Noncommutative Geometry and Conformal Geometry (joint work with Hang Wang)

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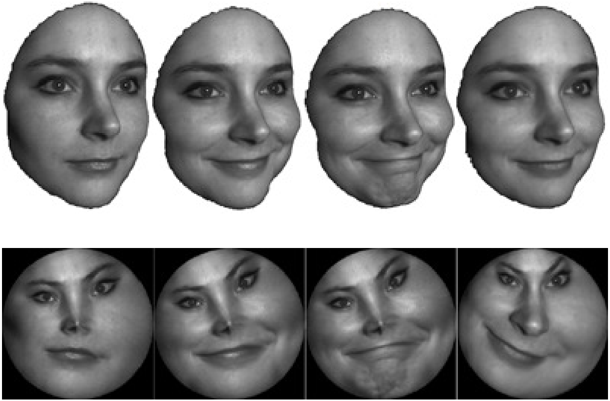
References

Main References

- RP+HW: *Noncommutative geometry, conformal geometry, and the local equivariant index theorem.* arXiv:1210.2032. Superceded by the 3 papers below.
- RP+HW: *Index map, σ-connections, and Connes-Chern character in the setting of twisted spectral triples.* arXiv:1310.6131.
- RP+HW: *Noncommutative geometry and conformal geometry. I. Local index formula and comformal invariants.* To be posted on arXiv soon.
- RP+HW: *Noncommutative geometry and conformal geometry. II. Connes-Chern character and the local equivariant index theorem.* To be posted on arXiv soon.
- RP+HW: *Noncommutative geometry and conformal geometry. III. Poincaré duality and Vafa-Witten inequality.* arXiv:1310.6138.

Conformal Geometry



Group Actions on Manifolds

Fact

If G is an arbitrary group of diffeomorphisms of a manifold M, then M/G need not be Hausdorff (unless Γ acts freely and properly).

Solution Provided by NCG

Trade the space M/G for the crossed product algebra,

$$C_c^\infty(M) \rtimes \Gamma = \left\{ \sum f_\phi u_\phi;\ f_\phi \in C_c^\infty(M) \right\},$$

$$u_\phi^* = u_\phi^{-1} = u_{\phi^{-1}}, \qquad u_\phi f = (f \circ \phi^{-1}) u_\phi.$$

Proposition (Green)

If G acts freely and properly, then $C_c^\infty(M/G)$ is Morita equivalent to $C_c^\infty(M) \rtimes G$.

The Noncommutative Torus

Example

Given $\theta \in \mathbb{R}$, let $\mathbb{Z}$ act on S^1 by

$$k.z := e^{2ik\pi\theta} z \qquad \forall z \in S^1 \ \forall k \in \mathbb{Z}.$$

Remark

If $\theta \notin \mathbb{Q}$, then the orbits of the action are dense in S^1.

The crossed-product algebra $\mathcal{A}_\theta := C^\infty(S^1) \rtimes_\theta \mathbb{Z}$ is generated by two operators U and V such that

$$U^* = U^{-1}, \qquad V^* = V^{-1}, \qquad VU = e^{2i\pi\theta} UV.$$

Remark

The algebra $\mathcal{A}_\theta$ is called the *noncommutative torus*.

Classical	**NCG**
Manifold M	Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$
Vector Bundle E over M	Projective Module $\mathcal{E}$ over $\mathcal{A}$ $\mathcal{E} = e\mathcal{A}^q, \;\; e \in M_q(\mathcal{A}), \; e^2 = e$
$\operatorname{ind} \not{D}_{\nabla^E}$	$\operatorname{ind} D_{\nabla^{\mathcal{E}}}$
de Rham Homology/Cohomology	Cyclic Cohomology/Homology
Atiyah-Singer Index Formula $\operatorname{ind} \not{D}_{\nabla^E} = \int \hat{A}(R^M) \wedge \mathrm{Ch}(F^E)$	Connes-Chern Character $\mathrm{Ch}(D)$ $\operatorname{ind} D_{\nabla^{\mathcal{E}}} = \langle \mathrm{Ch}(D), \mathrm{Ch}(\mathcal{E}) \rangle$
Characteristic Classes	Cyclic Cohomology for Hopf Algebras

Spectral Triples

Definition (Connes-Moscovici)

A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ consists of

1. A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.
2. An involutive algebra $\mathcal{A}$ represented in $\mathcal{H}$.
3. A selfadjoint unbounded operator D on $\mathcal{H}$ such that
    1. D maps $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$.
    2. $(D \pm i)^{-1}$ is compact.
    3. $[D, a]$ is bounded for all $a \in \mathcal{A}$.

Dirac Spectral Triple

Example

- (M^n, g) compact Riemannian spin manifold (n even) with spinor bundle $\displaystyle{\not}S = {\not}S^+ \oplus {\not}S^-$.
- $\not{D}_g : C^\infty(M, {\not}S) \to C^\infty(M, {\not}S)$ is the Dirac operator of (M, g).
- $C^\infty(M)$ acts by multiplication on $L^2_g(M, {\not}S)$.

Then $\left(C^\infty(M), L^2\left(M, {\not}S\right), \not{D}_g\right)$ is a spectral triple.

Remark

We also get spectral triples by taking

- $\mathcal{H} = L^2(M, \Lambda^\bullet T^*M)$ and $D = d + d^*$.
- $\mathcal{H} = L^2(M, \Lambda^{0,\bullet} T^*_{\mathbb{C}}M)$ and $D = \overline{\partial} + \overline{\partial}^*$ (when M is a complex manifold).

Diffeomorphism-Invariant Geometry

Setup

- M smooth manifold.
- $G = \mathrm{Diff}(M)$ full group diffeomorphism group of M.

Fact

The only G-invariant geometric structure of M is its manifold structure.

Theorem (Connes-Moscovici '95)

There is a spectral triple $\left(C_c^\infty(P) \rtimes G, L^2(P, \Lambda^\bullet T^*P), D\right)$, where

- $P = \left\{g_{ij}dx^i \otimes dx^j; (g_{ij}) > 0\right\}$ *is the metric bundle of M.*
- D *is a "mixed-degree" signature operator, so that*

$$D|D| = d_H + d_H^* + d_V d_V^* - d_V^* d_V.$$

Conformal Change of Metric

Example

- (M^n, g) compact Riemannian spin manifold (n even) with spinor bundle $\$ = \$^+ \oplus \$^-$.
- $\not{D}_g : C^\infty(M, \$) \to C^\infty(M, \$)$ is the Dirac operator of (M, g).
- $C^\infty(M)$ acts by multiplication on $L^2_g(M, \$)$.

Consider a conformal change of metric,

$$\hat{g} = k^{-2} g, \qquad k \in C^\infty(M), \; k > 0.$$

Then the Dirac spectral triple $\left(C^\infty(M), L^2_{\hat{g}}(M, \$), \not{D}_{\hat{g}}\right)$ is unitarily equivalent to $\left(C^\infty(M), L^2_g(M, \$), \sqrt{k}\not{D}_g\sqrt{k}\right)$ (i.e., the spectral triples are intertwined by a unitary operator).

Twisted Spectral Triples

Definition (Connes-Moscovici)

A twisted *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ $(\mathcal{A}, \mathcal{H}, D)_\sigma$ consists of

1. A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.
2. An involutive algebra $\mathcal{A}$ represented in $\mathcal{H}$ together with an automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ such that $\sigma(a)^* = \sigma^{-1}(a^*)$ for all $a \in \mathcal{A}$.
3. A selfadjoint unbounded operator D on $\mathcal{H}$ such that
    1. D maps $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$.
    2. $(D \pm i)^{-1}$ is compact.
    3. [D,a]$[D, a]_\sigma := Da - \sigma(a)D$ is bounded for all $a \in \mathcal{A}$.

Proposition (Connes-Moscovici)

Consider the following:

- *An ordinary spectral triple* $(\mathcal{A}, \mathcal{H}, D)$.
- *A positive element* $k \in \mathcal{A}$ *with associated inner automorphism* $\sigma(a) = k^2 a k^{-2}$, $a \in \mathcal{A}$.

Then $(\mathcal{A}, \mathcal{H}, kDk)_\sigma$ *is a twisted spectral triple.*

Pseudo-Inner Twistings

Proposition (RP+HW)

Consider the following data:

- *An ordinary spectral triple* $(\mathcal{A}, \mathcal{H}, D)$.
- *A positive even operator* $\omega = \begin{pmatrix} \omega^+ & 0 \\ 0 & \omega^- \end{pmatrix} \in \mathcal{L}(\mathcal{H})$ *so that there are inner automorphisms* $\sigma^\pm(a) = k^\pm a (k^\pm)^{-1}$ *associated positive elements* $k^\pm \in \mathcal{A}$ *in such way that*

$$k^+k^- = k^-k^+ \quad \text{and} \quad \omega^\pm a = \sigma^\pm(a)\omega^\pm \quad \forall a \in \mathcal{A}.$$

Set $k = k^+k^-$ *and* $\sigma(a) = kak^{-1}$. *Then* $(\mathcal{A}, \mathcal{H}, \omega D \omega)_\sigma$ *is a twisted spectral triple.*

Examples

1. Connes-Tretkoff's twisted spectral triples over NC tori associated to conformal weights.
2. Invertible doubles of conformal perturbations of spectral

Further Examples

- Conformal Dirac spectral triple (Connes-Moscovici).
- Twisted spectral triples over NC tori associated to conformal weights (Connes-Tretkoff).
- Twistings of ordinary spectral triples by scaling automorphisms (Moscovici).
- Twisted spectral triples associated to quantum statistical systems (e.g., Connes-Bost systems, supersymmetric Riemann gas) (Greenfield-Marcolli-Teh '13).

Connections over a Spectral Triple

Setup

- $(\mathcal{A}, \mathcal{H}, D)$ is an ordinary spectral triple.
- $\mathcal{E}$ finitely generated projective (right) module over $\mathcal{A}$.
- Space of differential 1-forms:

$$\Omega^1_D(\mathcal{A}) := \mathrm{Span}\{adb;\ a, b \in \mathcal{A}\} \subset \mathcal{L}(\mathcal{H}),$$

where $db := [D, b]$.

Definition

A *connection* on a $\mathcal{E}$ is a linear map $\nabla^{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$ such that

$$\nabla^{\mathcal{E}}(\xi a) = \xi \otimes da + \left(\nabla^{\mathcal{E}}\xi\right) a \qquad \forall a \in \mathcal{A}\ \forall \xi \in \mathcal{E}.$$

σ-Connections over a Twisted Spectral Triple

Setup/Notation

- $(\mathcal{A}, \mathcal{H}, D)_\sigma$ is a *twisted* spectral triple.
- $\mathcal{E}$ finitely generated projective (right) module over $\mathcal{A}$.
- Space of *twisted* differential 1-forms:

$$\Omega^1_{D,\sigma}(\mathcal{A}) = \text{Span}\{a d_\sigma b;\ a, b \in \mathcal{A}\} \subset \mathcal{L}(\mathcal{H}),$$

where $d_\sigma b := [D, b]_\sigma = Db - \sigma(b)D$.

Definition

A *σ-translate* of $\mathcal{E}$ is a finitely generated projective module $\mathcal{E}^\sigma$ together with a linear isomorphism $\sigma^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^\sigma$ such that

$$\sigma^{\mathcal{E}}(\xi a) = \sigma^{\mathcal{E}}(\xi)\sigma(a) \qquad \forall \xi \in \mathcal{E}\ \forall a \in \mathcal{A}.$$

σ-Connections

Definition (RP+HW)

A σ-connection on a finitely generated projective module $\mathcal{E}$ is a linear map $\nabla^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^{\sigma} \otimes_{\mathcal{A}} \Omega^1_{D,\sigma}(\mathcal{A})$ such that

$$\nabla^{\mathcal{E}}(\xi a) = \sigma^{\mathcal{E}}(\xi) \otimes d_{\sigma} a + \left(\nabla^{\mathcal{E}} \xi\right) a \qquad \forall a \in \mathcal{A}\ \forall \xi \in \mathcal{E}.$$

Example

If $\mathcal{E} = e\mathcal{A}^q$ with $e = e^2 \in M_q(\mathcal{A})$, then

1. $\mathcal{E}^{\sigma} = \sigma(e)\mathcal{A}^q$ is a σ-translate.
2. It is equipped with the *Grassmanian σ-connection*,

$$\nabla_0^{\mathcal{E}} = (\sigma(e) \otimes 1) d_{\sigma}.$$

Coupling with σ-connections

Proposition (RP+HW)

The datum of σ-connection on $\mathcal{E}$ defines a coupled operator,

$$D_{\nabla^{\mathcal{E}}} : \mathcal{E} \otimes_{\mathcal{A}} \operatorname{dom} D \to \mathcal{E}^{\sigma} \otimes_{\mathcal{A}} \mathcal{H},$$

of the form,

$$D_{\nabla^{\mathcal{E}}} = \begin{pmatrix} 0 & D^{-}_{\nabla^{\mathcal{E}}} \\ D^{+}_{\nabla^{\mathcal{E}}} & 0 \end{pmatrix},$$

where $D^{\pm}_{\nabla^{\mathcal{E}}} : \mathcal{E} \otimes \operatorname{dom} D^{\pm} \to \mathcal{E}^{\sigma} \otimes \mathcal{H}^{\mp}$ are Fredholm operators.

Example

For a Dirac spectral triple $(C^{\infty}(M), L^2_g(M), \not{S}, \not{D}_g)$ and $\mathcal{E} = C^{\infty}(M, E)$. Then

1. Any connection ∇^{E} on E defines a connection on $\mathcal{E}$.
2. The corresponding coupled operator agrees with $\not{D}_{\nabla^{E}}$.

Index Map

Definition

The index of $D_{\nabla^{\mathcal{E}}}$ is

$$\operatorname{ind} D_{\nabla^{\mathcal{E}}} := \frac{1}{2}\left(\operatorname{ind} D^{+}_{\nabla^{\mathcal{E}}} - \operatorname{ind} D^{-}_{\nabla^{\mathcal{E}}}\right).$$

where $\operatorname{ind} D^{\pm}_{\nabla^{\mathcal{E}}}$ is the usual Fredholm index of $D^{\pm}_{\nabla^{\mathcal{E}}}$.

Remark

$\operatorname{ind} D_{\nabla^{\mathcal{E}}}$ is actually an integer when $\sigma = \mathrm{id}$, and in all main examples when $\sigma \neq \mathrm{id}$.

Proposition (Connes-Moscovici, RP+HW)

1. $\operatorname{ind} D_{\nabla^{\mathcal{E}}}$ *depends only on the K-theory class of $\mathcal{E}$.*
2. *There is a unique additive map* $\operatorname{ind}_{D,\sigma} : K_0(\mathcal{A}) \to \frac{1}{2}\mathbb{Z}$ *such that*

$$\operatorname{ind}_D[\mathcal{E}] = \operatorname{ind} D_{\nabla^{\mathcal{E}}} \qquad \forall(\mathcal{E}, \nabla^{\mathcal{E}}).$$

Connes-Chern Character

Lemma (RP+HW)

Suppose that $(\mathcal{A}, \mathcal{H}, D)_\sigma$ *is p-summable, i.e.,* $\operatorname{Tr}|D|^{-p} < \infty$ *for some* $p \geq 1$*. Assume further that* D *is invertible and* $\mathcal{E} = e\mathcal{A}^q$, $e^2 = e \in M_q(\mathcal{A})$*. Then, for all* $k \geq \frac{1}{2}p$,

$$\operatorname{ind} D_{\nabla^\mathcal{E}} = \frac{1}{2}\operatorname{Tr}\left\{\gamma(D^{-1}[D, e]_\sigma)^{2k}\right\},$$

where $\gamma = \mathrm{id}_{\mathcal{H}^+} - \mathrm{id}_{\mathcal{H}^-}$.

Theorem (Connes-Moscovici, RP+HW)

Assume $(\mathcal{A}, \mathcal{H}, D)_\sigma$ *is p-summable. Then there is an even periodic cyclic class* $\mathrm{Ch}(D)_\sigma \in \mathrm{HP}^0(\mathcal{A})$ *, called Connes-Chern character, such that*

$$\operatorname{ind} D_{\nabla^\mathcal{E}} = \langle \mathrm{Ch}(D)_\sigma, \mathrm{Ch}(\mathcal{E})\rangle \qquad \forall(\mathcal{E}, \nabla^\mathcal{E}),$$

where $\mathrm{Ch}(\mathcal{E})$ *is the Chern character in periodic cyclic homology.*

Conformal Dirac Spectral Triple

Setup

1. M^n is a compact spin oriented manifold (n even).
2. $\mathcal{C}$ is a conformal structure on M.
3. G is a group of conformal diffeomorphisms preserving $\mathcal{C}$. Thus, given any metric $g \in \mathcal{C}$ and $\phi \in G$,
$$\phi_* g = k_\phi^{-2} g \text{ with } k_\phi \in C^\infty(M),\ k_\phi > 0.$$
4. $C^\infty(M) \rtimes G$ is the crossed-product algebra, i.e.,
$$C^\infty(M) \rtimes G = \left\{ \sum f_\phi u_\phi;\ f_\phi \in C_c^\infty(M) \right\},$$
$$u_\phi^* = u_\phi^{-1} = u_{\phi^{-1}}, \qquad u_\phi f = (f \circ \phi^{-1}) u_\phi.$$

Conformal Dirac Spectral Triple

Lemma (Connes-Moscovici)

For $\phi \in G$ *define* $U_\phi : L^2_g(M, \not{S}) \to L^2_g(M, \not{S})$ *by*

$$U_\phi \xi = k_\phi^{-\frac{n}{2}} \phi_* \xi \quad \forall \xi \in L^2_g(M, \not{S}).$$

Then U_ϕ *is a unitary operator, and*

$$U_\phi \not{D}_g U_\phi^* = \sqrt{k_\phi} \not{D}_g \sqrt{k_\phi}.$$

Proposition (Connes-Moscovici)

The datum of any metric $g \in \mathcal{C}$ *defines a twisted spectral triple* $\left(C^\infty(M) \rtimes G, L^2_g(M, \not{S}), \not{D}_g \right)_{\sigma_g}$ *given by*

1. *The Dirac operator* $\not{D}_g$ *associated to* g.
2. *The representation* $fu_\phi \to fU_\phi$ *of* $C^\infty(M) \rtimes G$ *in* $L^2_g(M, \not{S})$.
3. *The automorphism* $\sigma_g(fu_\phi) := k_\phi^{-1} fu_\phi$.

Conformal Connes-Chern Character

Theorem (RP+HW)

1. *The spectral triple* $\left(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\right)_{\sigma_g}$ *depends on the choice of* $g \in \mathcal{C}$ *only up to unitary equivalence and conformal deformations in the realm of twisted spectral triples.*
2. *The Connes-Chern character* $\mathrm{Ch}(\not{D}_g)_{\sigma_g} \in \mathrm{HP}^0(C^\infty(M) \rtimes G)$ *is an invariant of the conformal class* $\mathcal{C}$*.*

Definition

The conformal Connes-Chern character $\mathrm{Ch}(\mathcal{C}) \in \mathrm{HP}^0(C^\infty(M) \rtimes G)$ is the Connes-Chern character $\mathrm{Ch}(\not{D}_g)_{\sigma_g}$ for any metric $g \in \mathcal{C}$.

Computation of $\mathrm{Ch}(\mathcal{C})$

Proposition (Ferrand-Obata)

If the conformal structure $\mathcal{C}$ is non-flat, then $\mathcal{C}$ contains a G-invariant metric.

Fact

If $g \in \mathcal{C}$ is G-invariant, then $\left(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\right)_{\sigma_g}$ is an ordinary spectral triple (i.e., $\sigma_g = 1$).

Consequence

When $\mathcal{C}$ is non-flat, we are reduced to the compuation of the Connes-Chern character of $\left(C^\infty(M) \rtimes G, L^2_g(M, \$), \not{D}_g\right)$, where G is a group of isometries.

Computation of $\mathrm{Ch}(\mathcal{C})$

Remark

1. When G is a group of isometries, the Connes-Chern character of $\left(C^\infty(M) \rtimes G, L^2_g(M,\not{S}),\not{D}_g\right)$ is computed by using JLO or CM representatives and a differentiable version of the local equivariant index theorem (Azmi, Chern-Hu).
2. We produce a new approach to equivariant heat kernel asymptotics that proves the local equivariant index theorem and computes the JLO cocycle in the same shot.
3. This approach was subsequently used by Yong Wang in several papers (e.g., equivariant eta cochain).

Setup

- $\mathcal{C}$ is a nonflat conformal structure on M.
- g is a G-invariant metric in $\mathcal{C}$.

Notation

Let $\phi \in G$. Then

- M^ϕ is the fixed-point set of ϕ; this is a disconnected sums of submanifolds,
  $M^\phi = \bigsqcup M_a^\phi, \qquad \dim M_a^\phi = a.$
- $\mathcal{N}^\phi = (TM^\phi)^\perp$ is the normal bundle (vector bundle over M^ϕ).
- Over M^ϕ, with respect to $TM_{|M^\phi} = TM^\phi \oplus \mathcal{N}^\phi$, there are decompositions,

$$\phi' = \begin{pmatrix} 1 & 0 \\ 0 & \phi'_{|\mathcal{N}^\phi} \end{pmatrix}, \qquad \nabla^{TM} = \nabla^{TM^\phi} \oplus \nabla^{\mathcal{N}^\phi}.$$

Theorem (RP + HW)

For any G-invariant metric $g \in \mathcal{C}$, the conformal Connes-Chern character $\mathrm{Ch}(D\!\!\!/_g)_{\sigma_g}$ is represented by the periodic cyclic cocycle $\varphi = (\varphi_{2m})$ given by

$$\varphi_{2m}(f^0 U_{\phi_0}, \cdots, f^{2m} U_{\phi_{2m}}) =$$
$$\frac{(-i)^{\frac{n}{2}}}{(2m)!} \sum_a (2\pi)^{-\frac{a}{2}} \int_{M_a^\phi} \hat{A}(R^{TM^\phi}) \wedge \nu_\phi \left(R^{\mathcal{N}^\phi} \right) \wedge f^0 d\tilde{f}^1 \wedge \cdots \wedge d\tilde{f}^{2m},$$

where $\phi := \phi_0 \circ \cdots \circ \phi_{2m}$, and $\tilde{f}^j := f^j \circ \phi_0^{-1} \circ \cdots \circ \phi_{j-1}^{-1}$, and

$$\hat{A}\left(R^{TM^\phi}\right) := \det{}^{\frac{1}{2}} \left[\frac{R^{TM^\phi}/2}{\sinh\left(R^{TM^\phi}/2\right)} \right],$$
$$\nu_\phi\left(R^{\mathcal{N}^\phi}\right) := \det{}^{-\frac{1}{2}} \left[1 - \phi'_{|N^\phi} e^{-R^{\mathcal{N}^\phi}} \right].$$

Remark

The n-th degree component is given by

$$\varphi_n(f^0 U_{\phi_0}, \cdots, f^n U_{\phi_n}) = \begin{cases} \int_M f^0 d\tilde{f}^1 \wedge \cdots \wedge d\tilde{f}^n & \text{if } \phi_0 \circ \cdots \circ \phi_n = 1, \\ 0 & \text{if } \phi_0 \circ \cdots \circ \phi_n \neq 1. \end{cases}$$

This represents Connes' transverse fundamental class of M/G.

Cyclic Homology of $C^\infty(M) \rtimes G$

Theorem (Brylinski-Nistor, Crainic)

Along the conjugation classes of G,

$$\mathrm{HP}_0(C^\infty(M) \rtimes G) \simeq \bigoplus_{\langle\phi\rangle} \bigoplus_{a} H^{ev}_{G^\phi}(M^\phi_a),$$

where G^ϕ is the centralizer of ϕ and $H^{ev}_{G^\phi}(M^\phi_a)$ is the G^ϕ-invariant even de Rham cohomology of M^ϕ_a.

Lemma

Any closed form $\omega \in \Omega^\bullet_{G^\phi}(M^\phi_a)$ defines a cyclic cycle η_ω on $C^\infty(M) \rtimes G$ via the transformation,

$$f^0 df^1 \wedge \cdots \wedge df^k \longrightarrow U_\phi \tilde{f}^0 \otimes \tilde{f}^1 \otimes \cdots \otimes \tilde{f}^k, \quad f^j \in C^\infty(M^\phi_a)^{G^\phi},$$

where $\tilde{f}^j$ is a G^ϕ-invariant smooth extension of f^j to M.

Conformal Invariants

Theorem (RP+HW)

Assume that the conformal structure $\mathcal{C}$ is nonflat. Then

1. *For any closed even form $\omega \in \Omega^{ev}_{G^\phi}(M_a^\phi)$, the pairing $\langle \mathrm{Ch}(\mathcal{C}), \eta_\omega \rangle$ is a conformal invariant.*
2. *For any G-invariant metric $g \in \mathcal{C}$, we have*

$$\langle \mathrm{Ch}(\mathcal{C}), \eta_\omega \rangle = \int_{M_a^\phi} \hat{A}(R^{TM^\phi}) \wedge \nu_\phi \left(R^{\mathcal{N}^\phi} \right) \wedge \omega.$$

Remark

Branson-Ørsted proved that for $\omega = 1$ the above integral was independent of the choice of any metric $g \in \mathcal{C}$ preserved by ϕ.

Remark

The above invariants are not the type of conformal invariants appearing in the Deser-Swimmer conjecture solved by Alexakis.

Vaffa-Witten Inequality

Theorem (Vafa-Witten '84)

Let (M^n, g) be a compact spin Riemannian manifold. Then, there exists a constant $C > 0$ such that, for any Hermitian vector bundle E over M equipped with a Hermitian connection ∇^E, we have

$$|\lambda_1(\not{D}_{\nabla^E})| \leq C,$$

where $\lambda_1(\not{D}_{\nabla^E})$ is the smallest eigenvalue of the coupled Dirac operator $\not{D}_{\nabla^E}$.

Remark

The proof combines the max-min principle with some version of Poincaré duality in K-theory.

Remark

Moscovici extended Vafa-Witten inequality to ordinary spectral triples satisfying a suitable form of Poincaré duality.

Poincaré Duality

Definition (Connes, Moscovici)

Two ordinary spectral triples $(\mathcal{A}_1, \mathcal{H}, D)$ and $(\mathcal{A}_2, \mathcal{H}, D)$ are in *Poincaré duality* when

- $[a_1, a_2] = [[D, a_1], a_2] = 0$ for all $a_j \in \mathcal{A}_j$.
- The following bilinear form $(\cdot, \cdot)_D : K_0(\mathcal{A}_1) \times K_0(\mathcal{A}_2) \to \mathbb{Z}$ is *nondegenerate*,

$$(\mathcal{E}_1, \mathcal{E}_2)_D := \text{ind}\, D_{\nabla^{\mathcal{E}_1 \otimes \mathcal{E}_2}}, \quad \mathcal{E}_j \in K_0(\mathcal{A}_j).$$

Definition (RP+HW)

Two twisted spectral triples $(\mathcal{A}_1, \mathcal{H}, D)_{\sigma_1}$ and $(\mathcal{A}_2, \mathcal{H}, D)_{\sigma_2}$ are in *Poincaré duality* when

- $[a_1, a_2] = [[D, a_1]_{\sigma_1}, a_2]_{\sigma_2} = 0$ for all $a_j \in \mathcal{A}_j$.
- The following bilinear form $(\cdot, \cdot)_{D,\sigma} : K_0(\mathcal{A}_1) \times K_0(\mathcal{A}_2) \to \mathbb{Z}$ is *nondegenerate*,

$$(\mathcal{E}_1, \mathcal{E}_2)_{D,\sigma} := \text{ind}\, D_{\nabla^{\mathcal{E}_1 \otimes \mathcal{E}_2}}, \quad \mathcal{E}_j \in K_0(\mathcal{A}_j).$$

Poincaré Duality. Conformal Deformations

Example (RP+HW)

Consider the following data:

- Ordinary spectral triples $(\mathcal{A}_1, \mathcal{H}, D)$ and $(\mathcal{A}_2, \mathcal{H}, D)$ in Poincaré duality.
- Inner automorphisms $\sigma_j(a) = k_j^2 a k_j^{-2}$, $a_j \in \mathcal{A}$, with $k_j \in \mathcal{A}_j^+$.
- $k = k_1 k_2 \in \mathcal{A}_1 \otimes \mathcal{A}_2$.

Then $(\mathcal{A}_1, \mathcal{H}, kDk)_{\sigma_1}$ and $(\mathcal{A}_2, \mathcal{H}, kDk)_{\sigma_2}$ are in Poincaré duality.

Remark

When $k_1 = 1$, the *ordinary* spectral triple $(\mathcal{A}_1, \mathcal{H}, kDk)$ is in Poincaré duality with the *twisted* spectral triple $(\mathcal{A}_2, \mathcal{H}, kDk)_{\sigma_2}$.

Remark

The above results extend to pseudo-inner twistings of Poincaré dual pairs of ordinary spectral triples.

Further Examples

- Duals of duals for cocompact discrete subgroups of semisimple Lie groups satisfying the Baum-Connes conjecture (Connes).
- Ordinary and twisted spectral triples over noncommutative tori (Connes, RP+HW).
- Spectral triples describing the Standard Model of particle physics (Chamseddine, Connes, Marcolli).
- Quantum projective line (D'Andrea-Landi).
- Quantum Podleś spheres (Dąbrowski-Sitarz, Wagner).
- Conformal deformations of all the above (RP+HW).

σ-Hermitian Structures

Setup

- $(\mathcal{A}, \mathcal{H}, D)_\sigma$ is a twisted spectral triple.
- $\mathcal{E}$ is a finitely generated projective module with σ-translate $\mathcal{E}^\sigma$.

Definition (RP+HW)

A *σ-Hermitian* structure on $\mathcal{E}$ is given by

1. A Hermitian metric $(\cdot, \cdot) : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$.
2. A right $\mathcal{A}$-module isomorphism $\mathfrak{s} : \mathcal{E}^\sigma \to \mathcal{E}$ such that

$$(\mathfrak{s} \circ \sigma^{\mathcal{E}}(\xi), \eta) = \sigma\left[(\xi, \mathfrak{s} \circ \sigma^{\mathcal{E}}(\eta))\right] \quad \forall \xi, \eta \in \mathcal{E}.$$

Example

If $\sigma(a) = kak^{-1}$, then any Hermitian structure on $\mathcal{E}$ gives rise to a σ-Hermitian structure with

$$\mathfrak{s}(\xi) = \left(\sigma^{\mathcal{E}}\right)^{-1}(\xi)k^{-1} \qquad \forall \xi \in \mathcal{E}^\sigma.$$

σ-Hermitian σ-Connections

Definition (RP+HW)

A *σ-connection* $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ is σ-Hermitian when

$$(\xi, \mathfrak{s}\nabla^{\mathcal{E}}\eta) - (\mathfrak{s}\nabla^{\mathcal{E}}\xi, \eta) = d_\sigma(\mathfrak{s} \circ \sigma^{\mathcal{E}}(\xi), \eta) \quad \forall \xi, \eta \in \mathcal{E}.$$

Proposition (RP+HW)

Assume $\nabla^{\mathcal{E}}$ is σ-Hermitian σ-connection. Then

1. *The operator $\mathfrak{s}D_{\nabla^{\mathcal{E}}}$ is selfadjoint (and Fredholm).*
2. $|\mathfrak{s}D_{\nabla^{\mathcal{E}}}| = |D_{\nabla^{\mathcal{E}}}|$.
3. $\operatorname{ind} D^+_{\nabla^{\mathcal{E}}} = \dim\ker D^+_{\nabla^{\mathcal{E}}} - \dim\ker D^-_{\nabla^{\mathcal{E}}}$.

Definition

1. The eigenvalues of $\mathfrak{s}D_{\nabla^{\mathcal{E}}}$ are called the $\mathfrak{s}$-*eigenvalues* of $D_{\nabla^{\mathcal{E}}}$.
2. We order them into a sequence $\lambda_j(D_{\nabla^{\mathcal{E}}})$, $j = 1, 2, \ldots$, so that

$$|\lambda_1(D_{\nabla^{\mathcal{E}}})| \leq |\lambda_2(D_{\nabla^{\mathcal{E}}})| \leq \cdots.$$

Vafa-Witten Inequality for Twisted Spectral Triples

Theorem (RP+HW)

Let $(\mathcal{A}_1, \mathcal{H}, D)_{\sigma_1}$ *be a twisted spectral triple such that*

1. $(\mathcal{A}_1, \mathcal{H}, D)_{\sigma_1}$ *has a Poincaré dual* $(\mathcal{A}_2, \mathcal{H}, D)_{\sigma_2}$.
2. $\sigma_1(a) = kak^{-1}$ *for some positive element* $k \in \mathcal{A}_1$.
3. $\dim K_0(\mathcal{A}) \otimes \mathbb{Q} < \infty$.

Then there is a constant $C > 0$ *such that, for any Hermitian finitely generated projective module* $\mathcal{E}$ *over* $\mathcal{A}_1$ *and any* σ_1*-Hermitian connection* $\nabla^{\mathcal{E}}$ *on* $\mathcal{E}$*, we have*

$$|\lambda_1(D_{\nabla^{\mathcal{E}}})| \leq C\|k^{-1}\|.$$

Remark

- For *ordinary* spectral triples with *ordinary* Poincaré duality the inequality was obtained by Moscovici ('97).
- For $k = 1$ this extends Moscovici's inequality to ordinary spectral triples with twisted Poincaré duals.

Conformal Deformations of Spectral Triples

Theorem (RP+HW)

Consider the following data:

- *Ordinary spectral triples* $(\mathcal{A}_1, \mathcal{H}, D)$ *and* $(\mathcal{A}_2, \mathcal{H}, D)$ *in Poincaré duality.*
- *Inner automorphisms* $\sigma_j(a) = k_j^2 a k_j^{-2}$, $a_j \in \mathcal{A}$, *with* $k_j \in \mathcal{A}_j^+$.
- $k = k_1 k_2 \in \mathcal{A}_1 \otimes \mathcal{A}_2$.

Then, there is a constant $C > 0$ *independent of* k_1 *and* k_2*, such that, for any Hermitian module* $\mathcal{E}$ *over* $\mathcal{A}_1$ *equipped with a* σ_1*-Hermitian connection* $\nabla^{\mathcal{E}}$*, we have*

$$|\lambda_1\left((k_1 D k_1)_{\nabla^{\mathcal{E}}}\right)| \leq C \|k_1^{-1}\| \|k_1 k_2\|^2.$$

Remark

There is also a version of the above results for pseudo-inner twistings of Poincaré dual pairs of ordinary spectral triples.

Vafa-Witten Inequality in Conformal Geometry

Theorem (RP+HW)

Let (M, g) be an even dimensional compact Riemannian spin manifold. Then there is a constant $C > 0$ such that, for any conformal factor $k \in C^\infty(M)$, $k > 0$, and any Hermitian vector bundle E equipped with a Hermitian connection ∇^E, we have

$$\left|\lambda_1(D_{\hat{g},\nabla^E})\right| \leq C\|k\|_\infty, \qquad \hat{g} := k^{-2}g,$$

where $\|k\|_\infty$ is the maximum value of k.

Further Results

Remark (RP+HW)

There are also versions of Vafa-Witten inequality for:

1. Twisted spectral triples over noncommutative tori associated to conformal weights (with control of the dependence on the conformal weight).
2. Conformal deformations by group elements of spectral triples associated to duals of cocompact discrete subgroups of semisimple Lie groups satisfying the Baum-Connes conjecture (with control of the dependence on the conformal factor).