

Levi-Civita connections for noncommutative tori

Jonathan Rosenberg



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In honor of Henri, a long-time friend

Connections

One of the most basic notions in differential geometry is that of a **connection**. There are many equivalent points of view, but for our purposes we'll define them this way. Let M be a C^∞ manifold and $p: E \rightarrow M$ a smooth vector bundle. Recall that a **section** of E is a (smooth) map $s: M \rightarrow E$ with $p \circ s = \text{id}_M$. If E is a trivial bundle, then a section s is just a C^∞ (vector-valued) function on M and we can take directional derivatives of s . A **connection** is a way of doing this on a nontrivial bundle.

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$$\nabla: \mathcal{X}(M) \times \Gamma(E) \rightarrow \Gamma(E), \quad (X, s) \mapsto \nabla_X(s),$$

which is $C^\infty(M)$ -linear in the variable X , i.e., $\nabla_{fX}(s) = f\nabla_X(s)$, and satisfies the **Leibniz rule** for derivatives.

Connections (cont'd)

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Now suppose that a **metric** is given on E , i.e., a smoothly varying family of inner products on the fibers $p^{-1}(x)$ of E so that we have a pairing

$$\langle \cdot, \cdot \rangle: \Gamma(E) \times \Gamma(E) \rightarrow C^\infty(M), \quad (s, s') \mapsto \langle s, s' \rangle.$$

We say ∇ is **compatible with the metric** if

$$X \cdot \langle s, s' \rangle = \langle \nabla_X s, s' \rangle + \langle s, \nabla_X s' \rangle. \quad (2)$$

This means that the inner product of **parallel** sections ($\nabla_X s = 0 \ \forall X$) is constant.

Levi-Civita's Theorem

Now suppose $E = TM$ is the **tangent bundle** of M . That means $\Gamma(E) = \mathcal{X}(M)$, so we can define the **torsion** of a connection ∇ ,

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \quad (3)$$

This is a bilinear map $\mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)$.

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Theorem (Levi-Civita, 1917)

On a Riemannian manifold M , there is one and only one torsion-free connection on TM compatible with the metric.

The connection in this theorem is called the **Levi-Civita connection**.

Riemannian Curvature

Levi-Civita's Theorem gives an easy way to define curvature. On a Riemannian manifold, we let ∇ be the Levi-Civita connection, and then the **Riemann curvature tensor** is

$$R(X, Y) = \nabla_Y \nabla_X - \nabla_X \nabla_Y - \nabla_{[X, Y]}. \quad (4)$$

Thus $R \equiv 0$, i.e., the metric is **flat** $\Leftrightarrow \nabla$ is a Lie algebra homomorphism. It's a nontrivial fact that R is a **tensor**, i.e., $(X, Y, Z) \mapsto R(X, Y)Z$ is a section of $\text{Hom}(TM^{\otimes 3}, TM)$.

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- 1 What is a non-commutative manifold?
- 2 Assuming we know what a non-commutative manifold is, what is a vector field on such an object?

To answer ①, we'll define a (compact) noncommutative manifold to be given by a “nice” Fréchet subalgebra A^∞ of a unital C^* -algebra A . (The prototypes are noncommutative tori, to be discussed shortly.) The sections of a (smooth) vector bundle are replaced by a finitely generated projective A^∞ -module. This is motivated by the fact that in the commutative case, $\Gamma(E)$ is such a module over $C^\infty(M)$.

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- 2 (local) derivations of $C^\infty(M)$, and
- 3 sections of the tangent bundle.

In the noncommutative case we could consider derivations of A^∞ , the analogue of ②, but there is no reason why this should agree with ③. In any event, the space of derivations is a Lie algebra, but not necessarily a projective A^∞ -module. So here we propose a novel solution: use **both** definitions at once!

Connes' Definition of Connections

In his famous 1980 *Comptes Rendus* paper, Connes proposed using (1) as a definition of connection in the noncommutative case. Here we replace s by an element of a projective A^∞ -module (we're using left modules; Connes used right modules) and take for X an element of \mathfrak{g} , the Lie algebra of a group G acting on A^∞ . Such X 's are of course derivations, and can be viewed as *very special* vector fields. But we have to toss aside the analogue of $C^\infty(M)$ -linearity in the variable X , since the space of X 's isn't an A^∞ -module. Connes also showed that (4) still works as a definition of curvature, and still has tensorial properties.

Noncommutative Tori

For any reasonable definition of noncommutative manifold, basic examples should be the **noncommutative tori**. Fix an $n \times n$ skew-symmetric matrix Θ and let A_Θ be the universal C^* -algebra on unitaries U_j , $j = 1, \dots, n$, with $U_j U_k = \exp(2\pi i \Theta_{jk}) U_k U_j$. This algebra carries a **gauge action** of \mathbb{T}^n given by $t \cdot U_j = t_j U_j$, and the smooth vectors A_Θ^∞ for this action look like $\mathcal{S}(\mathbb{Z}^n)$ with convolution twisted by a 2-cocycle. The algebra A_Θ^∞ is our NC substitute for $C^\infty(\mathbb{T}^n)$. The infinitesimal generators of the gauge action are ∂_j with $\partial_j(U_k) = \delta_{jk} 2\pi i U_k$. From now on we'll fix Θ and drop the ∞ notation from A_Θ^∞ , since we only care about "smooth functions."

Vector Fields on NC Tori

We are now working on A_Θ with the basic $*$ -derivations ∂_j . Since tori are **parallelizable**, we would expect the “tangent bundle” on A_Θ to be trivial, so define

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These correspond to **definitions 2 and 3** of a vector field on an ordinary manifold.

The Basic Problem

We're now ready to define a **Riemannian metric** on A_Θ . We define this to be an A_Θ -valued inner product on \mathcal{X} making it into a (pre)Hilbert C^* -module. (The idea of doing this is due to Rieffel.) But we also want the inner product to be “real” on “real” vector fields, so we add the requirement

$$\langle \partial_j, \partial_k \rangle = \langle \partial_j, \partial_k \rangle^* = \langle \partial_k, \partial_j \rangle. \quad (5)$$

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The big problem is that the axiom (1) for a connection only makes sense when X is a derivation, i.e., when $X \in \mathcal{D}$, though we need s to lie in an A_Θ -module, i.e., $s \in \mathcal{X}$.

Inner Derivations

What if $X = \text{ad } a$ is an inner derivation? The axiom for a connection would require

$$\nabla_{\text{ad } a}(bs) = [a, b]s + b\nabla_{\text{ad } a}(s),$$

so

$$\nabla_{\text{ad } a} \circ b - b\nabla_{\text{ad } a} = [a, b].$$

This forces $\nabla_{\text{ad } a}$ to be multiplication by a , up to something central.

The Theorem of Bratteli-Elliott-Jørgensen

Theorem (Bratteli-Elliott-Jørgensen)

Let Θ be “generic” (in a specific number-theoretic sense). Then $\mathcal{D}/\{\text{inner derivations}\}$ is just the linear span of $\partial_1, \dots, \partial_n$. (Also, the center of A_Θ is just the scalars.)

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Note: In this case, there is a canonical splitting $\text{Inn } A_\Theta \rightarrow A_\Theta$ given by $\text{ad } a \mapsto a - \tau(a)$, where τ is the (unique) normalized trace.

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Note: In this case, there is a canonical splitting $\text{Inn } A_\Theta \rightarrow A_\Theta$ given by $\text{ad } a \mapsto a - \tau(a)$, where τ is the (unique) normalized trace. Therefore in the situation of the Theorem we will define $\nabla_{\text{ad } a}$ to be multiplication by $a - \tau(a)$, and then ∇ is determined once

$$\nabla_1 = \nabla_{\partial_1}, \dots, \nabla_n = \nabla_{\partial_n}$$

are given.

Statement of the Theorem

Theorem (NC Levi-Civita)

Let Θ be generic in the sense of the *B-E-J Theorem*. Fix any Riemannian metric satisfying (5) on $\mathcal{X}_\Theta = \mathcal{X}(A_\Theta)$. Then there is a unique connection

$$\nabla: \mathcal{D}_\Theta \times \mathcal{X}_\Theta \rightarrow \mathcal{X}_\Theta$$

compatible with the metric, normalized as we've explained on inner derivations, and satisfying the symmetry condition

$$\nabla_j \partial_k = \nabla_k \partial_j.$$

(This is the “*torsion-free*” condition applied to $\partial_1, \dots, \partial_n$. Torsion doesn't make sense for inner derivations.)

Sketch of Proof

Define

$$\langle \nabla_j \partial_k, \partial_\ell \rangle = \frac{1}{2} [\partial_j \langle \partial_k, \partial_\ell \rangle + \partial_k \langle \partial_\ell, \partial_j \rangle - \partial_\ell \langle \partial_j, \partial_k \rangle]. \quad (6)$$

Then the axioms are all satisfied. In the other directions, the axioms force (6).

The Curvature Tensor

Now that we have an analogue of Levi-Civita's Theorem, we can define the curvature for a Riemannian metric just as in the classical case, using the **standard definition**.

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Proposition

$R(X, Y) \equiv 0$ if either X or Y is an inner derivation.

Proof.

Direct calculation. □

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Thus the curvature is completely determined by the

$$R_{i,j,k,\ell} = \langle R(\partial_i, \partial_j)\partial_k, \partial_\ell \rangle.$$

Bianchi Identities, etc.

Theorem

The curvature satisfies the identities:

- 1 $R_{j,k,l,m} + R_{k,l,j,m} + R_{l,j,k,m} = 0$ (*Bianchi identity*)
- 2 $R_{j,k,l,m} = -R_{k,j,l,m}$.
- 3 $R_{j,k,l,m} = -R_{j,k,m,l}$.
- 4 $R_{j,k,l,m} = R_{l,m,j,k}$.

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- 4 $R_{j,k,l,m} = R_{l,m,j,k}$.

Proof.

Exactly as in the classical case. □

Metrics on A_θ Conformal to a Flat Metric

The simplest nontrivial example is the case of a noncommutative 2-torus, or an irrational rotation algebra A_θ . This is simple for θ irrational and satisfies the B-E-J condition for generic θ . Let's consider metrics "conformal" to the simplest flat metric $\langle \partial_j, \partial_k \rangle = \delta_{j,k}$. In other words, we assume

$$\langle \partial_j, \partial_k \rangle = e^h \delta_{j,k}, \quad h = h^* \in A_\theta.$$

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Then direct calculation gives

$$\left\{ \begin{array}{l} \nabla_1 \partial_1 = -\nabla_2 \partial_2 = \frac{1}{2}(k_1 \partial_1 - k_2 \partial_2), \\ \nabla_2 \partial_1 = \nabla_1 \partial_2 = \frac{1}{2}(k_2 \partial_1 + k_1 \partial_2), \\ k_j = \partial_j(e^h)e^{-h}, \\ R_{1,2,1,2} = -\frac{1}{2}(\Delta(e^h) - \partial_1(e^h)e^{-h}\partial_1(e^h) - \partial_2(e^h)e^{-h}\partial_2(e^h)). \end{array} \right.$$

A Version of Gauss-Bonnet for A_θ

These formulas are the same as what one has in the classical case $\theta = 0$ for a metric on T^2 conformal to the flat metric on $\mathbb{R}^2/\mathbb{Z}^2$. But in the commutative case, k_j further simplifies to $\partial_j(h)$ and $R_{1,2,1,2}$ reduces to $-\frac{1}{2}e^h\Delta h$. This is not quite the Gaussian curvature since ∂_1 and ∂_2 are orthogonal but not normalized. Hence the Gaussian curvature in the commutative case is $e^{-2h}R_{1,2,1,2} = -\frac{1}{2}e^{-h}\Delta h$. Since the Riemannian volume form involves a factor of e^h , we see that the NC analogue of Gauss-Bonnet is this:

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Theorem (Gauss-Bonnet)

In the case of A_θ with metric $\langle \partial_j, \partial_k \rangle = e^h \delta_{j,k}$, $\tau(R_{1,2,1,2}e^{-h}) = 0$, regardless of the value of h .

More General Metrics on A_θ

A more complicated case is the one where the metric is given by $\langle \partial_j, \partial_k \rangle = e^{h_j} \delta_{j,k}$, i.e., the metric is given by

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Now the formulas are not as nice. One finds now that

$$\nabla_1 \partial_1 = \frac{1}{2}(k_1 \partial_1 - k_2' \partial_2)$$

$$\nabla_2 \partial_2 = \frac{1}{2}(-k_1' \partial_1 + k_2 \partial_2)$$

$$\nabla_1 \partial_2 = \nabla_2 \partial_1 = \frac{1}{2}(k_1'' \partial_1 + k_2'' \partial_2)$$

$$k_j = \partial_j(e^{h_j})e^{-h_j}, \quad j = 1, 2,$$

$$k_1' = \partial_1(e^{h_2})e^{-h_1}, \quad k_2' = \partial_2(e^{h_1})e^{-h_2},$$

$$k_1'' = \partial_2(e^{h_1})e^{-h_1}, \quad k_2'' = \partial_1(e^{h_2})e^{-h_2}.$$

The Curvature

One finds in this situation that

$$R_{1,2,1,2} = \frac{1}{2}[-\partial_2(k_2') - \partial_1(k_2'')]e^{h_2} \\ + \frac{1}{4}[k_1\partial_1(e^{h_2}) - k_2'\partial_2(e^{h_2}) + k_1''\partial_2(e^{h_1}) - k_2''\partial_1(e^{h_2})].$$

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“Gauss-Bonnet” in this case would be the statement that $\tau(e^{-h_1/2}R_{1,2,1,2}e^{-h_2/2}) = 0$. I haven't been able to verify this in general but it's true in many special cases.

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Another thing one could do is compute the “Laplacian” for this metric and apply spectral analysis to it as in Connes-Moscovici and Fathizadeh-Khalkhali.