A higher index theorem for proper cocompact actions

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Happy Birthday Henry!

In this talk, we will explain Connes' noncommutative geometry approach to equivariant index theory of proper actions. This leads to some interesting new understanding.

This is joint work with Markus Pflaum and Hessel Posthuma.

- The localized longitudinal index theorem for Lie groupoids and the van Est map, arXiv:1112.4857.

- The transverse index theorem for proper cocompact actions of Lie groupoids, arXiv:1301.0479.

Part I: Equivariant Index theory

The study of equivariant index theory was started when Atiyah and Singer proved their seminal index theorem. We review some of the developments related to operator algebras and noncommutative geometry.

Invariant elliptic operators

Let G be a Lie group (or discrete group) acting properly on a manifold M.

Assume that the action is cocompact, i.e. the quotient space M/G is compact.

Let D be a G-invariant elliptic operator M on G-equivariant vector bundles, i.e. $gD = Dg$.

Examples:

- The de Rham operator on M.

- The lift of an elliptic differential operator on X to the universal covering space $M = \widetilde{X}$ with the deck transformation of the fundamental group.

Lefschetz fixed point theorem

Let $f : M \to M$ be a diffeomorphism on a closed manifold M. $f^* : \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$ descends to

$$f^* : H^\bullet(M) \to H^\bullet(M).$$

The Lefschetz number $L(f)$ of f is defined to be

$$\sum_i (-1)^i \operatorname{tr}(f^*|_{H^i(M)}).$$

Theorem: (Lefschetz) When f only has isolated nondegenerated fixed points, then

$$L(f) = \sum_{p=f(p)} \pm 1.$$

L^2-index

Assume G is unimodular. Let G act properly and cocompactly on a smooth manifold M. Let D be a G-invariant elliptic operator on M.

The kernel and cokernel of D are equipped with unitary G-representations. The formal difference

$$\mathrm{ind}_a^G(D) := [\ker(D)] - [\mathrm{coker}(D)]$$

defines an element in $K_0(C_r^*(G))$.

Define the L^2-index $\mathrm{ind}_{L^2}(D)$ to be

$$\mathrm{tr}_G\left(\mathrm{ind}_a^G(D)\right) \in \mathbb{R}.$$

Covering space

Let X be a closed manifold, and D_X be an elliptic differential operator on X.

Let M be the universal covering of X, and G be the fundamental group of X. G acts on M properly, freely, and cocompactly.

The differential operator D_X lifts to a G-invariant elliptic differential operator D_M on M.

Theorem: (Atiyah)

$$\mathrm{ind}_{L^2}(D_M) = \mathrm{ind}_a(D_X).$$

Higher index theorem

Theorem: (Connes-Moscovici) Let G be a countable discrete group acting properly and freely on a manifold M and D a G-invariant elliptic differential operator on M. For any $[c] \in H^{2k}(G, \mathbb{C})$,

$$\langle \mathrm{ind}^G(D), [c] \rangle = \frac{1}{(2\pi\sqrt{-1})^k (2k)!} \int_{T^*X} \mathrm{ch}(\sigma(D)) \widehat{A}(T^*X) \Psi^*([c]),$$

where $X = M/G$, $\Psi : X \to BG$ is the classifying map, and $\Psi^*([c]) \in H^{2k}(X, \mathbb{C})$ is the pull-back of the class $[c]$.

When $D_{\widetilde{X}}^{\mathrm{sign}}$ is the signature operator on $M = \widetilde{X}$ (the universal covering space of X),

$$\langle \mathrm{ind}_{\widetilde{X}}^{\mathrm{sign}}(D), [c] \rangle = \int_X L(X) \Psi^*([c]).$$

Homogeneous spaces

Let G be a unimodular Lie group, and H be a compact subgroup of G. Consider the homogeneous space $M = G/H$. G acts properly and cocompactly on M from the left.

Let D be a G-invariant elliptic differential operator on M. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H.

Theorem: (Connes-Moscovici)

$$\mathrm{ind}_{L^2}(D) = \left\langle \hat{A}(\mathfrak{g}, H) \wedge \mathrm{ch}(\sigma(D))_{\mathfrak{m}^*}, [V] \right\rangle,$$

where $\mathfrak{m}^* \subset \mathfrak{g}^*$ is the conormal space of $\mathfrak{h}$ in $\mathfrak{g}$, $[V]$ is the fundamental class of $\mathfrak{m}^*$.

This result was recently generalized to the L^2-index of a G-invariant elliptic operator on a manifold with a proper and cocompact action by H. Wang.

Example

Let Σ_g to be the closed Riemann surface of genus $g \geq 2$ with the $\bar{\partial}_{\Sigma_g}$ operator.

The universal covering space $\mathbb{H}$ is the upper half plane with the $\bar{\partial}_{\mathbb{H}}$ operator.

The index of $\bar{\partial}_{\Sigma_g}$ is $g-1>0$. The Atiyah covering index theorem implies that $\bar{\partial}_{\mathbb{H}}$ has nontrivial L^2 kernel.

The kernel of $\bar{\partial}_{\mathbb{H}}$ gives a Hilbert space representation of $SL(2,\mathbb{R})$, belonging to the discrete series representations.

Part II: An equivariant index theorem

In this part, we will explain an index formula computing the pairing between the index of a G-invariant elliptic operator and differentiable cohomology of G.

Differentiable group cohomology

Let G be a Lie group. Let $C^\infty(G^{\times k})$ be the space of smooth functions on

$$\underbrace{G \times \cdots \times G}_{k}.$$

Define a differential $\delta : C^\infty(G^{\times k}) \to C^\infty(G^{\times k})$ by

$$\begin{aligned} & \delta(\varphi)(g_1, \cdots, g_{k+1}) \\ = \; & \varphi(g_2, \cdots, g_k) \\ - \; & \varphi(g_1 g_2, \cdots, g_{k+1}) + \cdots + (-1)^k \varphi(g_1, \cdots, g_k g_{k+1}) \\ + \; & (-1)^{k+1} \varphi(g_1, \cdots, g_k). \end{aligned}$$

The differentiable group cohomology $H^\bullet_{\text{diff}}(G)$ is defined to be the cohomology of $(C^\infty(G^{\times \bullet}), \delta)$.

Index pairing

Assume G to be unimodular. Fix a Haar measure on G.

There is a natural pairing between $C^\infty(G^{\times k})$ and $C_c^\infty(G)^{\hat{\otimes}(k+1)}$ by

$$\begin{aligned} < \widehat{\varphi}, f_0 \otimes \cdots \otimes f_k > \; := & \int f_0(g_k^{-1} \cdots g_1^{-1}) f_1(g_1) \cdots f_k(g_k) \\ & \varphi(g_1, \cdots, g_k) dg_1 \cdots dg_k \end{aligned}$$

The above pairing descends to define a pairing between $H^{\bullet}_{\text{diff}}(G)$ and $K_\bullet(C_c^\infty(G))$.

Let D be a G-invariant elliptic operator on M. For $[\varphi] \in H^{\bullet}_{\text{diff}}(G)$, define $\text{ind}_{[\varphi]}(D)$ to be

$$< [\varphi], \text{ind}^G(D) > .$$

An index theorem

Theorem: (Pflaum-Posthuma-Tang) Let G be a Lie group acting properly and cocompactly on a manifold M. Suppose that D is an elliptic G-invariant differential operator on M, and $[\varphi] \in H^{2k}_{\text{diff}}(G; L)$. The index pairing evaluated on these elements is given by

$$\text{ind}_{[\varphi]}(D) = \frac{1}{(2\pi\sqrt{-1})^k (2k)!} \int_{T^*M} c\Phi([\varphi]) \wedge \hat{A}(T^*M) \wedge \text{ch}(\sigma(D)),$$

where

i) $c \in C^\infty_{\text{cpt}}(M)$ is a cut-off function, i.e.

$$\int_G c(g^{-1}x)dg = 1, \ \forall x \in M.$$

ii) Φ is a map from $H^\bullet_{\text{diff}}(G; L)$ to the de Rham cohomology of G-invariant differential forms on M.

Examples:

1. When G is the fundamental group of X with $M = \widetilde{X}$, $H^{\bullet}_{\text{diff}}(G) = H^{\bullet}(G)$. The index formula is the Connes-Moscovici higher index theorem.

2. When G is a unimodular Lie group and H is a compact subgroup, choose $M = G/H$. The index formula for $\varphi = 1 \in H^0_{\text{diff}}(G)$ is the Connes-Moscovici index theorem for homogeneous spaces.

3. When $\mathsf{H}_{\mathcal{F}}$ is the holonomy groupoid of a regular foliation $\mathcal{F}$ on X, choose M to be $\mathsf{H}_{\mathcal{F}}$. Assume that $\mathsf{H}_{\mathcal{F}}$ is unimodular. The index formula for $[\varphi] = 1 \in H^0_{\text{diff}}(\mathsf{H}_{\mathcal{F}})$ is the Connes index theorem for measured foliations.

Part III: Localized index theory

We explain the ideas of the proof of the main theorem. They are inspired by the several recent developments in noncommutative geometry.

-Connes' tangent groupoid

-Hopf cyclic theory

-Controlled topology

-Local index theory

Hopf algebra and differentiable group cohomology

For a Lie group G, $H(G) := C^\infty(G)$ has a natural Hopf algebra structure. We can consider its associated Hopf cyclic (co)homology.

The Hopf cyclic cohomology of $C^\infty(G)$ can be identified with the differentiable groupoid cohomology $H^\bullet_{\mathsf{diff}}(G, \mathbb{C})$.

The (smooth) group algebra $C_c^\infty(G)$ is a module of $H(G)$. The characteristic map provides a map

$$H^\bullet_{\mathsf{diff}}(\mathsf{G}, L) \longrightarrow HP^\bullet(C_c^\infty(\mathsf{G})).$$

This provides the pairing between $H^\bullet_{\mathsf{diff}}(\mathsf{G}, L)$ with $\mathrm{ind}^{\mathsf{G}}(D)$.

Jets Hopf algebra and localized cohomology

The space $J^\infty(G)$ of ∞-jets of $C^\infty(G)$ at the identity element is also a Hopf algebra. Its Hopf cyclic cohomology is computed using the following complex.

Let $C^k_{\mathrm{loc}}(G,\mathbb{C})$ denote the space of ∞-jets of $C^\infty(G^{\times k})$ at the identity. The differential d naturally induces a differential on $C^\bullet_{\mathrm{loc}}(G,\mathbb{C})$. The cohomology groups are denoted by $H^\bullet_{\mathrm{loc}}(G,\mathbb{C})$.

The cohomology $H^\bullet_{\mathrm{loc}}(G,\mathbb{C})$ can be identified with the Lie algeba cohomology of $\mathfrak{g}$. The localization map from $H^\bullet_{\mathrm{diff}}(G,\mathbb{C})$ to $H^\bullet(\mathfrak{g},\mathbb{C})$ is identified with the van Est morphism.

Van Est morphism

Let M be a proper cocompact G-manifold. Let $H^\bullet_G(M,\mathbb{C})$ be the de Rham cohomology of G-invariant differential forms on M. The localization map gives

$$\Phi : H^\bullet_{\text{diff}}(G,\mathbb{C}) \to H^\bullet_G(M,\mathbb{C}).$$

For the proper cocompact G action on its homogeneous space $M = G/H$. Then $H^\bullet_G(M,\mathbb{C})$ can be identified with $H^\bullet(\mathfrak{g},H;\mathbb{C})$, and the map Φ coincides with the van Est morphism

$$v : H^\bullet_{\text{diff}}(\mathsf{G},\mathbb{C}) \to H^\bullet(\mathfrak{g},H).$$

Localized K-theory after Moscovici-Wu, I

Denote $\Psi^{-\infty}(M)$ to be the space of smoothing operators on M. Let $\mathcal{U}$ be any finite open covering of M. Define

$$\begin{aligned} \Psi^{-\infty}(M,\mathcal{U}) &:= \{K \in \Psi^{-\infty}(M) \mid \mathrm{supp}(K) \subset \mathcal{U}^2\}, \\ \Psi^{-\infty}(M,\mathcal{U})^{\sim} &:= \Psi^{-\infty}(M,\mathcal{U}) \oplus \mathbb{C} \end{aligned}$$

where $\mathcal{U}^k := \bigcup_{U\in\mathcal{U}} U^{\times k} \subset M^{\times k}$ for $k \in \mathbb{N}$.

Let $M_\infty\big(\Psi^{-\infty}(M,\mathcal{U})\big)$ be the inductive limit of all $N \times N$-matrices with entries in $\Psi^{-\infty}(M,\mathcal{U})$. Define $K^0(M,\mathcal{U})$ to be

$$\begin{aligned} K^0\big(M,\mathcal{U}\big) := \Big\{(P,e) \in M_\infty\Big(\Psi^{-\infty}(M,\mathcal{U})^{\sim}\Big) \times M_\infty(\mathbb{C})\Big) \mid \\ P^2 = P,\ P^* = P, \quad e^2 = e,\ e^* = e \ \text{ and} \\ P - e \in M_\infty\Big(\Psi^{-\infty}(M,\mathcal{U})\Big)\Big\} \Big/ \text{path equivalence.} \end{aligned}$$

Localized K-theory for proper cocompact G-manifolds

A (finite) refinement $\mathcal{V} \subset \mathcal{U}$ obviously leads to an inclusion $\Psi^{-\infty}(M,\mathcal{V}) \hookrightarrow \Psi^{-\infty}(M,\mathcal{U})$ which induces a map $K^0(M,\mathcal{V}) \to K^0(M,\mathcal{U})$. With these maps, the localized K-theory of M is defined as

$$K^0_{\mathrm{loc}}(M) := \varprojlim_{\mathcal{U} \in \mathrm{Cov}^{\mathrm{fin}}(M)} K^0(M,\mathcal{U}).$$

Given an elliptic operator D on M, one can define an element in the localized K-theory of M, which is denoted by $\mathrm{ind}_{\mathrm{loc}}(D) \in K^0_{\mathrm{loc}}(M)$.

Extending this idea to a proper cocompact G-manifold M, we can define the localized K-theory group $K^\bullet_{\mathrm{loc}}(M;G)$.

Localized index pairing

Let M be a proper cocompact G-manifold, and D be an elliptic G-invariant differential operator on M. The localization of $\mathrm{ind}^G(D)$ defines an element

$$\mathrm{ind}^G_{\mathrm{loc}}(D) \in K^0_{\mathrm{loc}}(M;G).$$

We have a localized index pairing between $\mathrm{ind}^G_{\mathrm{loc}}(D)$ and $[\varphi] \in H^\bullet_G(M)$,

$$\langle -, - \rangle_{\mathrm{loc}} : H^\bullet_G(M) \times K^\bullet_{\mathrm{loc}}(M,G) \longrightarrow \mathbb{C}.$$

Theorem: For $[\varphi] \in H^\bullet_{\mathrm{diff}}(\mathrm{G})$,

$$\langle [\varphi], \mathrm{ind}^{\mathrm{G}}(D) \rangle = \langle \Phi([\varphi]), \mathrm{ind}^{\mathrm{G}}_{\mathrm{loc}}(D) \rangle_{\mathrm{loc}}.$$

Equivariant localized index theorem

Theorem: Let D be a G-invariant elliptic differential operator on M. For $\alpha \in H^k_G(M)$,

$$\langle \alpha, \mathrm{ind}^G_{\mathrm{loc}}(D) \rangle_{\mathrm{loc}} = \frac{1}{(2\pi\sqrt{-1})^k} \int_{T^*M} \pi^*(c \cdot \alpha) \wedge \hat{A}(T^*M)\, \mathrm{ch}(\sigma(D)).$$

Our result generalizes with full generality to a Lie groupoid proper and cocompact action. In particular,

Theorem: Let Ω be a transverse measure on $(M, \mathcal{F})$. For a longitudinal elliptic operator D and a foliated cocycle $c \in H^{2k}_{\mathcal{F}}(M, \mathbb{C})$,

$$\langle c, \mathrm{ind}^{\mathcal{H}_{\mathcal{F}}}_{\mathrm{loc}}(D) \rangle_{\mathrm{loc}} = \frac{1}{(2\pi\sqrt{-1})^k} \int_{\mathcal{F}^*} \hat{A}(\mathcal{F}^*)\mathrm{ch}(\sigma(D)) \wedge c \wedge \Omega.$$

The $\bar{\partial}$-operator on $\mathbb{C}$ (after Connes and Moscovici)

Consider the abelian group $G = \mathbb{R}^2$ and the trivial subgroup H. The homogeneous space $M = G/H$ is $\mathbb{R}^2$ on which $\mathbb{R}^2$ acts by translation.

Consider the $\bar{\partial}$-operator on $M \cong \mathbb{C}$. $\bar{\partial} + \bar{\partial}^*$ is an $\mathbb{R}^2$-invariant elliptic operator on M. As there are no L^2 (anti)holomorphic functions on $\mathbb{C}$, the kernel $\ker(\bar{\partial})$ and cokernel $\mathrm{coker}(\bar{\partial})$ are trivial. Hence the L^2-index $\mathrm{ind}_{L^2}(\bar{\partial})$ of $\bar{\partial}$ is 0.

Higher index of the $\bar\partial$-operator

The Baum-Connes map defines a K-theory index $\mathrm{Ind}(\bar\partial)$ of $\bar\partial$ in $K_0(C^*(\mathbb{R}^2)) \cong K^0(\mathbb{R}^2)$.

On $\mathbb{R}^2$, consider the group 2-cocycle $\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{C}$ defined by

$$\alpha\Big((x, y), (x', y')\Big) = xy' - x'y.$$

A careful computation of the higher index associated to α in the main theorem gives

$$\mathrm{ind}_{[\alpha]}(\bar\partial) = 1.$$

Outlook

1. We are working on using the index theorem to extract more representation theory information.

2. We are applying the index theorem to study geometric properties of leaves of (singular) foliations.

3. The assumption about the ellipticity of D may be weakened, i.e. transversely elliptic to G-orbits.

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