Expanders, exactness, and exotic completions

Rufus Willett

University of Hawai‘i

Texas A&M University, April 2014

X: metric space.

Assume bounded geometry: for all $r > 0$, there is a uniform bound on the cardinality of all r-balls.

Examples: Cayley graph of group, net in Riemannian manifold...

X: bounded geometry metric space.

$T = (T_{xy})_{x,y \in X}$: X-by-X indexed complex matrix.

T is *finite propagation*: there exists $r > 0$ such that

$$d(x,y) > r \quad \Rightarrow \quad T_{xy} = 0.$$

$\mathbb{C}_u[X]$: $*$-algebra of all finite propagation matrices with bounded entries.

Complete in natural representation on $l^2(X)$

$\rightsquigarrow$ *(uniform) Roe algebra* $C_u^*(X)$.

$P_r(X)$: simplicial complex with vertex set X, and

$$(x_0, \dots, x_n) \text{ a simplex iff } \mathrm{diam}(\{x_0, \dots, x_n\}) \leqslant r.$$

$K_*^u(P_r(X))$: (uniform) K-homology of $P_r(X)$, built from generalized elliptic operators on X.

There is a *higher index / assembly* map

$$\mu : \lim_{r \to \infty} K_*^u(P_r(X)) \to K_*(C_u^*(X)).$$

(Uniform) coarse Baum-Connes conjecture: this is an isomorphism.

(Uniform) coarse Baum-Connes assembly map:

$$\mu : \lim_{r\to\infty} K_*^u(P_r(X)) \to K_*(C_u^*(X)).$$

Idea: topological data lives in left hand side ... but has better properties under image by μ.

$\rightsquigarrow$ Applications to Novikov conjecture, existence of positive scalar curvature metrics...

(X_n): sequence of (vertex sets of finite) graphs such that:

- each X_n is connected;
- $|X_n|$ tends to infinity;
- there is a uniform bound on all vertex degrees.

$X := \sqcup_{n \in N} X_n$.

Metric on X: restricts to edge metric on each X_n, and satisfies $d(X_n, X \backslash X_n) \to \infty$ as $n \to \infty$.

Study coarse Baum-Connes for such X as:

- general case reduces to this;
- understand counterexamples...

(X_n): sequence of finite graphs, bounded vertex degree, getting bigger.

$\Delta_n : l^2(X_n) \to l^2(X_n)$ graph Laplacian:

$$\Delta_n : \delta_x \mapsto \sum_{d(x,y)=1} \delta_x - \delta_y.$$

Define

$$\Delta := \prod_n \Delta_n : l^2(X) \to l^2(X).$$

(X_n) is an *expander*: there exists $c > 0$ such that

$$\text{spectrum}(\Delta) \subseteq \{0\} \sqcup [c, \infty).$$

Note: Δ is finite propagation, so in $\mathbb{C}_u[X]$

$\rightsquigarrow$ $e^{-t\Delta} \in C^*_u(X)$ for all $t > 0$

$\rightsquigarrow$ if (X_n) an expander,

$$p := \lim_{t\to\infty} e^{-t\Delta}$$

exists in $C^*_u(X)$.

p is a *ghost projection*: $p_{xy} \to 0$ as $x, y \to \infty$.

Theorem (Higson)

$$\mu : \lim_{r\to\infty} K^u_*(P_r(X)) \to K_*(C^*_u(X))$$

coarse assembly map.

For some expanders, $[p] \notin Image(\mu)$.
(Expect: never in $Image(\mu)$).

Expanders are generic among such sequences...

What can we say about coarse Baum-Connes for generic sequences?

(Uniform) coarse Baum-Connes assembly map

$$\mu : \lim_{r\to\infty} K_*^u(P_r(X)) \to K_*(C_u^*(X)).$$

Theorem

For generic sequences of graphs as above (ignoring some technicalities):

1. μ *is not surjective;*
2. μ *is injective;*
3. μ_{max} *is an isomorphism.*

$\mu_{\max}$: change norm on $\mathbb{C}_u[X]$ to

$$\|T\| := \sup\{\|\pi(T)\| \mid \pi \text{ a } *\text{-representation of } \mathbb{C}_u[X]\}.$$

$\rightsquigarrow$ maximal (uniform) coarse Baum-Connes assembly map

$$\mu_{\max} : \lim_{r\to\infty} K_*^u(P_r(X)) \to K_*(C_{u,\max}^*(X)).$$

$\mathbb{C}_u[X]$: $*$-algebra of finite propagation matrices with bounded entries.

$$\mathbb{C}_f[X] := \{T \in \mathbb{C}_u[X] \mid \{(x, y) \mid T_{xy} \neq 0\} \text{ is finite.}\},$$

$\rightsquigarrow$ Short exact sequence

$$0 \to \mathbb{C}_f[X] \to \mathbb{C}_u[X] \to \mathbb{C}_\infty[X] := \frac{\mathbb{C}_u[X]}{\mathbb{C}_f[X]} \to 0.$$

Exactness issues: does this stay exact on completion?

Want to complete ... need representations of $\mathbb{C}_\infty[X]$.

(Y, y_0): pointed bounded geometry metric space.

(Y, y_0) is an *ultralimit* of X if there is a sequence (x_n) in X, tending to infinity, such that

$$B_X(x_n; r) \text{ is isometric to } B_Y(y_0; r)$$

for all $r > 0$, and all but finitely many n.

Example: $\mathrm{girth}(X_n) \to \infty \rightsquigarrow$ all ultralimits of $X = \sqcup X_n$ are trees.

Each gives rise to $*$-homomorphism

$$\mathbb{C}_\infty[X] \twoheadrightarrow \mathbb{C}_u[Y] \hookrightarrow \mathcal{B}(l^2(Y)).$$

Consider again

$$0 \to \mathbb{C}_f[X] \to \mathbb{C}_u[X] \to \mathbb{C}_\infty[X] := \frac{\mathbb{C}_u[X]}{\mathbb{C}_f[X]} \to 0.$$

Completing

$$0 \to \mathcal{K}(l^2(X)) \xrightarrow{\iota} C_u^*(X) \xrightarrow{\pi} C_\infty^*(X) \to 0.$$

Ghost projection p in kernel of π, but not image of ι.
For generic graphs:

$$\begin{array}{ccccccccc} \longrightarrow & K_*^u(pt) & \longrightarrow & \lim_{r\to\infty} K_*^u(P_r(X)) & \longrightarrow & \lim_{r\to\infty} \widetilde{K_*^u}(P_r(X)) & \longrightarrow & . \\ & \cong \downarrow \mu & & \downarrow \mu & & \cong \downarrow \mu & & \\ \longrightarrow & K_*(\mathcal{K}) & \xrightarrow{\iota} & K_*(C_u^*(X)) & \xrightarrow{\pi} & K_*(C_\infty^*(X)) & \longrightarrow & \end{array}$$

If we use maximal completions, corresponding bottom row *is* exact.

The maximal assembly map

$$\mu : \lim_{r \to \infty} K_*^u(P_r(X)) \to K_*(C^*_{u,max}(X))$$

is 'generically' an isomorphism. Is it *always* an isomorphism?

Note: 'yes' $\Rightarrow$ Novikov conjecture, Gromov-Lawson-Rosenberg conjecture...

Theorem (W.-Yu)

'No.'

Obstruction: geometric property (T).

G: locally compact group.

Baum-Connes conjecture with coefficients: the higher index map

$$\mu : K_*^{top}(G; A) \to K_*(A \rtimes_r G)$$

is an isomorphism.

Gromov: can 'embed' some expanders into groups

$\rightsquigarrow$ counterexamples based on failures of exactness (Higson-Lafforgue-Skandalis).

There are counterexamples to isomorphism of the Baum-Connes assembly map

$$\mu : K_*^{top}(G; A) \to K_*(A \rtimes_r G)$$

based on exactness.

Theorem (W.-Yu)

Maximal completion fixes these.

Maximal completion is not right in general: other property (T) obstructions exist.

Theorem (Baum-Guentner-W.)

There exists a crossed product functor $\rtimes_{\mathcal{E}}$ which is minimal subject to:

- *it is exact;*
- *it takes Morita equivalences to Morita equivalences.*

There are no known counterexamples to the BC conjecture for $\rtimes_{\mathcal{E}}$, and some counterexamples become confirming examples.

More in Paul Baum's talk...