x \in X\}$.
4. $L(\gamma, n + 2j)$ is the $\mathbb{C}$ vector bundle on $X \times S^1$ obtained by doing a clutching construction using $(n + 2j)I_r - \gamma\colon X \to GL(r, \mathbb{C})$.
5. Similarly, $L(\gamma, -n - 2j)$ is obtained by doing a clutching construction using $(-n - 2j)I_r - \gamma\colon X \to GL(r, \mathbb{C})$.

Restriction of E_γ to $X \times S^1$

Let N be any positive integer such that :

$$n + 2N > \sup\{||\gamma(x)||, x \in X\}$$

The restriction of E_γ to $X \times S^1$ is:

$$E_\gamma \mid X \times S^1 = \bigoplus_{j=0}^{j=N} L(\gamma, n+2j) \otimes \pi^* \mathrm{Sym}^j(H)$$

Restriction of E_γ to $\overline{X \times S^1}$

The restriction of E_γ to $\overline{X \times S^1}$ is:

$$E_\gamma \mid \overline{X \times S^1} = \bigoplus_{j=0}^{j=N} L(\gamma, -n-2j) \otimes \pi^* \mathrm{Sym}^j(H^*)$$

Here H^* is the dual vector bundle of H:

$$H_x^* = \mathrm{Hom}_{\mathbb{C}}(H_x, \mathbb{C}) \qquad\qquad x \in X$$

$$E_\gamma = \Big(\bigoplus_{j=0}^{j=N} L(\gamma, n{+}2j)\otimes\pi^*\mathrm{Sym}^j(H)\Big)\bigsqcup\Big(\bigoplus_{j=0}^{j=N} L(\gamma, -n{-}2j)\otimes\pi^*\mathrm{Sym}^j(H^*)\Big)$$

Theorem (PB and Erik van Erp)

$$\mu(X \times S^1 \sqcup \overline{X \times S^1},\ E_\gamma\,,\ \pi) = [P_\gamma]$$