Lie-Hopf algebras and their Hopf cyclic cohomology

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Local Index Formula in NCG

Theorem (Connes-Moscovici, 1995)

For an (odd) spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ *such that the residues*

$$\int\!\!\!\!\!\!-_{\ell} T \ := \ \mathrm{Res}_{s=0}\, \mathrm{Tr}(s^{\ell}\, T|D|^{-2s}), \qquad T \in \{\mathcal{A}, [D,c], |D|^{-z};\ z \in \mathbb{C}\}$$

make sense, one has:

$[(\varphi_n)_{n=1,3,\ldots}]$ *is a cocycle in the* (b,B)*-bicomplex of* $\mathcal{A}$*,*

$$\varphi_n(a^0, \ldots, a^n) = \sum_{\mathbf{k},\ell} c_{n,\mathbf{k},\ell} \int\!\!\!\!\!\!-_{\ell} a^0 [D, a^1]^{(k_1)} \ldots [D, a^n]^{(k_n)} |D|^{-n-2|k|}$$

$$\nabla(T) = [D^2, a],\ T^{(k)} = \nabla^k(T),\ |k| = k^1 + \ldots + k^n\,,$$

$$c_{n,\mathbf{k},\ell} = \frac{(-1)^{|k|}\,\Gamma^{(\ell)}\left(|k| + \frac{n}{2}\right)}{k_1! \ldots k_n!(k_1+1)\ldots(k_1+\cdots+k_n+n)}.$$

Theorem (Connes-Moscovici 1998)

- *For any* $n \geq 1$ *and any oriented flat manifold* M^n*, there is a canonical Hopf algebra* $\mathcal{H}_n$ *acting on the algebra* $\mathcal{A}_n := C_c^\infty(FM) \rtimes \mathrm{Diff}(M)$.

- *There is a canonical cyclic cohomology theory, associated to* $\mathcal{H}_n$*, canonically isomorphic to the Gelfand Fuks cohomology of the Lie algebra of formal vector fields on* $\mathbb{R}^n$.

- *There is a characteristic map from the mentioned cyclic cohomology of* $\mathcal{H}_n$ *to the cyclic cohomology of the algebra* $\mathcal{A}_n$ *such that the index cocycle is trapped in its image.*

Modular pair in involution

Let H be a Hopf algebra.

- An algebra map $\delta : H \to \mathbb{C}$ is called a character
- An element $\sigma \in H$ is called a group-like if $\Delta(\sigma) = \sigma \otimes \sigma$
- The pair (δ, σ) is called modular pair in involution (MPI) if $\delta(\sigma) = 1$ and

$$\widetilde{S}_\delta^2(h) = \sigma h \sigma^{-1}$$

Here $\widetilde{S}_\delta(h) = \sum \delta(h_{(1)}) S(h_{(2)})$

Hopf cyclic cohomology, $HC(H, {}^{\sigma}\mathbb{C}_{\delta})$

We define operators b and B, $\quad H^{\otimes q} \underset{B}{\overset{b}{\rightleftarrows}} H^{\otimes q+1}$

$$b := \sum_{i=0}^{q+1} (-1)^i d_i, \qquad B := \left(\sum_{i=0}^{q} (-1)^{qi} t^i \right) s_{q-1}(1-t).$$

where

$$d_0(h^1 \otimes \ldots \otimes h^q) = 1 \otimes h^1 \otimes \ldots \otimes h^q,$$
$$d_i(h^1 \otimes \ldots \otimes h^q) = \sum h^1 \otimes \ldots \otimes \Delta(h^i) \otimes \ldots \otimes h^q,$$
$$d_{q+1}(h^1 \otimes \ldots \otimes h^q) = h^1 \otimes \ldots \otimes h^q \otimes \sigma,$$
$$s_j(h^1 \otimes \ldots \otimes h^q) = h^1 \otimes \ldots \otimes \varepsilon(h^{j+1}) \otimes \ldots \otimes h^q,$$
$$t(h^1 \otimes \ldots \otimes h^q) = \sum \widetilde{S}_{\delta}({h^1}_{(2)}) \cdot (h^2 \otimes \ldots \otimes h^q \otimes \sigma),$$

where H acts on $H^{\otimes q}$ diagonally.

Action of Hopf algebras on algebras

The same way that groups or Lie algebras act on algebras by automorphisms or derivations respectively, we want Hopf algebras act on algebras. In this case we say an algebra A is H-module algebra if

$$h \triangleright (ab) = \sum (h_{(1)} \triangleright a)(h_{(2)} \triangleright b), \qquad h \triangleright 1_A = \varepsilon(h) 1_A$$

Characteristic map

Suppose a Hopf algebra H acts on an algebra A. Provided for a MPI (δ, σ) the algebra A obtains a δ-invariant σ-trace, $\tau : A \to \mathbb{C}$, i.e

$$\tau\big(h(a)\big) = \delta(h)\tau(a), \qquad \tau(ba) = \tau\big(a\sigma(b)\big)$$

there is a characteristic map

$$\chi_\tau : HC^\bullet(H, {}^\sigma\mathbb{C}_\delta) \to HC^\bullet(A)$$

$$\chi_\tau(h^1 \otimes \ldots \otimes h^n)(a^0 \otimes \ldots \otimes a^n)) = \tau(a_0 h^1(a_1) \cdots h^n(a_n))$$

Crossed product algebra

Let M be a manifold and $\Gamma \leq \mathrm{Diff}\, M$.
One defines the left action of Γ on $C^\infty(M)$ by

$$\phi \triangleright f = f \circ \phi^{-1}$$

Define the crossed product algebra

$$C^\infty(M) \rtimes \Gamma$$

A typical element is a finite sum of $\sum_i f^i \, U^*_{\phi_i}$ where fU^*_ϕ stands for $f \rtimes \phi^{-1}$.
Multiplication reads as

$$f^1 \, U^*_{\phi_1} f^2 \, U^*_{\phi_2} = f^1(\phi_2^{-1} \triangleright f^2) U^*_{\phi_2 \phi_1}$$

The Hopf algebra $\mathcal{H}_n$

- Let $M = \mathbb{R}^n$. Vector fields on FM are generated by

$$Y_j^i := y_j^k \frac{\partial}{\partial y_k^i}, \qquad X_k = y_k^i \frac{\partial}{\partial x^i}$$

- We lift them on $\mathcal{A}_n := C_c^\infty(FM) \rtimes \Gamma$ by
$$X_k(fU_\varphi^*) = X_k(f)U_\varphi^*, \quad Y_j^i(fU_\varphi^*) = Y_j^i(f)U_\varphi^*$$
- We have $Y_j^i(ab) = Y_j^i(a)b + aY_j^i(b)$,
- However, for X_k we have

$$X_k(ab) = X_k(a)b + aX_k(b) + \delta_{j,k}^i(a)Y_i^j(b)$$

,

- Here $\delta_{jk}^i(fU_\varphi^*) = \gamma_{jk}^i(\varphi)fU_\varphi^*$, and

$$\gamma_{jk}^i(\varphi)(x, y) = \left[y^{-1} \cdot \varphi'(x)^{-1} \cdot \partial_\mu \varphi'(x) \cdot y\right]_j^i y_k^\mu$$

continued ...

One defines the higher order operators $\delta^{l}_{j,k|l_1,\dots,l_m}$ by

$$\delta^{l}_{j,k|l_1,\dots,l_m} = [X_{l_m}, \delta^{l}_{j,k|l_1,\dots,l_{m-1}}]$$

They satisfy the Bianchi identities $\delta^{i}_{jk|l} - \delta^{i}_{jl|k} = \delta^{i}_{\mu l}\delta^{\mu}_{jk} - \delta^{i}_{\mu k}\delta^{\mu}_{jl}$

Definition

As an algebra $\mathcal{H}_n$ is the subalgebra of $\mathcal{L}(\mathcal{A}_n)$ generated by X_k, Y^i_j and all $\delta^{l}_{j,k|l_1,\dots,l_m}$. The comultiplication of $\mathcal{H}_n$ is obtained by the Leibniz rule $h(ab) = h_{(1)}(a)\, h_{(2)}(b), \quad h \in \mathcal{H}_n,\ a, b \in \mathcal{A}_n$.
So we see that

$$\begin{aligned}
\Delta(X_l) &= X_l \otimes 1 + 1 \otimes X_l + \delta^{i}_{j,l} \otimes Y^{j}_{i} \\
\Delta(Y^{i}_{j}) &= Y^{i}_{j} \otimes 1 + 1 \otimes Y^{i}_{j} \\
\Delta(\delta^{i}_{j,k}) &= \delta^{i}_{j,k} \otimes 1 + 1 \otimes \delta^{i}_{j,k}
\end{aligned}$$

Actions of $\mathcal{H}_n$ on $\mathcal{A}_\Gamma(FM)$

- $\mathcal{H}_n$ acts on $\mathcal{A}_\Gamma(FM)$
- $\mathcal{A}_\Gamma(FM)$ possesses the trace

$$\tau(fU_\varphi^*) = \begin{cases} \int_{FM} f\varpi_{FM} & \text{if} \quad \varphi = \mathrm{id} \\ 0 & \text{otherwise} \end{cases}$$

- The character on $\mathcal{H}_n$ defined by

$$\delta(Y_j^i) = \delta_j^i, \quad \delta(X_l) = 0, \quad \delta(\delta^i_{j_0 j_1 | j_2, \ldots, j_m}) = 0$$

Structure of $\mathcal{H}_n$

- X_k, Y_j^i form a representation of $\mathfrak{g} := g\ell_n^{\mathrm{affine}}$
- $\delta_{j,k|\ell_1,\ldots,\ell_m}^i$ generates $\mathcal{F} := U(\mathfrak{a}_n^+)^*$
- $\big(U(\mathfrak{g}), \mathcal{F}\big)$ forms a matched pair of Hopf algebras

$$\mathcal{H}_n = \big(U(\mathfrak{g}) \blacktriangleright\!\!\triangleleft \mathcal{F}\big)^{\mathrm{cop}}$$

Past progress ([Moscovici-R])

$$
\begin{array}{ccccc}
 & & C^\bullet(\mathcal{H}, \mathbb{C}_\delta) & \xrightarrow{\chi_\tau} & C^\bullet(\mathcal{A}_\Gamma) \\
 & & \uparrow{\scriptstyle \mathcal{I}} & & \uparrow{\scriptstyle \Phi} \\
C^\bullet_{\mathrm{top}}(\mathfrak{a}) & \xrightarrow{\mathcal{E}_{\mathrm{LH}}} & C^{\bullet,\bullet}_{\mathrm{c-w}}(\mathfrak{g}^*, \mathcal{F}) & \xrightarrow{\Theta} & C^\bullet_{\mathrm{Bott}}(\Omega)
\end{array}
$$

$C^{\bullet,\bullet}_{\mathrm{c-w}}(\mathfrak{g}^*, \mathcal{F})$

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle \partial_{\mathrm{c-w}}} & & \uparrow{\scriptstyle \partial_{\mathrm{c-w}}} & & \uparrow{\scriptstyle \partial_{\mathrm{c-w}}} & & \\
\wedge^2\mathfrak{g}^* & \xrightarrow{b_{\mathrm{c-w}}} & (\wedge^2\mathfrak{g}^* \otimes \wedge^2\mathcal{F})^{\mathcal{F}} & \xrightarrow{b_{\mathrm{c-w}}} & (\wedge^2\mathfrak{g}^* \otimes \wedge^3\mathcal{F})^{\mathcal{F}} & \xrightarrow{b_{\mathrm{c-w}}} & \dots \\
\uparrow{\scriptstyle \partial_{\mathrm{c-w}}} & & \uparrow{\scriptstyle \partial_{\mathrm{c-w}}} & & \uparrow{\scriptstyle \partial_{\mathrm{c-w}}} & & \\
\mathfrak{g}^* & \xrightarrow{b_{\mathrm{c-w}}} & (\mathfrak{g}^* \otimes \wedge^2\mathcal{F})^{\mathcal{F}} & \xrightarrow{b_{\mathrm{c-w}}} & (\mathfrak{g}^* \otimes \wedge^3\mathcal{F})^{\mathcal{F}} & \xrightarrow{b_{\mathrm{c-w}}} & \dots \\
\uparrow{\scriptstyle \partial_{\mathrm{c-w}}} & & \uparrow{\scriptstyle \partial_{\mathrm{c-w}}} & & \uparrow{\scriptstyle \partial_{\mathrm{c-w}}} & & \\
\mathbb{C} & \xrightarrow{b_{\mathrm{c-w}}} & (\mathbb{C} \otimes \wedge^2\mathcal{F})^{\mathcal{F}} & \xrightarrow{b_{\mathrm{c-w}}} & (\mathbb{C} \otimes \wedge^3\mathcal{F})^{\mathcal{F}} & \xrightarrow{b_{\mathrm{c-w}}} & \dots
\end{array} ,
$$

$$b_{\mathrm{c-w}}(\alpha \otimes f^0 \wedge \cdots \wedge f^q) = \alpha \otimes 1 \wedge f^0 \wedge \cdots \wedge f^q,$$

$$\partial_{\mathrm{c-w}}(\alpha \otimes f^0 \wedge \cdots \wedge f^q) =$$
$$\partial\alpha \otimes f^0 \wedge \cdots \wedge f^q - \sum_i \theta^i \wedge \alpha \otimes X_i \triangleright (f^0 \wedge \cdots \wedge f^q).$$

We now define a multiplication on the bicompplex

$$C^{p,q}_{\mathrm{coinv}}(\mathfrak{g}^*,\mathcal{F}) := (\wedge^p\mathfrak{g}^*\otimes\mathcal{F}^{\otimes q+1})^{\mathcal{F}}.$$

$$C^{p,q}_{\mathrm{coinv}}\otimes C^{r,s}_{\mathrm{coinv}}\to C^{p+r,q+s}_{\mathrm{coinv}},$$
$$(\omega_1\otimes f^0\otimes\ldots\otimes f^q)*(\omega_2\otimes g^0\otimes\ldots\otimes g^s)$$
$$=\omega_1\wedge\omega_2\otimes f^0\otimes\ldots\otimes f^{q-1}\otimes f^q g^0\otimes g^1\otimes\ldots\otimes g^s$$

Let us now define the graded multiplication

$$(\omega_1\otimes\tilde{f})\cdot(\omega_2\otimes\tilde{g})=(-1)^{qr}(\omega_1\otimes\tilde{f})*(\omega_2\otimes\tilde{g})$$

and eventually we use the canonical projection $\pi:\mathcal{F}^{\otimes(q+1)}\to\wedge^{q+1}\mathcal{F}$ to define the multiplication on $C^{\bullet,\bullet}_{\mathrm{c-w}}(\mathfrak{g}^*,\mathcal{F})$, by which it becomes a commutative DG algebra.

Hopf version of universal connection and curvature

$$\omega^i_j := \theta^i_j \otimes 1 + \theta^k \otimes \eta^i_{j,k} \in C^{1,0}_{\mathrm{c-w}}(\mathfrak{g}^*, \mathcal{F})$$

$$\Omega^i_j := \theta^k \otimes 1 \wedge \eta^i_{j,k} \in C^{1,1}_{\mathrm{c-w}}(\mathfrak{g}^*, \mathcal{F})$$

$$\partial_T(\Omega^i_j) = -\theta^l \wedge \theta^k_l \otimes 1 \wedge \eta^i_{j,k} - \theta^l \wedge \theta^k \otimes 1 \wedge \eta^i_{j,k|l}$$
$$- \theta^r_j \wedge \theta^k \otimes 1 \wedge \eta^i_{r,k} + \theta^r_k \wedge \theta^k \otimes 1 \wedge \eta^i_{j,r} + \theta^i_s \wedge \theta^k \otimes 1 \wedge \eta^s_{j,k}.$$

$$\omega^i_k \cdot \Omega^k_j$$
$$= \theta^i_k \wedge \theta^p \otimes 1 \wedge \eta^k_{j,p} + \theta^l \wedge \theta^p \otimes \eta^i_{k,l} \otimes \eta^k_{j,p} - \theta^l \wedge \theta^p \otimes \eta^i_{k,l}\eta^k_{j,p} \otimes 1.$$

$$\Omega^i_k \cdot \omega^k_j$$
$$= -\theta^l \wedge \theta^k_j \otimes 1 \wedge \eta^i_{k,l} - \theta^l \wedge \theta^p \otimes 1 \otimes \eta^i_{k,l}\eta^k_{j,p} + \theta^l \wedge \theta^p \otimes \eta^i_{k,l} \otimes \eta^k_{j,p}.$$

In other words

$$\partial_T(\Omega^i_j) = \Omega^i_k \cdot \omega^k_j - \omega^i_k \cdot \Omega^k_j$$

Similarly we have,

$$\partial_T(\omega^i_j) = -\theta^k_j \wedge \theta^i_k \otimes 1 - \theta^\ell \wedge \theta^k_\ell \otimes \eta^i_{jk} - \theta^\ell \wedge \theta^k \otimes \eta^i_{j,k|\ell}$$
$$- \theta^p_j \wedge \theta^k \otimes \eta^i_{p,k} + \theta^i_q \wedge \theta^k \otimes \eta^q_{j,k} - \theta^p_k \wedge \theta^k \otimes \eta^i_{j,p} - \theta^k \otimes 1 \wedge \eta^i_{j,k}$$

On the other hand we have

$$\omega^i_k \cdot \omega^k_j =$$
$$\theta^i_k \wedge \theta^k_j \otimes 1 + \theta^\ell \wedge \theta^k_j \otimes \eta^i_{k,\ell} + \theta^i_k \wedge \theta^p \otimes \eta^k_{j,p} + \theta^\ell \wedge \theta^p \otimes \eta^i_{k,l}\eta^k_{j,p}$$

We see that

$$\boxed{\partial_T(\omega^i_j) = -\Omega^i_j + \omega^i_k \cdot \omega^k_j}$$

Weil Algebra

The truncated Weil algebra $\hat{W} := \oplus_{p,q\geq 0}\hat{W}^{p,2n}$ is recalled as follows.

$$\hat{W}^{p,2q} = A^p(gl_n) \otimes S^q_{2n}(gl_n)$$

It is the commutative DG algebra generated by the connection elements T^i_j of degree 1 and the curvature elements R^i_j of degree 2.

$$dT^i_j = -1 \otimes R^i_j + T^i_k \wedge T^k_j \otimes 1, \quad dR^i_j = T^k_j \otimes R^i_k - T^i_k \otimes R^k_j$$

Hopf-Weil basis

By the universal property of $\hat{W}(g\ell_n)$ we define the following DG algebra map

$$L : \hat{W}^{(r,2k)}(g\ell_n) \to C^{r+k,k}_{\mathrm{c-w}}(\mathfrak{g}^*, \mathcal{F}) \quad (1)$$

$$L(T^i_j) = \omega^i_j, \qquad L(R^i_j) = \Omega^i_j \quad (2)$$

Theorem

The map $L : \hat{W}^{(r,2k)}(g\ell_n) \to C^{r+k,k}_{\mathrm{c-w}}(\mathfrak{g}^*, \mathcal{F})$ defined above is a quasi-isomorphism.

Lie-Hopf Algebras

We first introduce the setting.

- $\mathcal{F}$ is a commutative Hopf algebra.
- $\mathfrak{g}$ is a finite dimensional Lie algebra.
- $\mathfrak{g}$ acts on $\mathcal{F}$ by derivations, $\triangleright : \mathfrak{g} \otimes \mathcal{F} \to \mathcal{F}$.
- $\mathcal{F}$ coacts on $\mathfrak{g}$, $\blacktriangledown : \mathfrak{g} \to \mathfrak{g} \otimes \mathcal{F}$.

For a fixed basis $\{X_1, \cdots, X_N\}$ of $\mathfrak{g}$, we write the coaction as

$$\blacktriangledown : X_i \mapsto X_j \otimes f_i^j.$$

Lie-Hopf Algebras

For the element $f^j_{i,k} := X_k \triangleright f^j_i$, we say the coaction $\blacktriangledown : \mathfrak{g} \to \mathfrak{g} \otimes \mathcal{F}$ satisfies the structure identity of $\mathfrak{g}$ if

$$f^k_{j,i} - f^k_{i,j} = \sum_{s,r} C^k_{s,r} f^r_i f^s_j + \sum_l C^l_{i,j} f^k_l$$

Finally we introduce an action of $\mathfrak{g}$ on $\mathcal{F}^{\otimes 2}$ as

$$X \bullet (f \otimes g) := X_{<0>} \triangleright f \otimes X_{<1>} g + f \otimes X \triangleright g.$$

Lie-Hopf Algebras

Definition

We say $\mathcal{F}$ is a $\mathfrak{g}-$Hopf algebra if

(A) Coaction $\blacktriangledown : \mathfrak{g} \to \mathfrak{g} \otimes \mathcal{F}$ satisfies the structure identity of $\mathfrak{g}$,

(B) Coalgebra structure of $\mathcal{F}$ is $\mathfrak{g}-$equivariant, i.e.

$$\Delta(X \triangleright f) = X \bullet \Delta(f), \quad \text{and} \quad \varepsilon(X \triangleright f) = 0.$$

Theorem

$\mathcal{F}$ is a $\mathfrak{g}-$Hopf algebra if and only if $(\mathcal{F}, U(\mathfrak{g}))$ is a matched pair of Hopf algebras meach means $\mathcal{F} \blacktriangleright\!\!\triangleleft U(\mathfrak{g})$ is a Hopf algebra.

Coinvariant Lie subalgebra and Hopf algebra

Thanks to (A)

$$\mathfrak{g}_0 := \mathfrak{g}^{\mathcal{F}} := \{X \in \mathfrak{g} \mid \blacktriangledown(X) = X \otimes 1\},$$

is a Lie algebra.
And because of (B)

$$\mathcal{F}_0 := \mathcal{F}_{\mathfrak{g}} := \frac{\mathcal{F}}{\langle \mathfrak{g} \triangleright \mathcal{F} \rangle}$$

is a Hopf algebra

Cartan calculus for Lie-Hopf algebras

For any $Y \in \mathfrak{g}_0$ we define the contraction ι_Y and the Lie derivative $\mathcal{L}_Y$ on $C^{\bullet,\bullet}(\mathfrak{g}^*, \mathcal{F})$.

$$\iota_Y(\omega \otimes \tilde{f}) = \iota_Y(\omega) \otimes \tilde{f}, \qquad \mathcal{L}_Y(\omega \otimes \tilde{f}) = \mathcal{L}_Y(\omega) \otimes \tilde{f} + \omega \otimes Y \triangleright \tilde{f}. \quad (3)$$

(a) The contraction ι_Y is a derivation of degree -1.

(b) The Lie derivative $\mathcal{L}_Y$ is a derivation of degree 0.

(c) $\mathcal{L}_X = \partial_T \iota_Y + \iota_Y \partial_T$

(d) $[\partial_T, \mathcal{L}_Y] = 0$

(e) $\mathcal{L}_{[Y_1, Y_2]} = [\mathcal{L}_{Y_1}, \mathcal{L}_{Y_2}]$

(f) $[\iota_{Y_1}, \mathcal{L}_{Y_2}] = \iota_{[Y_1, Y_2]}$

Weil homomorphism

Let $\alpha : \mathfrak{g}_0^* \to \mathfrak{g}^*$ be a (algebraic) Cartan connection i.e.

$$\alpha \circ \mathrm{ad}_Y = \mathrm{ad}_Y \circ \alpha, \quad \iota_Y(\alpha(\omega)) = \omega(Y).$$

We extend α to $\alpha : \mathfrak{g}_0^* \to C^{1,0}_{\mathrm{c-w}}(\mathfrak{g}^*, \mathcal{F})$ by

$$\alpha(\omega) = (\alpha(\omega))_{<0>} \otimes (\alpha(\omega))_{<-1>}.$$

The extension of α defines a Cartan connection on $C^{\bullet,\bullet}_{\mathrm{c-w}}(\mathfrak{g}^*, \mathcal{F})$.
As a result we get a map of $\mathfrak{g}_0$-DG algebras

$$C_\alpha : \wedge^\bullet \mathfrak{g}_0^* \otimes S^\bullet(\mathfrak{g}_0^*)_{[2q]} \to C^{\bullet,\bullet}_{c-w}(\mathfrak{g}^*, \mathcal{F})$$

Here $q = \dim \mathfrak{g} - \dim \mathfrak{g}_0$.

Classical geometries

Geometry	Automorphism group	Structure group
1. General	$\mathrm{Diff}(\mathbb{R}^n)$	GL_n
2. Vol preserving	$\mathrm{Diff}_{\mathrm{Vol}}(\mathbb{R}^n)$	SL_n
3. Symplectic	$\mathrm{Diff}_{\mathrm{Sp}}(\mathbb{R}^{2n})$	Sp_{2n}
4. Contact	$\mathrm{Diff}_{\mathrm{Cn}}(\mathbb{R}^{2n+1})$	Cn_{2n+1}

Geometries and the corresponding Hopf algebras

$$C^{\infty}_{\Gamma}(M)) := C^{\infty}_{c}(M) \rtimes \Gamma$$

Geometry	Lie alg.	Hopf alg.	Iso. Lie alg.	Algebra
General	$\mathfrak{a}_n$	$\mathcal{H}_n$	$g\ell_n$	$C^{\infty}_{\Gamma}(F^+(\mathbb{R}^n))$
Vol pres.	$\mathfrak{sa}_n$	$S\mathcal{H}_n$	$s\ell_n$	$C^{\infty}_{\Gamma}(F^+_{\rm s}(\mathbb{R}^n))$
Sympl.	$\mathfrak{sp}_{2n}$	$Sp\mathcal{H}_{2n}$	sp_{2n}	$C^{\infty}_{\Gamma}(F^+_{\rm sp}(\mathbb{R}^{2n}))$
Contact	$\mathcal{F}\mathfrak{n}_{2n+1}$	$Cn\mathcal{H}_{2n+1}$	cn_{2n+1}	$C^{\infty}_{\Gamma}(F^+_{\rm cn}(\mathbb{R}^{2n+1}))$

The quantum symmetry of $\mathcal{A}_\Gamma := C_c^\infty(\mathbb{R}^n) \rtimes \Gamma$

- $X_k = \frac{\partial}{\partial x^i}, \quad X_k(fU_\phi^*) = X_k(f)U_\phi^*,$
- For X_k we have

$$X_k(ab) = X_k(a)b + \sum \sigma_k^i(a)X_i(b)$$

- Here

$$\sigma_k^i(fU_\phi^*) = \frac{\partial \phi^i}{\partial x^k} fU_\phi^*$$

and higher derivatives

$$\sigma_{j_1,\dots,j_k}^i(fU_\phi^*) = \frac{\partial^k \phi^i}{\partial x^{j_1} \dots \partial x^{j_k}} fU_\phi^*$$

The Hopf algebra $\mathcal{K}_n$

We let the Hopf algebra $\mathcal{K}_n$ be the subalgebra of $\mathcal{L}(\mathcal{A}_\Gamma$ generated by

$$X_\ell, \quad \sigma^i_{j_1,\ldots,j_m}, \quad \sigma^{-1}$$

here $\sigma = \det[\sigma^i_j]$ is the Jacobi automorphism of $\mathcal{A}_\Gamma$.
Its Hopf algebra structure is

$$\Delta(X_l) = X_l \otimes 1 + \sigma^k_l \otimes X_k,$$
$$\Delta(\sigma^i_j) = \sigma^k_j \otimes \sigma^i_k,$$
$$\Delta(\sigma) = \sigma \otimes \sigma, \quad \Delta(\sigma^{-1}) = \sigma^{-1} \otimes \sigma^{-1},$$
$$\Delta(\sigma^i_{j_1,\ldots,j_k}) = [\Delta(X_{j_k}), \Delta(\sigma^i_{j_1,\ldots,j_{k-1}})],$$
$$\varepsilon(\sigma) = \varepsilon(\sigma^{-1}) = 1, \qquad \varepsilon(\sigma^i_j) = \delta^i_j, \qquad \varepsilon(X_l) = \varepsilon(\sigma^i_{j_1,\ldots,j_k}) = 0.$$

Action of $\mathcal{K}_n$ on $\mathcal{A}_\Gamma$

- $\mathcal{K}_n$ acts on $\mathcal{A}_\Gamma$
- $\mathcal{A}_\Gamma$ possesses the trace

$$\tau(fU_\varphi^*) = \begin{cases} \int_{\mathbb{R}^n} f\varpi_{\mathbb{R}^n} & \text{if } \varphi = \mathrm{id} \\ 0 & \text{otherwise} \end{cases}$$

- ${}^{\sigma^{-1}}\mathbb{C}$ defines a MPI module on $\mathcal{K}_n$
- τ is a σ^{-1} trace
- $$\chi_\tau : HC^\bullet(\mathcal{K}_n, {}^{\sigma^{-1}}\mathbb{C}) \to HC^\bullet(\mathcal{A}_\Gamma)$$

$HC(\mathcal{K}_n, {}^{\sigma^{-1}}\mathbb{C})$

There is a map of algebras

$$C^{\bullet,\bullet}_{c-w}(V^*, \mathcal{F}_{\mathcal{K}}) \to C^{\bullet,\bullet}_{c-w}(g^*, g\ell_n, \mathcal{F}_{\mathcal{K}})$$

That induces

$$HP^{\bullet}(\mathcal{K}_n; {}^{\sigma^{-1}}\mathbb{C}) \cong HP^{\bullet}(\mathcal{H}_n, g\ell_n; \mathbb{C}_{\delta})$$

Theorem
Hopf cyclic cohomology of $\mathcal{K}_n$ with coefficients in ${}^{\sigma^{-1}}\mathbb{C}$ consists of all Chern classes.

$$\hat{\Omega}^i_j = \theta^k \otimes 1 \wedge S(\gamma^p_j)\gamma^i_{p,k} \in C^{1,1}_{\mathrm{c-w}}(V^*, \mathcal{F}_{\mathcal{K}})$$

$$HP^m\big(C^{\bullet,\bullet}_{\mathrm{c-w}}(V^*, \mathcal{F}_{\mathcal{K}})\big) = \bigoplus_{i_1+\cdots+i_k=m} \langle \mathrm{Tr}(\hat{\Omega}^{i_1}) \cdots \mathrm{Tr}(\hat{\Omega}^{i_k})\rangle$$

For $n = 1$ we have the following $b + B$ cocycles in $HC^1(\mathcal{K}_1, {}^{\sigma^{-1}}\mathbb{C})$

$$C_0 := \mathbf{1} \otimes \sigma^{-1} X_1 \in {}^{\sigma^{-1}}\mathbb{C} \otimes \mathcal{K}_1,$$

$$C_1 := \mathbf{1} \otimes \sigma^{-2} \sigma_{1,1}^1 \in {}^{\sigma^{-1}}\mathbb{C} \otimes \mathcal{K}_1$$

For $n = 2$ we have the following $b + B$ cocycles in $HC^2(\mathcal{K}_2, {}^{\sigma^{-1}}\mathbb{C})$

$$C_0 := \mathbf{1} \otimes \sigma^{-1}\sigma_2^i X_1 \otimes \sigma^{-1} X_i - \sigma^{-1}\sigma_1^i X_2 \otimes \sigma^{-1} X_i,$$

$$C_1 := \mathbf{1} \otimes \sigma^{-1}\sigma_2^i S(\sigma_s^j)\sigma_{j,1}^s \otimes \sigma^{-1} X_i - \sigma^{-1}\sigma_1^i S(\sigma_s^j)\sigma_{j,2}^s \otimes \sigma^{-1} X_i,$$

$$C_2 := \mathbf{1} \otimes \sigma^{-1} S(\sigma_r^i)\sigma_1^t \sigma_{i,2}^s \otimes \sigma^{-1} S(\sigma_k^r)\sigma_{s,t}^k \otimes \\ - \sigma^{-1} S(\sigma_r^i)\sigma_2^t \sigma_{i,1}^s \otimes \sigma^{-1} S(\sigma_k^r)\sigma_{s,t}^k,$$

$$(C_1)^2 := \mathbf{1} \otimes \sigma^{-1} S(\sigma_s^i)\sigma_1^t \sigma_{i,2}^s \otimes \sigma^{-1} S(\sigma_k^r)\sigma_{r,t}^k \\ - \sigma^{-1} S(\sigma_s^i)\sigma_2^t \sigma_{i,1}^s \otimes \sigma^{-1} S(\sigma_k^r)\sigma_{r,t}^k.$$

A surprise

$$\chi_\tau : HC^\bullet(\mathcal{K}_n, {}^{\sigma^{-1}}\mathbb{C}) \to HC^\bullet(\mathcal{A}_\Gamma)$$

is not injective.

For n=1 $\quad \chi_\tau(C_1) = 0$

For n=2 $\quad \chi_\tau(C_1) = \chi_\tau(C_1^2) = 0$