Hopf Cyclic Cohomology and Characteristic Classes

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Origins and motivation

- The version of cyclic cohomology adapted to Hopf algebras emerged from the joint work with of A. CONNES on the local index formula for the hypoelliptic signature operator on "spaces of leaves" of foliations. For foliations of codimension n, we have found a Hopf algebra $\mathcal{H}_n$ which plays the role of a "quantum" structure group for their "spaces of leaves". While the characteristic classes of foliations are described in terms of Gelfand-Fuks Lie algebra cohomology, the appropriate tool for above problem turned out to be Hopf cyclic cohomology.
- The two cohomologies were shown to be canonically isomorphic by an explicit, but quite intricate, quasi-isomorphism.
- The transplantation of the characteristic classes in the Hopf cyclic cohomological framework broadened the scope of their applicability.
- Thus, the issue of finding more explicit constructions of the Hopf cyclic characteristic classes becomes relevant.

Chern-Weil construction of characteristic classes

- $\nabla =$ torsion-free connection on M^n, with connection form on $\omega = (\omega^i_j) \in \Omega^1(FM) \otimes \mathfrak{gl}_n$ and curvature $\Omega = d\omega + \omega \wedge \omega$, $\Omega = (\Omega^i_j) \in \Omega^2(FM) \otimes \mathfrak{gl}_n$;
- $\forall\, P \in I(\mathfrak{gl}_n) = S(\mathfrak{gl}_n^*)^{\mathrm{GL}_n}$, the form $P(\Omega)$ is closed and basic, i.e. $P(\Omega) \in \Omega^*(M)$, hence $[P(\Omega)] \in H^*(M, \mathbb{R})$.
- In particular,

$$\det\left(\mathrm{Id} - \frac{t}{2\pi i} A\right) = \sum_{k=1}^{n} t^k c_k(A), \qquad A \in \mathfrak{gl}_n(\mathbb{C})$$

give the classical Chern forms $c_k(\Omega) \in \Omega^{2k}(M)$, and the Pontryagin classes $[p_k(\Omega)] = [c_{2k}(\Omega)] \in H^{4k}(M, \mathbb{R})$.

Local Index Formula in Noncommutative Geometry

Theorem (A. Connes & HM, 1995)

Assume $(\mathcal{A}, \mathfrak{H}, D)$ = *spectral triple, such that* $\exists$ *residue*

$$⨍ T := \operatorname{Res}_{s=0} \operatorname{Tr}(T|D|^{-2s}), \quad T \in \Psi\{\mathcal{A}, [D, \mathcal{A}], |D|^{-z}; z \in \mathbb{C}\}.$$

1. $[(\varphi_n)_{n=1,3,\ldots}]$ *is a cocycle in the* (b, B)*-bicomplex of* $\mathcal{A}$,

$$\varphi_n(a^0, \ldots, a^n) = \sum_{\mathbf{k}} c_{n,\mathbf{k}} ⨍ a^0[D, a^1]^{(k_1)} \ldots [D, a^n]^{(k_n)} |D|^{-n-2|k|}$$

$$\nabla(T) = [D^2, a], \quad T^{(k)} = \nabla^k(T), \qquad |k| = k^1 + \ldots + k^n,$$

$$c_{n,\mathbf{k}} = \frac{(-1)^{|k|}\, \Gamma\left(|k| + \frac{n}{2}\right)}{k_1! \ldots k_n!(k_1 + 1) \ldots (k_1 + \cdots + k_n + n)}.$$

2. $[(\varphi_n)_{n=1,3,\ldots}] = ch^*(\mathfrak{H}, F) \in HC^*(\mathcal{A})$.

Relation with Atiyah-Singer Local Index Formula

1. The zeta functions associated to the Dirac spectral triple $(C^\infty(M^m),\ L^2(S), \not{D})$ are meromorphic with simple poles.
2. $$\int\!\!\!\!\!- P \;=\; \frac{1}{(2\pi)^n}\int_{S^*M}\sigma_{-n}(P). \qquad \forall\, P \in \Psi DO(M^n);$$
   (Guillemin-Wodzicki residue)
3. $$\int\!\!\!\!\!- f^0[\not{D}, f^1]^{(k_1)}\dots[\not{D}, f^n]^{(k_n)}\ |\not{D}|^{-(n+2|k|)} = 0\ , \quad \text{if} \quad |k| > 0\ ;$$
4. $$\int\!\!\!\!\!- f^0[\not{D}, f^1]\dots[\not{D}, f^n]|\not{D}|^{-n} =$$
   $$= c_n\int_M \det\left(\frac{\nabla^2/4\pi i}{\sinh \nabla^2/4\pi i}\right)^{\frac{1}{2}} \wedge f^0\ df^1\wedge\dots\wedge df^n;$$
5. under the isomorphism (Connes, HKR)
   $$HP^*(C^\infty(M^m)) \cong H_*^{\mathrm{dR}}(M, \mathbb{C}),$$
   $$ch^*(\mathfrak{H}, \not{D}) \equiv [(\varphi_n)\,] \cong\ [\hat{A}(R)]\ \equiv\ Ch_*\,(\not{D}).$$

Local index formula for codimension 1

- $M = S^1$, $\Gamma \subset \mathbf{G} = \mathrm{Diff}(S^1)^\delta$,
  $\mathfrak{H} = L^2(FS^1 \times S^1, y^{-1}\, dx\, dy\, d\alpha) \otimes \mathbb{C}^2$

$$Q = 2y\partial_y\, \partial_\alpha \gamma_1 + \frac{1}{i}\, y\partial_x\, \gamma_2 + \left((y\partial_y)^2 - \partial_\alpha^2 - \frac{1}{4} \right) \gamma_3 \,,$$

where $\gamma_1, \gamma_2, \gamma_3 \in M_2(\mathbb{C})$ are the Pauli matrices.

- $$\frac{1}{\sqrt{2i}} \varphi_1(a^0, a^1) = \Gamma\left(\frac{1}{2}\right) ⨍ a^0[Q, a^1](Q^2)^{-1/2}$$

$$- \frac{1}{2}\Gamma\left(\frac{3}{2}\right) ⨍ a^0 \nabla[Q, a^1](Q^2)^{-3/2} + \frac{1}{2 \cdot 3}\Gamma\left(\frac{5}{2}\right) ⨍ a^0 \nabla^2[Q, a^1](Q^2)^{-5/2}$$

$$- \frac{1}{2 \cdot 3 \cdot 4}\Gamma\left(\frac{7}{2}\right) ⨍ a^0 \nabla^3[Q, a^1](Q^2)^{-7/2} \quad \equiv 0$$

Local index cocycle

- $$\frac{1}{\sqrt{2i}}\varphi_3(a^0,a^1,a^2,a^3) = \frac{1}{3i}\Gamma\left(\frac{3}{2}\right) ⨍ a^0[Q,a^1][Q,a^2][Q,a^3](Q^2)^{-3/2}$$
$$-\frac{1}{2\cdot 3\cdot 4}\Gamma\left(\frac{5}{2}\right) ⨍ a^0\nabla[Q,a^1][Q,a^2][Q,a^3](Q^2)^{-5/2}$$
$$-\frac{1}{3\cdot 4}\Gamma\left(\frac{5}{2}\right) ⨍ a^0[Q,a^1]\nabla[Q,a^2][Q,a^3](Q^2)^{-5/2}$$
$$-\frac{1}{2\cdot 4}\Gamma\left(\frac{5}{2}\right) ⨍ a^0[Q,a^1][Q,a^2]\nabla[Q,a^3](Q^2)^{-5/2}$$
$=$ transverse fundamental cocycle $+$ boundary.

While the computation is purely symbolical, it requires the symbol σ'_{-4}, hence about 10^3 terms!

Deciphering the transverse local index formula

1. The Chern character cocycle of the transverse signature operator D is a sum of cochains of the form

$$\sum \int\!\!\!\!\!- a^0[Q, a^1]^{(k_1)} \dots [Q, a^q]^{(k_q)} \, |Q|^{-q-|2k|}, \quad a^i = f^i \, U^*_{\varphi^i} \in \mathcal{A}.$$

2. By the residue formula, these can be expressed as

$$\sum \tau \left(a^0 \, h^1(a^1) \cdots h^q(a^q) \right),$$

where $\tau \left(f \, U^*_\varphi \right) = \int f \, vol_P$, if $\varphi = \mathrm{Id}$ and 0 otherwise,

h^i are transverse differential operators, e.g. $X(f \, U^*_\varphi) = X(f) \, U^*_\varphi$.

3. The operators h generate a Hopf algebra $\mathcal{H}$, which acts on $\mathcal{A}$ and gives rise to a characteristic map

$$\chi^q_\Gamma : C^q(\mathcal{H}) \equiv \mathcal{H} \otimes \ldots \otimes \mathcal{H} \longrightarrow C^q(\mathcal{A}) \, .$$

4. The Chern character of D is in the range of the induced map in cyclic cohomology $\chi^*_\Gamma : HC^*(\mathcal{H}) \longrightarrow HC^*(\mathcal{A}_\Gamma)_{(1)}$.

Hopf algebra $\mathcal{H}_1$

- Action on $\mathcal{A} = C^\infty(F^+S^1) \rtimes \mathbf{G}$, $\qquad \tilde{\varphi}(x,y) = (\varphi(x), \varphi'(x)\, y)$,

$$Y(fU_\varphi^*) = y\,\frac{\partial f}{\partial y}\, U_\varphi^*, \quad X(fU_\varphi^*) = y\,\frac{\partial f}{\partial x}\, U_\varphi^*$$

- $Y(f \circ \tilde{\varphi}) \; = \; Y(f) \circ \tilde{\varphi} \quad \Longrightarrow \quad Y(ab) = Y(a)\, b \, + \, a\, Y(b)$.

- $X(f \circ \tilde{\varphi}) = (X(f) \circ \tilde{\varphi}) + y \dfrac{\varphi''(x)}{\varphi'(x)} (Y(f) \circ \tilde{\varphi})(x,y)$

$$\Longrightarrow X(ab) = X(a)\, b \, + \, a\, X(b) + \delta_1(a)\, Y(b).$$

- $\delta_n := [X, \delta_{n-1}] \Longrightarrow \delta_n(fU_\varphi^*) = y^n \frac{d^n}{dx^n}\left(\log \frac{d\varphi}{dx}\right) fU_\varphi^*$.

- As algebra= generated by $\{X,\, Y,\, \delta_1,\, \delta_2, \ldots\}$ subject to relations:

$$[Y, X] = X, \quad [Y, \delta_k] = k\delta_k, \quad [X, \delta_k] = \delta_{k+1}, \quad [\delta_k, \delta_\ell] = 0.$$

- As coalgebra: $\Delta(Y) = Y \otimes 1 + 1 \otimes Y$,

$$\begin{aligned}
\Delta(X) &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \\
\Delta(\delta_1) &= \delta_1 \otimes 1 + 1 \otimes \delta_1, \\
\Delta(\delta_2) &= \delta_2 \otimes 1 + \delta_1 \otimes \delta_1 + 1 \otimes \delta_2, \\
\Delta(\delta_3) &= \delta_3 \otimes 1 + \delta_2 \otimes \delta_1 + 3\delta_1 \otimes \delta_2 + \delta_1^2 \otimes \delta_1 + 1 \otimes \delta_3, \text{ etc.}
\end{aligned}$$

- Counit: $\epsilon(X) = \epsilon(Y) = \epsilon(\delta_k) = 0, \quad \epsilon(1) = 1.$
- Antipode: $S(1) = 1\,, \quad S(X) = -X + \delta_1 Y,\ S(Y) = -Y,$
  $S(\delta_1) = -\delta_1, \quad S(\delta_2) = \delta_1^2 - \delta_1, \quad \ldots$
- Character $\delta \in \mathcal{H}_1^*$: $\delta(Y) = 1, \quad \delta(X) = 0, \quad \delta(\delta_n) = 0.$
- Twisted antipode $S_\delta(h) = \delta(h_{(1)})\, S(h_{(2)})$ is involutive:

$$S_\delta^2 = \mathrm{Id}\,.$$

Hopf cyclic cohomology

- Cylic structure: $\mathcal{H}^{\natural}_{(\delta,1)} = \{C^n(\mathcal{H};\mathbb{C}_\delta) = \mathcal{H}^{\otimes n}\}_{n\geq 0}$

$$\partial_0(h^1 \otimes \ldots \otimes h^{n-1}) = 1 \otimes h^1 \otimes \ldots \otimes h^{n-1},$$
$$\partial_j(h^1 \otimes \ldots \otimes h^{n-1}) = h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^{n-1}$$
$$\partial_n(h^1 \otimes \ldots \otimes h^{n-1}) = h^1 \otimes \ldots \otimes h^{n-1} \otimes 1$$
$$\sigma_i(h^1 \otimes \ldots \otimes h^{n+1}) = h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^{n+1}$$
$$\tau_n(h^1 \otimes \ldots \otimes h^n) = S_\delta(h^1) \cdot h^2 \otimes \ldots \otimes h^n \otimes 1$$

Bicomplex : $$b = \sum_{i=0}^{n+1}(-1)^i \partial_i, \quad B = \left(\sum_{i=0}^{n}(-1)^{ni}\tau_n^i\right)\sigma_{n-1}\tau_n.$$

Example ($HP^*(\mathcal{H}_1;\mathbb{C}_\delta)$)

$TF = X \otimes Y - Y \otimes X - \delta_1\, Y \otimes Y$ (fundamental class)
$GV = \delta_1$ (Godbillon-Vey class)

Relative Hopf cyclic cohomology

$\mathcal{K}$ = Hopf subalgebra of $\mathcal{H}$, $\mathcal{C} := \mathcal{H} \otimes_{\mathcal{K}} \mathbb{C}$, with $\mathcal{K}$ acting on $\mathcal{H}$ by right multiplication and on $\mathbb{C}$ by the counit.
As left $\mathcal{H}$-module $\mathcal{C} \simeq \mathcal{H}/\mathcal{H}\mathcal{K}^+$, where $\mathcal{K}^+ = \mathrm{Ker}\,\varepsilon|\mathcal{K}$, via

$$h + \mathcal{H}\mathcal{K}^+ \mapsto \dot{h} = h \otimes_{\mathcal{K}} 1 \in \mathcal{H} \otimes_{\mathcal{K}} \mathbb{C}.$$

Cylic structure: $\{C^n(\mathcal{H}, \mathcal{K}; \mathbb{C}_\delta^\sigma) = \mathcal{C}^{\otimes n}\}_{n \geq 0}$

$$\partial_0(c^1 \otimes \ldots \otimes c^{n-1}) = \dot{1} \otimes c^1 \otimes \ldots \otimes \ldots \otimes c^{n-1},$$
$$\partial_i(c^1 \otimes \ldots \otimes c^{n-1}) = c^1 \otimes \ldots \otimes c^i_{(1)} \otimes c^i_{(2)} \otimes \ldots \otimes c^{n-1},$$
$$\partial_n(c^1 \otimes \ldots \otimes c^{n-1}) = c^1 \otimes \ldots \otimes c^{n-1} \otimes \dot{1};$$
$$\sigma_i(c^1 \otimes \ldots \otimes c^{n+1}) = c^1 \otimes \ldots \otimes \varepsilon(c^{i+1}) \otimes \ldots \otimes c^{n+1},$$
$$\tau_n(\dot{h}^1 \otimes c^2 \otimes \ldots \otimes c^n) = S_\delta(h^1) \cdot (c^2 \otimes \ldots \otimes c^n \otimes \dot{1}).$$

Transverse Index Theorem

Theorem (A. Connes & HM, 1998)

There are canonical constructions for the following entities:

1. *a Hopf algebra* $\mathcal{H}_n$ *associated to* $\mathrm{Diff}(\mathbb{R}^n)$*, with modular character* δ*, and modular pair* $(\delta, 1)$*;*
2. *an isomorphism* κ_* *between the Gelfand-Fuks cohomology* $H^*_{\mathrm{GF}}(\mathfrak{a}_n, O_n)$ *and* $HP^*(\mathcal{H}_n, O_n; \mathbb{C}_\delta)$*;*
3. *an action of* $\mathcal{H}_n$ *on* $\mathcal{A}_{\mathbf{G}}(F\mathbb{R}^n)$*, and a characteristic map*

$$\chi(h^1 \otimes \ldots \otimes h^n)(a^0, \ldots, a^n) = \tau(a^0\, h^1(a^1) \ldots h^n(a^n))$$

$$\chi_* : HP^*(\mathcal{H}_n, O_n; \mathbb{C}_\delta) \to HP^*_{(1)}(\mathcal{A}_{\mathbf{G}}(P\mathbb{R}^n)), \quad P\mathbb{R}^n := F\mathbb{R}^n / O_n$$

4. $ch_*(D)_{(1)} = \chi_*(\mathcal{L})$, $\quad \mathcal{L} \in HP^*(\mathcal{H}_n, O_n; \mathbb{C}_\delta) \cong H^*_{\mathrm{GF}}(\mathfrak{a}_n, O_n)$.

Diff-invariant de Rham cohomology

- de Rham complex of invariant forms on ∞-jet bundle
  $\{\Omega^*(P^\infty M)^{\mathbf{G}}, d\}$ for M^n = smooth manifold, $\mathbf{G} = \mathrm{Diff}(M)^\delta$

$$F^k M = k - \text{jets at } 0 \quad \text{of local diffeos} \quad \rho : \mathbb{R}^n \to M$$

$$M \leftarrow F^1 M \leftarrow F^2 M \leftarrow \cdots \qquad F^\infty M := \varprojlim F^k M$$

$$P^k M = F^k M / O_n \qquad M \leftarrow P^1 M \leftarrow P^2 M \leftarrow \cdots \quad P^\infty M := \varprojlim P^k M$$

$$\mathbf{G} - \text{action} : \qquad \phi \in \mathbf{G},\ \rho \in F^k M \Longrightarrow \phi \cdot j_0^\infty(\rho) := j_0^\infty(\phi \circ \rho)$$

- Gelfand-Fuks Lie algebra cohomology complex of formal vector fields
  $\{C^*(\mathfrak{a}_n), d\}$ $\qquad \mathfrak{a}_n = \{v = j_0^\infty\left(\frac{d}{dt}\,|_{t=0}\,\rho_t\right),\ t \mapsto \rho_t : \mathbb{R}^n \to \mathbb{R}^n\}$

$$\tilde{v}\,|_{j_0^\infty(\phi)} = j_0^\infty\left(\frac{d}{dt}\,|_{t=0}\,(\phi \circ \rho_t)\right); \quad \tilde{\omega}(\tilde{v}_1, \ldots, \tilde{v}_m) = \omega(v_1, \ldots, v_m)$$

- DGA-isomorphism $\qquad \omega \in C^\bullet(\mathfrak{a}_n, O_n) \mapsto \tilde{\omega} \in \Omega^\bullet(P^\infty M)^{\mathbf{G}}$

Diff-equivariant de Rham cohomology

- Simplicial manifold $\bar{\triangle}_{\mathbf{G}} M = \{\bar{\triangle}_{\mathbf{G}} M[p] = \mathbf{G}^{p+1} \times M\}_{p \geq 0}$,

$$\bar{\partial}_i(\rho_0, \ldots, \rho_p, x) = (\rho_0, \ldots, \check{\rho}_i, \ldots, \rho_p), \quad 0 \leq i \leq p,$$
$$\bar{\sigma}_i(\rho_0, \ldots, \rho_p, x) = (\rho_0, \ldots, \rho_i, \rho_i, \ldots, \rho_p, x), \quad 0 \leq i \leq p.$$

- Geometric realization $|\bar{\triangle}_{\mathbf{G}} M| = \coprod \Delta^p \times \bar{\triangle}_{\mathbf{G}} M[p] / \sim$
- Dupont complex of (covariant) compatible forms $\{\Omega^* |\bar{\triangle}_{\mathbf{G}} M|, d\}$

$$\omega = \{\omega_p\}_{p \geq 0}, \qquad \omega_p \in \Omega^\bullet(\Delta^p \times \triangle_{\mathbf{G}} M[p])$$
$$(\mu_\bullet \times \mathrm{Id})^* \omega_q = (\mathrm{Id} \times \mu^\bullet)^* \omega_p \in \Omega^\bullet (\Delta^p \times \triangle_{\mathbf{G}} M[q])$$
$$\omega(\rho_0 \rho, \ldots, \rho_p \rho) = \rho^* \omega(\rho_0, \ldots, \rho_p), \qquad \forall\, \rho,\ \rho_i \in \mathbf{G}$$

- Bott complex $\{\bar{C}^\bullet(\mathbf{G}, \Omega^\bullet(M)), \bar{\delta}, d\}$ $\bar{c}(\rho_0, \ldots, \rho_p) \in \Omega^q(M)$

$$\bar{c}(\rho_0 \rho, \ldots, \rho_p \rho) = \rho^* \bar{c}(\rho_0, \ldots, \rho_p), \quad \forall\, \rho, \rho_i \in \mathbf{G}$$
$$\bar{\delta}\bar{c}(\rho_0, \ldots, \rho_p) = \sum_{i=0}^{p} (-1)^i \bar{c}(\rho_0, \ldots, \check{\rho}_i, \ldots, \rho_p).$$

Differentiable cohomology (à la Haefliger)

- Differentiable cochain $\omega \in \bar{C}^p_{\mathrm{d}}(\mathbf{G}, \Omega^q(M))$ if locally,

$$\omega(\rho_0, \ldots, \rho_p, x) = \sum f_I\left(x, j^k_x(\rho_0), \ldots, j^k_x(\rho_p)\right) dx^I.$$

- Differentiable compatible form $\omega = \{\omega_p\}_{p\geq 0} \in \Omega^\bullet_{\mathrm{d}}(|\bar{\triangle}_{\mathbf{G}} M|)$ if

$$\omega_p(\mathbf{t}; \rho_0, \ldots, \rho_p, x) = \sum f_{I,J}\left(\mathbf{t}; x, j^k_x(\rho_0), \ldots, j^k_x(\rho_p)\right) dt^I \wedge dx^J.$$

Theorem (Differentiable analogue of Dupont's Theorem)

The chain map $\oint_{\Delta^\bullet} : \Omega^\bullet_{\mathrm{d}}(|\bar{\triangle}_{\mathbf{G}} M|) \to \bar{C}^\bullet_{\mathrm{d}}(\mathbf{G}, \Omega^*(M))$ *induces an isomorphism* $H^\bullet_{\mathrm{d}}(|\bar{\triangle}_{\mathbf{G}} M|, \mathbb{R}) \cong H^\bullet_{\mathrm{d},\mathbf{G}}(M, \mathbb{R})$.

Explicit van Est-Haefliger quasi-isomorphism

$\nabla =$ torsion free connection $\Longrightarrow$ cross-section to $\pi_1 : F^\infty M \to FM$

$$\sigma_\nabla(u) = j_0^\infty(\exp_x^\nabla \circ u); \qquad \sigma_{\nabla^\phi} = \phi^{-1} \circ \sigma_\nabla \circ \phi, \quad \phi \in \mathbf{G}\,,$$
$$\sigma_p(\mathbf{t}; \rho_0, \ldots, \rho_p, u) \; = \; \sigma_{\sum_0^p \; t_i \, \nabla^{\rho_i}}(u);$$
$$\hat{\sigma} = \{\sigma_p\}_{p \geq 0} : |\bar{\triangle}_{\mathbf{G}} FM| \to F^\infty M.$$

Theorem

The chain map $\mathcal{C}_\nabla(\omega) = \hat{\sigma}^*(\tilde{\omega}) \in \Omega_{\mathrm{d}}^\bullet(|\bar{\triangle}_{\mathbf{G}} FM|)$ *induces quasi-iso of DG-algebras* $\mathcal{C}_\nabla : C^\bullet(\mathfrak{a}_n, \mathrm{O}_n) \to \Omega_{\mathrm{d}}^\bullet(|\bar{\triangle}_{\mathbf{G}}(PM, \mathrm{O}_n)|)$.

Corollary

The composition $\mathcal{D}_\nabla = \oint_{\Delta^\bullet} \circ \, \mathcal{C}_\nabla : C^\bullet(\mathfrak{a}_n, \mathrm{O}_n) \to \bar{C}_{\mathrm{d}}^{\mathrm{tot}\,\bullet}(\mathbf{G}, \Omega^*(PM))$

is quasi-isomorphism.

Hopf cyclic analogue of van Est isomorphism

This involves Connes' map $\Phi : \bar{C}^\bullet(\mathbf{G}, \Omega^p(F\mathbb{R}^n)) \to CC^\bullet(C_c^\infty(F\mathbb{R}^n) \ltimes \mathbf{G})$.
If $\lambda \in \mathrm{Im}(\mathcal{D}_\nabla)$ where now $M = \mathbb{R}^n$, and $\nabla =$ flat connection, then $\Phi(\lambda)$ is of the form

$$\Phi(\lambda)(a^0, \ldots, a^\ell) = \sum_\alpha \tau(a^0 h_\alpha^1(a^1) \ldots h_\alpha^q(a^q)), \qquad h_\alpha^i \in \mathcal{H}_n,$$

with the tensor $\sum_\alpha h_\alpha^1 \otimes \ldots \otimes h_\alpha^q \in \mathcal{H}_n^{\otimes q}$ uniquely determined by λ.
One obtains a chain map $\Upsilon : \mathrm{Im}(\mathcal{D}_\nabla) \to CC^{\mathrm{tot}\bullet}(\mathcal{H}_n, \mathbb{C}_\delta)$,

$$\Upsilon(\lambda) = \sum_\alpha h_\alpha^1 \otimes \ldots \otimes h_\alpha^q \in \mathcal{H}_n^{\otimes q}.$$

Theorem (AC & HM 1998, 2001)

The composition $\Upsilon \circ \mathcal{D}_\nabla : C^\bullet(\mathfrak{a}_n, \mathrm{O}_n) \to CC^{\mathrm{tot}*}(\mathcal{H}_n, O_n; \mathbb{C}_\delta)$ *is quasi-isomorphism.*

Characteristic cocycles by simplicial Chern-Weil

- The universal connection $\vartheta = (\vartheta^i_j)$, where $\vartheta^i_j(\sum_{k=1}^n \xi^k \partial_k) = \partial_j \xi^i$, and curvature forms $R = (R^i_j)$, where $R^i_j = d\vartheta^i_j + \vartheta^i_k \wedge \vartheta^k_j$ in $C^\bullet(\mathfrak{a}_n)$ generate a DG-subalgebra $CW^\bullet(\mathfrak{a}_n)$.
- By Gelfand-Fuks Thm. $CW^\bullet(\mathfrak{a}_n) \hookrightarrow C^\bullet(\mathfrak{a}_n)$ is quasi-isomorphism.
- $CW^\bullet(\mathfrak{a}_n) \cong \hat{W}(\mathfrak{gl}_n) = W(\mathfrak{gl}_n)/\mathcal{I}_{2n}$, where $W(\mathfrak{gl}_n) = \wedge^\bullet \mathfrak{gl}_n^* \otimes S(\mathfrak{gl}_n^*)$ and $\mathcal{I}_{2n}$ = ideal generated by elements of $S(\mathfrak{gl}_n^*)$ of deg $> 2n$.

Lemma

For any torsion-free connection, $\sigma^*_\nabla(\widetilde{\vartheta}^i_j) = \omega^i_j$ *and* $\sigma^*_\nabla(\widetilde{R}^i_j) = \Omega^i_j$.

- Simplicial connection and curvature:

$$\hat{\omega}_p(\mathbf{t}; \rho_0, \ldots, \rho_p) := \sum_{i=0}^p t_i \rho_i^*(\omega) \quad \in \Omega^1_{\mathrm{d}}(|\bar{\triangle}_{\mathbf{G}} FM|)$$
$$\hat{\Omega} := d\hat{\omega} + \hat{\omega} \wedge \hat{\omega} \quad \in \Omega^2_{\mathrm{d}}(|\bar{\triangle}_{\mathbf{G}} FM|),$$
$$\hat{\Omega}_p(\mathbf{t}; \rho_0, \ldots, \rho_p) = \sum_{i=0}^p dt_i \wedge \rho_i^*(\omega) + \sum_{i=0}^p t_i \big(\rho_i^*(\Omega) - \rho_i^*(\omega) \wedge \rho_i^*(\omega)\big) + \sum_{i,j=0}^p t_i t_j \, \rho_i^*(\omega) \wedge \rho_j^*(\omega).$$

Vey basis in differentiable Dupont algebra

- The forms $\hat{\omega}^i_j$ and $\hat{\Omega}^i_j$ generate a DG-subalgebra $CW^\bullet_{\mathrm{d}}(|\bar{\triangle}_{\mathbf{G}}FM|)$, and $\mathcal{C}_\nabla$ gives isomorphism between $CW^\bullet(\mathfrak{a}_n) \equiv \hat{W}(\mathfrak{gl}_n)$ and $CW^\bullet_{\mathrm{d}}(|\bar{\triangle}_{\mathbf{G}}FM|)$.
- $c_k(\hat{\Omega}) = d(Tc_k(\hat{\omega}))$, with $Tc_k(\hat{\omega}) = k\int_0^1 c_k\left(\hat{\omega}, \hat{\Omega}_t, \ldots, \hat{\Omega}_t\right) dt$, $\hat{\Omega}_t = t\hat{\Omega} + (t^2 - t)\hat{\omega}\wedge\hat{\omega}$.
- By restriction to O_n-basic elements, $\mathcal{C}_\nabla$ induces an isomorphism of $CW^\bullet(\mathfrak{a}_n, \mathrm{O}_n)) \equiv \hat{W}(\mathfrak{gl}_n, \mathrm{O}_n)$ onto $CW^\bullet_{\mathrm{d}}(|\bar{\triangle}_{\mathbf{G}}PM|)$.
- $c_{2k}(\hat{\Omega})$ remain but $c_{2k-1}(\hat{\Omega}) = d(Tc_{2k-1}(\hat{\omega}))$, with $Tc_{2k-1}(\hat{\omega}) = (2k-1)\int_0^1 c_{2k-1}\left(\mathfrak{s}(\hat{\omega}), \hat{\Omega}_t, \ldots, \hat{\Omega}_t\right) dt$, $\hat{\Omega}_t = t\mathfrak{s}(\hat{\Omega}) + \mathfrak{o}(\hat{\Omega}) + (t^2 - 1)\mathfrak{s}(\hat{\omega})\wedge\mathfrak{s}(\hat{\omega})$.
- $\{Tc_I(\hat{\omega})\wedge c_J(\hat{\Omega})\}_{(I,J)\in\mathcal{V}_n}$, resp. $\{Tc_I(\hat{\omega})\wedge c_J(\hat{\Omega})\}_{(I,J)\in\mathcal{V}O_n}$, represents a basis in cohomology.

Vey basis in differentiable Bott complex

Corollary

The cocycles obtained by their integration along fibers,

$$C_{I,J}(\nabla) := \oint_{\Delta^\bullet} Tc_I(\hat{\omega}) \wedge c_J(\hat{\Omega}), \quad (I, J) \in \mathcal{V}O_n,$$

form a complete set of representatives for a basis of $H^\bullet_{\mathrm{d},\mathbf{G}}(PM, \mathbb{R})$.

For example, $C_{\emptyset,\{2k\}}(\nabla) = C_{2k}(\hat{\Omega}) = \oint_{\Delta^\bullet} c_{2k}(\hat{\Omega})$ is the cocycle

$C_{2k}(\hat{\Omega}) = \{C_{2k}^{(p)}(\hat{\Omega})\}_{p \geq 0}$ with components

$$C_{2k}^{(p)}(\hat{\Omega})(\phi_0, \ldots, \phi_p) = (-1)^p \oint_{\Delta^p} c_{2k}\left(\hat{\Omega}(\mathbf{t}; \phi_0, \ldots, \phi_p)\right) =$$

$$= (-1)^p \sum_{1 \leq i_1 < \ldots < i_{2k} \leq n} \sum_{\mu \in S_{2k}} (-1)^\mu \int_{\Delta^p} \hat{\Omega}^{i_1}_{\mu(i_1)} \wedge \cdots \wedge \hat{\Omega}^{i_q}_{\mu(i_{2k})}(\mathbf{t}; \phi_0, \ldots, \phi_p).$$

Generators of the Hopf algebra $\mathcal{H}_n$

$\mathcal{H}_n$ acts on $\mathcal{A} = C_c^\infty(F\mathbb{R}^n) \ltimes \mathbf{G}$ where $\mathbf{G} = \mathrm{Diff}(\mathbb{R}^n)^\delta$ by

$$X_k = y_k^\mu \frac{\partial}{\partial x^\mu}, \qquad Y_i^j = y_i^\mu \frac{\partial}{\partial y_j^\mu};$$

$$Y_i^j(f\, U_\phi^*) = Y_i^j(f)\, U_\phi^*, \qquad U_\phi^*\, Y_i^j\, U_\phi = Y_i^j;$$

$$X_k(f\, U_\phi^*) = X_k(f)\, U_\phi^*, \qquad U_\phi^*\, X_k\, U_\phi = X_k - \gamma_{jk}^i(\phi)\, Y_i^j,$$

$$\delta_{jk}^i(f\, U_\phi^*) = \gamma_{jk}^i(\phi)\, f\, U_\phi^*,$$

$$\delta_{jk\,\ell_1\ldots\ell_r}^i := [X_{\ell_r}, \ldots [X_{\ell_1}, \delta_{jk}^i] \ldots];$$

$$\delta_{jk\,\ell_1\ldots\ell_r}^i(f\, U_\phi^*) := \gamma_{jk\,\ell_1\ldots\ell_r}^i(\phi)\, f\, U_\phi^*,$$

where $\qquad \gamma_{jk\,\ell_1\ldots\ell_r}^i(\phi) := X_{\ell_r} \cdots X_{\ell_1}\big(\gamma_{jk}^i(\phi)\big)$

Basic algebra generators : $\quad \{X_k,\ Y_i^j,\ \delta_{jk}^i\}$

Transfer to Hopf cyclic cohomology

Let $M = \mathbb{R}^n$, $\mathbf{G} = \mathrm{Diff}(\mathbb{R}^n)$ and $\nabla =$ trivial connection. Identify $F\mathbb{R}^n \cong \mathbb{R}^n \ltimes \mathrm{GL}_n(\mathbb{R})$. One has

$$\omega^i_j \;:=\; (\mathbf{y}^{-1})^i_\mu \, d\mathbf{y}^\mu_j \;=\; \left(\mathbf{y}^{-1}\, d\mathbf{y}\right)^i_j , \qquad i,j = 1,\dots,n$$

$$\phi^*(\omega^i_j) \;=\; \omega^i_j \,+\, \gamma^i_{jk}(\phi)\,\theta^k, \qquad \phi \in \mathbf{G}$$

$$\gamma^i_{j\,k}(\phi)(x,\mathbf{y}) \;=\; \left(\mathbf{y}^{-1}\cdot \phi'(x)^{-1}\cdot \partial_\mu \phi'(x)\cdot \mathbf{y}\right)^i_j \,\mathbf{y}^\mu_k$$

$$\hat{\omega}(\mathbf{t};\phi_0,\dots,\phi_p)^i_j = \sum_{r=0}^{p} t_r \phi_r^*(\omega^i_j) = \sum_{r=0}^{p} t_r \gamma^i_{jk}(\phi_r)\,\theta^k$$

$$\hat{\Omega}(\mathbf{t};\phi_0,\dots,\phi_p) = \sum_{r=0}^{p} dt_r \wedge \phi_r^*(\omega) - \sum_{i=0}^{p} t_r\, \phi_r^*(\omega)\wedge\phi_r^*(\omega)$$

$$+ \sum_{r,s=0}^{p} t_r t_s\, \phi_r^*(\omega)\wedge\phi_s^*(\omega).$$

Vey basis in Hopf cyclic complex

Theorem

(1) *The cocycles* $\Upsilon(C_{I,J}(\nabla))$, *with* $(I, J) \in \mathcal{V}_n$, *form a complete set of representatives for the periodic Hopf cyclic cohomology* $HP^{\bullet}(\mathcal{H}_n; \mathbb{C}_{\delta})$.
(2) *The cocycles* $\Upsilon(C_{I,J}(\nabla))$, *with* $(I, J) \in \mathcal{V}O_n$, *form a complete set of representatives for* $HP^{\bullet}(\mathcal{H}_n, \mathrm{O}_n; \mathbb{C}_{\delta})$.
(3) *Every cohomology class in* $HP^{\bullet}(\mathcal{H}_n; \mathbb{C}_{\delta})$ *and* $HP^{\bullet}(\mathcal{H}_n, \mathrm{O}_n; \mathbb{C}_{\delta})$, *can be represented by cocycles manufactured out of the algebra generators* $\{X_k, Y^i_j, \delta^i_{jk}\}$ *of* $\mathcal{H}_n$.

Proof.

In addition to the δ^i_{jk} from the previous slide, the map Φ, when applied to monomials $a = f\, U^*_{\phi} \in \mathcal{A}$, brings in the operators X_k and Y^i_j, via
$df \;= \sum_{k=1}^n X_k(f)\theta^k \,+\, \sum_{i,j=1}^n Y^i_j(f)\,\omega^j_i$.

□

Example: Chern cocycle

The top component of $\{C_q^{(p)}(\hat{\Omega})\}_{p\geq 0}$, $q = 2k$, in the simplified cyclic model

$$\kappa_q^{(q)}(\hat{\Omega}) = \sum_{1\leq i_s, j_t \leq n} \sum_{\mu \in S_q} (-1)^\mu \theta^{j_0^1} \wedge \cdots \wedge \theta^{j_q} \otimes \eta^{i_1}_{\mu(i_1), j_0^1} \wedge \cdots \wedge \eta^{i_q}_{\mu(i_q), j_q}.$$

The lower components $\kappa_q^{(p)}(\hat{\Omega})$ are given by similar expressions, with coefficients of the form

$$\int_{\Delta^p} t_1^{k_1} \cdots t_p^{k_p} dt_1 \wedge \cdots \wedge dt_p = \frac{1}{(k_1+1)\cdots(k_1+\ldots+k_p+p)};$$

note the resemblance with the coefficients appearing in the local index formula.