Higher analytic indices and symbolic index pairing

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Connes-Moscovici localized analytic indices

Let D be an elliptic operator on a closed manifold M. Then D is Fredholm.

$$\operatorname{ind} D = \dim \operatorname{Ker} D - \dim \operatorname{Coker} D \in \mathbb{Z}.$$

Atiyah-Singer index theorem computes $\operatorname{ind} D$ in topological terms.
Connes-Moscovici: for $[\phi] \in H^{\bullet}(M)$ define

$$\operatorname{ind}_{[\phi]} D \in \mathbb{C}.$$

This contains the Fredholm index: $\operatorname{ind}_1 D = \operatorname{ind} D$.
Connes-Moscovici also prove the corresponding index theorem.

Alexander-Spanier complex

$$C^k(M) = C^\infty(M^{k+1})$$

$\delta\colon C^k(M) \to C^{k+1}(M)$ is defined by

$$\delta\phi(x_0, x_1, \ldots, x_{k+1}) = \sum (-1)^i \phi(x_0, \ldots, \widehat{x_i}, \ldots, x_{k+1}).$$

The complex $(C^\bullet(M), \delta)$ is acyclic in positive degrees.
The locally zero subcomplex:

$$C_0^k(M) = \{\phi \in C^k(M) \mid \phi = 0 \text{ near diagonal of } M^{k+1}\}.$$

$$C_{AS}^\bullet(M) = (C^\bullet(M)/C_0^\bullet(M), \delta)$$

$$H^\bullet\left(C_{AS}^\bullet(M)\right) = H^\bullet(M).$$

Variants of the definition

- Consider only antisymmetric cochains:

$$\phi(x_{\lambda(0)}, x_{\lambda(1)}, \ldots, x_{\lambda(k)}) = \mathrm{sgn}(\lambda)\phi(x_0, x_1, \ldots, x_k)$$

  for all $\lambda \in S_{k+1}$.

- Consider (locally finite sums of) decomposable cochains.
  For $f_i \in C^\infty(M)$
  $(f_0 \otimes f_1 \otimes \ldots f_k)(x_0, x_1, \ldots, x_k) = f_0(x_0)f_1(x_1)\ldots f_k(x_k)$.
  Finite sum of such cochains form a subcomplex of $(C^\bullet, \delta)$. Note:

$$\delta(f_0 \otimes f_1 \otimes \ldots f_k) = \sum (-1)^i f_0 \otimes f_1 \ldots \otimes f_{i-1} \otimes 1 \otimes f_i \otimes \ldots f_k$$

From now on $C^\bullet(M)$ denotes the subcomplex of the antisymmetric decomposable cochains. $C_0^\bullet(M)$ its locally zero subcomplex, $C_{AS}^\bullet(M) = C^\bullet(M)/C_0^\bullet(M)$.

Cochains on smoothing opertors

For $\phi = f_0 \otimes f_1 \otimes \dots f_k$ and $A_i \in \Psi^{-\infty}(M)$ define

$$\tau_\phi(A_0, A_1, \dots, A_k) = \operatorname{Tr} A_0 f_0 A_1 f_1 \dots A_k f_k,$$

extend by linearity to antisymmetric decomposable ϕ. Viewed as a cyclic cochain on $\Psi^{-\infty}(M)$, τ_ϕ satisfies

$$b\tau_\phi = \tau_{\delta\phi},\ \ B\tau_\phi = 0$$

and hence $\phi \mapsto \tau_\phi$ defines a morphism of complexes $C^\bullet(M) \to CC^\bullet(\Psi^{-\infty}(M))$.

Index and K-theory

Let Q be a parametrix for D, and define

$$S_0 = 1 - QD, \text{ and } S_1 = 1 - DQ \in \Psi^{-\infty}(M)$$

Then

$$U_D = \begin{bmatrix} S_0 & -(1+S_0)Q \\ D & S_1 \end{bmatrix} \in M_2(\Psi(M))$$

is an invertible operator with the inverse given by

$$U_D^{-1} = \begin{bmatrix} S_0 & (1+S_0)Q \\ -D & S_1 \end{bmatrix}.$$

Using U_D form the idempotent

$$P_D = U_D \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U_D^{-1} = \begin{bmatrix} S_0^2 & S_0(1+S_0)Q \\ DS_0 & 1 - S_1^2 \end{bmatrix} \tag{1}$$

The index class $P_D - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in K_0(\Psi^{-\infty}(M))$ is well defined.

Definition of localized analytic indices

Notice that $P_D - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is the matrix of smoothing operators, and $\operatorname{ind} D = \operatorname{Tr}\left(P_D - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$.

To define higher indices:

- Choose a representative $\phi \in C^\bullet(M)$ of $[\phi] \in H^k(M)$. $\delta\phi = 0$ in a neighborhood U of the diagonal in M^{k+2}.
- Chose a parametrix Q for D with Schwartz kernel supported in a sufficiently small (depending on U) neighborhood of the diagonal in M^2.
- Form P_D and define

$$\operatorname{ind}_{[\phi]} D = c_k \tau_\phi(P_D, P_D, \ldots, P_D)$$

The result is well-defined.

Definition of $\widehat{\mathrm{Tr}}_\phi$

Let $R \in \Psi^1(M)$, $R \geq 0$, invertible. For $A \in \Psi(M)$ form

$$\zeta(s) = \mathrm{Tr}\, AR^{-s}.$$

$\zeta(s)$ has a meromorphic continuation and near 0

$$\zeta(s) = \frac{1}{s}\,\mathrm{Res}\, A + \widehat{\mathrm{Tr}}\, A + O(s).$$

$\mathrm{Res}\, A$ is the noncommutative residue, does not depend on R. $\widehat{\mathrm{Tr}}\, A$ depends on R, for A trace class $\widehat{\mathrm{Tr}}(A) = \mathrm{Tr}\, A$.
For $\phi = f_0 \otimes f_1 \otimes \dots f_k$ and $A_i \in \Psi(M)$ define

$$\widehat{\mathrm{Tr}}_\phi(A_0, A_1, \dots, A_k) = \widehat{\mathrm{Tr}}(A_0 f_0 A_1 f_1 \dots A_k f_k)$$

$\widehat{\mathrm{Tr}}_\phi$ is a (discontinuous) cyclic cochain on $\Psi(M)$.

Definition of χ_ϕ

In the notations above set:

$$\chi_\phi = (b + B)\,\widehat{\mathrm{Tr}}_\phi - \widehat{\mathrm{Tr}}_{\delta\phi}$$

Lemma

- *If* $\phi \in C_0^\bullet(M)$ *then* $\chi_\phi = 0$
- *If* $A_i \in \Psi^{-\infty}(M)$ *for some* i*, then* $\chi_\phi(A_0, A_1, \ldots, A_{k+1}) = 0$.

Corollary

The map $\phi \mapsto \chi_\phi$ *defines a morphism of complexes*

$$C^\bullet_{AS}(M) \to CC^{\bullet+1}(\mathcal{S}(M)),$$

where $\mathcal{S}(M) = \Psi(M)/\Psi^{-\infty}(M)$ *is the algebra of the complete symbols.*

Index pairing

To simplify notations, assume that D is an elliptic system (i.e. operates on a sections of a trivial bundle). $\sigma(D) \in \mathcal{S}(M)$ is an invertible element, therefore defines an element in $K_1(\mathcal{S}(M))$, and $\mathrm{Ch}(\sigma(D)) \in HC_\bullet(\mathcal{S}(M))$.

Theorem

Let $[\phi] \in H^\bullet(M)$.

$$\mathrm{ind}_{[\phi]} D = \langle \chi_\phi, \mathrm{Ch}(\sigma(D)) \rangle$$

Proof of the Theorem

Recall that

$$P_D = U_D \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U_D^{-1}$$

From the (proof of) invariance of cyclic homology under inner automorphisms,

$$\operatorname{Ch} P_D = \operatorname{Ch} 1 + (b + B) T_D \in CC_\bullet(\Psi(M))$$

for some explicit chain $T_D \in CC_\bullet(\Psi(M))$. Complete symbol map induces a morphism

$$\sigma \colon CC_\bullet(\Psi(M)) \to CC_\bullet(\mathcal{S}(M)),$$

and $(b + B)\sigma(T_D) = 0$.

Lemma

$$[\sigma(T_D)] = \operatorname{Ch}(\sigma(D)) \in HC_\bullet(\mathcal{S}(M)).$$

Proof of the Theorem (contd.)

Proof of the Theorem.

Assume $k > 0$.

$$\langle \chi_\phi, \mathrm{Ch}(\sigma(D)) \rangle = \widehat{\mathrm{Tr}}_\phi((b+B)T_D) - \widehat{\mathrm{Tr}}_{\delta\phi}(T_D) =$$
$$\widehat{\mathrm{Tr}}_\phi(\mathrm{Ch}\, P_D - \mathrm{Ch}\, 1) - \widehat{\mathrm{Tr}}_{\delta\phi}(T_D) = c_k \tau_\phi(P_D, P_D, \ldots, P_D) - \widehat{\mathrm{Tr}}_{\delta\phi}(T_D)$$

If the parametrix Q used in construction of P_D is supported in sufficiently small neighborhood of the diagonal, depending on ϕ, then $\widehat{\mathrm{Tr}}_{\delta\phi}(T_D) = 0$. □

Corollary

For any choice of the parametrix Q

$$\mathrm{ind}_{[\phi]} D = c_k \tau_\phi(P_D, P_D, \ldots, P_D) - \mathrm{Tr}_{\delta\phi}(T_D)$$

The odd case

Assume that D is self-adjoint elliptic. One can define its index class in $K_1(\Psi^{-\infty}(M))$ as follows. Let $\rho \in C^\infty(\mathbb{R})$ be such that $\rho(\lambda) = 1$ for large λ, $\rho(-\lambda) = -\rho(\lambda)$.
Then

$$U = e^{\pi i \rho(D)}$$

is invertible operator, representing $\operatorname{ind} D$. U is homotopic to 1 in $\Psi^0(M)$ via the path $t \mapsto e^{\pi i t \rho(D)}$. Therefore

$$\operatorname{Ch} U = (b + B)T$$

for certain chain $T \in CC_\bullet(\Psi^0(M))$.
Let ϕ be Alexander-Spanier cocycle. Define then

$$\operatorname{ind}_{[\phi]} D = \operatorname{Tr}_\phi \operatorname{Ch} U - \operatorname{Tr}_{\delta\phi} T$$

The odd case (contd.)

Let P be the spectral projection on the positive spectrum for D.

$$\operatorname{ind}_{[\phi]} D = \langle \chi_\phi, \operatorname{Ch}(\sigma(P)) \rangle$$

Toeplitz picture: for $f \in C^\infty(M)$ set

$$T_f = P\ f\ P$$

Then if $\phi = \sum_i f_0^i \otimes f_1^i \ldots \otimes f_k^i$

$$\operatorname{ind}_\phi D = c_k \operatorname{Tr} \left(\sum_i \sum_{\lambda \in S_{k+1}} \operatorname{sgn} \lambda \, T_{f_{\lambda(0)}} T_{f_{\lambda(1)}} \ldots T_{f_{\lambda(k)}} \right)$$

Cyclic cohomology of symbols

$\mathcal{S}^0(M) = \Psi^0(M)/\Psi^{-\infty}(M)$.

Theorem (Wodzicki)

The principal symbol map

$$\sigma_{pr}: \mathcal{S}^0(M) \to C^\infty(S^*M)$$

induces an isomorphism in periodic cyclic cohomology.

Theorem (Wodzicki, Brylinski-Getzler)

There exists an isomorphism

$$HC^\bullet(\mathcal{S}(M)) \cong H^\bullet(S^*M \times S^1)$$

Zakharevich map

There exists a canonical isomorphism

$$Z\colon H^\bullet(S^*M \times S^1) \to HC^\bullet(\mathcal{S}(M))$$

(induced by a morphism of complexes). It has the following properties:

- If $\phi = f_0 \otimes f_1 \ldots \otimes f_k$ is Alexander-Spanier cochain on M then

  $$Z(\pi^*\phi)(A_0, A_1, \ldots A_k) = \operatorname{Res} A_0 f_0 A_1 f_1 \ldots A_k f_k =: \operatorname{Res}_\phi(A_0, \ldots, A_k),$$

  where $\pi\colon S^*M \times S^1 \to M$ is the natural projection.
- If ϕ is a cocycle

  $$[\chi_\phi] = Z(\pi^*\phi \cup [S^1])$$

The diagram

Theorem

The following diagram commutes:

$$\begin{array}{ccc} HC^{\bullet}_{per}(\mathcal{S}(M)) & \xrightarrow{i^*} & HC^{\bullet}_{per}(\mathcal{S}^0(M)) \\ Z\uparrow & & \uparrow\sigma^*_{pr} \\ H^{\bullet}(S^*M \times S^1) & \xrightarrow{\mathcal{I}} & H_{\bullet}(S^*M) \end{array}$$

Here $i\colon \mathcal{S}^0(M) \hookrightarrow \mathcal{S}(M)$ *is the inclusion map and* $\mathcal{I}$ *is given by*

$$\mathcal{I}(\alpha) = \left(\int_{S^1} \alpha\right) \cap [Td]$$

where $[Td] \in H_{\bullet}(S^*M)$ is the dual of the pullback of the Todd class.

Index calculations

Applying this result to $[S^1] \in H^\bullet(S^*M \times S^1)$ we obtain

$$\chi_1 = Z([S^1]) = \sigma_{pr}^*([Td]) \in HC^\bullet_{per}(\mathcal{S}^0(M)).$$

Hence

$$\operatorname{ind} D = \langle \chi_1, \mathrm{Ch}(\sigma(D)) \rangle = \langle \mathrm{Ch}(\sigma_{pr}(D)), [Td] \rangle$$

More generally, in the even case

$$\operatorname{ind}_{[\phi]} D = \langle \chi_\phi, \mathrm{Ch}(\sigma(D)) \rangle = \langle \pi^*\phi \cup \mathrm{Ch}(\sigma_{pr}(D)), [Td] \rangle.$$

The odd case

For $\phi = \sum_i f_0^i \otimes f_1^i \ldots \otimes f_k^i$ let $DR(\phi) = \sum_i f_0^i df_1^i \ldots df_k^i$. DR is a morphism from Alexander-Spanier complex to de Rham complex.

Corollary

Assume $\phi = \sum_i f_0^i \otimes f_1^i \ldots \otimes f_k^i$, $\delta\phi$ *locally* 0*. Then*

$$\operatorname{Tr}\left(\sum_i \sum_{\lambda \in S_{k+1}} \operatorname{sgn}\lambda\, T_{f^i_{\lambda(0)}} T_{f^i_{\lambda(1)}} \ldots T_{f^i_{\lambda(k)}}\right) = \int_M DR(\phi) \wedge \ Td \wedge \pi_* \left(\operatorname{Ch}(\sigma_{pr}(D))\right)$$

If $k = \dim M$ then $\delta\phi$ locally 0 for any ϕ (after modifying the notion of locally 0). This extends result of Helton-Howe .

Vanishing results

Apply the diagram to $1 \in H^\bullet(S^*M \times S^1)$. We obtain

$$[\mathrm{Res}] = Z(1) = 0 \in HC^\bullet_{per}(\mathcal{S}^0(M)).$$

Corollary (Wodzicki)

Let $P \in \Psi^0$ be an idempotent. Then

$$\mathrm{Res}\, P = 0$$

More generally, let $\phi = \sum_i f_0^i \otimes f_1^i \ldots \otimes f_k^i$ be an Alexander-Spanier cocycle. Then

$$[\mathrm{Res}_\phi] = Z(\pi^*\phi) = 0 \in HC^\bullet_{per}(\mathcal{S}^0(M)).$$

Vanishing results (contd.)

Corollary

- *Let k be even and let $P \in \mathcal{S}^0$ be an idempotent, $\phi = \sum_i f_0^i \otimes f_1^i \ldots \otimes f_k^i$ – an Alexander-Spanier cocycle. Then*

$$\operatorname{Res}_\phi \operatorname{Ch}(P) = const. \cdot \sum_i \operatorname{Res} Pf_0^i Pf_1^i \ldots Pf_k^i = 0$$

- *Let k be odd, $U \in \mathcal{S}$ be a complete symbol of an elliptic operator (i.e. invertible). Then*

$$\operatorname{Res}_\phi \operatorname{Ch}(U) = const. \cdot \sum_i \operatorname{Res} Uf_0^i U^{-1} f_1^i \ldots U^{-1} f_k^i = 0$$

Here $const.$ is a nonzero constant.

Manifolds with boundary

$(M, \partial M)$ – compact connected manifold with connected boundary.
Relative Alexander-Spanier complex:

$$C^{\bullet}_{AS}(M, \partial M) = \{(\phi, \psi) \mid \phi \in C^{\bullet}_{AS}(M), \psi \in C^{\bullet -1}_{AS}(\partial M)\}$$

$$\delta(\phi, \psi) = (\delta\phi, \phi|_{\partial M} - \delta\psi).$$

Algebra of operators: ${}^{\mathrm{b}}\Psi$ operators of R. Melrose. *b*-trace:

$${}^{\mathrm{b}}\mathrm{Tr} \colon {}^{\mathrm{b}}\Psi(M) \to \mathbb{C}.$$

It is not a trace, even on ${}^{\mathrm{b}}\Psi^{-\infty}$. But coincides with the operator trace on operators in ${}^{\mathrm{b}}\Psi^{-\infty}$ vanishing near the boundary.

Suspended operators

Let Y be a compact manifold without boundary. $\Psi \mathrm{sus}(Y)$ is the space of order pseudodifferential operators on $Y \times \mathbb{R}$ which are translation invariant and have Schwartz kernel rapidly decreasing off-diagonal.
Let t be the variable on $\mathbb{R}$, and τ the dual variable Given such an operator A we can construct a family of operators $\widehat{A}(\tau)$ on Y:

$$\widehat{A}(\tau) = e^{-it\tau} \circ A \circ e^{it\tau}$$

The linear functional $\overline{\mathrm{Tr}}$: for operators of sufficiently small order

$$\overline{\mathrm{Tr}}(A) = \int_{-\infty}^{\infty} \widehat{A}(\tau) d\tau.$$

then extended.

Indicial family: homomorphism

$$I\colon\ {}^{\mathrm{b}}\Psi(M) \to \Psi_{\mathrm{sus}}(\partial M).$$

If $A \in {}^{\mathrm{b}}\Psi^{-\infty}(M)$, $B \in {}^{\mathrm{b}}\Psi(M)$

$${}^{\mathrm{b}}\mathrm{Tr}\,[A, B] = \overline{\mathrm{Tr}}\, I(A)[t, I(B)].$$

Define for $\phi = f_0 \otimes f_1 \ldots \otimes f_k$ – Alexander-Spanier cochain on M – then

$${}^{\mathrm{b}}\mathrm{Tr}_{\phi}(A_0, A_1, \ldots A_k) = {}^{\mathrm{b}}\mathrm{Tr}\, A_0 f_0 A_1 f_1 \ldots A_k f_k,\ A_i \in {}^{\mathrm{b}}\Psi$$

Similarly for $\psi = g_0 \otimes g_1 \ldots \otimes g_k$ – Alexander-Spanier cochain on ∂M

$$\overline{\mathrm{Tr}}_{\psi}(B_0, B_1, \ldots B_k, B_{k+1}) = \overline{\mathrm{Tr}} B_0 g_0 B_1 g_1 \ldots B_k g_k [B_{k+1}, t],\ B_i \in \Psi_{\mathrm{sus}}(\partial M)$$

For $\phi \in C^{\bullet}_{AS}(M)$

$$\rho_\phi = (b+B)\,{}^{\mathrm{b}}\mathrm{Tr}_\phi - {}^{\mathrm{b}}\mathrm{Tr}_{\delta\phi} - \overline{\mathrm{Tr}}_{\phi|_{\partial M}}.$$

For $\psi \in C^{\bullet}_{AS}(\partial M)$

$$\lambda_\psi = (b+B)\overline{\mathrm{Tr}}_\psi - \overline{\mathrm{Tr}}_{\delta\psi}$$

and

$$\chi_{(\phi,\psi)} = \rho_\phi + I^*\lambda_\psi.$$

Then $(\phi,\psi) \mapsto \chi_{(\phi,\psi)}$ defines a morphism of complexes

$$C^{\bullet}_{AS}(M,\partial M) \to CC^{\bullet+1}({}^{\mathrm{b}}\Psi(M)/{}^{\mathrm{b}}\Psi^{-\infty}(M)).$$

Index Theorem

Let D be an elliptic first order b operator, $\sigma(D) \in {}^{\mathrm{b}}\Psi(M)/{}^{\mathrm{b}}\Psi^{-\infty}(M)$ its complete symbol. Assume that near the boundary $D = \frac{d}{dx} + D^{\partial}$, x-normal coordinate, D^{∂} – first-order invertible self-adjoint operator on the boundary.

Theorem

For (ϕ, ψ) – relative Alexander-Spanier cocycle

$$\langle \chi_{(\phi,\psi)}, \mathrm{Ch}(\sigma(D)) \rangle = \int_M DR(\phi) \wedge Td(M) \wedge \pi_* \,\mathrm{Ch}(\sigma_{pr}(D)) - \int_{\partial M} DR(\psi) \wedge Td(\partial M) \wedge \pi_* \,\mathrm{Ch}(\sigma_{pr}(D^{\partial})).$$