Quizz 1 (Notes, books, and calculators are not authorized)

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Find a parametric representation of the line in $\mathbb{R}^4$ that passes through $P = (4, -2, 3, 1)$ in the direction of $u = (2, 5, -7, 8)$.

The line in question consists of the points $X = (x_1, x_2, x_3, x_4)$ that are such that there is $t \in \mathbb{R}$ so that $(X - P) = tu$. This means that

$$X - P = tu = (2t, 5t, -7t, 8t).$$

The parametric representation of the line is

$$x_1 = 4 + 2t, \quad x_2 = -2 + 5t, \quad x_3 = 3 - 7t, \quad x_4 = 1 + 8t.$$

Question 2: Find a unit vector orthogonal to $u = (2, 1, 3)$ and $v = (4, -1, 2)$.

It is a known property of the cross product that the following vector $w := u \times v$ is orthogonal to u and v. By definition

$$w = (1 \times 2 - 3 \times (-1), 3 \times 4 - 2 \times 2, 2 \times (-1) - 1 \times 4) = (5, 8, -6). \tag{1}$$

By normalizing w we obtain the desired vector:

$$\frac{w}{\|w\|} = \frac{1}{\sqrt{125}}(5, 8, -6).$$

Question 3: Find an equation of the hyperplane in $\mathbb{R}^3$ that passes through $P = (1, -3, -4)$ and is parallel to the hyperplane H' determined by the equation $3x_1 - 6x_2 + 5x_3 = 2$.

The definition of H' implies that $n = (3, -6, 5)$ is normal to H'. Since H and H' are parallel, $n = (3, -6, 5)$ is also a normal vector to H. By definition H is the collection of points $X = (x_1, x_2, x_3)$ so that $(X - P)\cdot n = 0$. This means

$$\begin{aligned} 0 =& (X - P)\cdot(3, -6, 5) = (x_1 - 1, x_2 + 3, x_3 + 4)\cdot(3, -6, 5) = \\ &= 3x_1 - 3 - 6x_2 - 18 + 5x_3 + 20) = 3x_1 - 6x_2 + 5x_3 - 1. \end{aligned}$$

The equation of the hyperplane in question is

$$3x_1 - 6x_2 + 5x_3 = 1.$$

Question 4: Prove the Schwarz inequality: $|u\cdot v| \leq \|u\|\|v\|$ for all vectors u and v in $\mathbb{R}^n$.

Let u and v be two vectors in $\mathbb{R}^n$. The following holds for every $t \in \mathbb{R}$,

$$0 \leq \|u + tv\|^2 := (u + tv)\cdot(u + tv) = u\cdot u + tu\cdot v + tv\cdot u + t^2 v\cdot v = \|u\|^2 + 2t(u\cdot v) + t^2\|v\|^2.$$

This means that the quadratic polynomial $\|u\|^2 + 2t(u\cdot v) + t^2\|v\|^2$ in t cannot have two disctinct real roots. This implies that the discriminant $4(u\cdot v) - 4\|v\|^2\|u\|^2$ is negative, or equivalently

$$4(u\cdot v) \leq 4\|v\|^2\|u\|^2.$$

Dividing by 4 and taking the square root gives the Schwarz inequality

$$|(u\cdot v)| \leq \|v\|\|u\|.$$
