Circle on the Road
Houston 2011

Multiplication Graphs

Lesson Plan by Sam Vandervelde

Warm-Up Announce to students that multiplication is going to play a central role today, so we had better make sure that everyone remembers how to multiply. Solicit a three-digit number from the audience in some creative or entertaining manner. Display the chosen number and identify any students who already know about the 7–11–13 mystery. Explain that we are going to multiply by 7, then 11, then 13 to review multiplication. (Have students who know the trick immediately predict the final answer and write it down on a sheet of paper to be placed in the front of the room, then they keep quiet.) Invite three volunteers to work out successive products on board, others double-check answers, discover the surprise at end, namely that the original number is just "doubled," in the sense that 274 becomes 274274. (Compare answers with any students who already knew the trick and have one of them explain it.) Leave it to students to figure out why this works, or perhaps guide them through an explanation, depending on the mood of the group. If a hint is required, figure out what we have to multiply 274 by in order to get 274274.

Intro Activity Announce that today students will get a chance to be mathematicians—investigate new ideas, make conjectures, even prove some results if we're lucky. Distribute 12×12 multiplication table handout and have students circle all the multiples of 12 within the table. Guide them to ignore the obvious ones (right column and bottom row) and focus on the pairs of numbers that are less than 12 but whose product is nonetheless a multiple of 12. Have students call out such pairs of numbers, writing them on the board and connecting them with a straight line segment in a haphazard manner. (Reuse existing numbers, so that there are no repeats.) Once they are all on display challenge students to clean things up by arranging the numbers and edges in a more appealling manner; see what they come up with. This serves as an informal introduction to graph isomorphism.

Now ask what students notice thus far. Some numbers never get used, for instance. What is different about these numbers? Answer: no common factor with 12. A debate might also arise regarding whether or not to draw an edge from 6 to itself; is this allowed in a graph? Answer: it is up to us. (But see later remark.) Wrap up this segment by formalizing the definition of a multiplication graph for any N; namely, label vertices of a graph with all positive integers less than N which share a common factor with N, then draw edges between pairs whose product is divisible by N. We'll call N the "master number" used to generate the graph.

Main Investigation To become more familiar with this construction have students create multiplication graphs for master numbers $N = 6$ through $N = 11$, double-check with one another, then display all these graphs on the board, but leave off the numbers at the vertices, so that we have an unlabeled graph associated with each master number. Ask for observations and follow up on their ideas. For instance, students will probably observe that primes give empty graphs; have someone explain why. They might also note that the graphs for $N = 6$ and $N = 8$ are the same unless we allow "loops" from a vertex to itself, which may be a good reason to permit them.

At this point produce two multiplication graphs with unlabeled vertices; a star graph with one hub vertex connected to 60 satellite vertices, and a complete graph on 10 vertices. Challenge students to figure out which master number was used to generate these two multiplication graphs. Invite them to offer thoughts on how to even approach this question. (They will need to study lots of multiplication graphs, notice patterns, and make conjectures.) So divide and conquer—have students create graphs for the master numbers $N = 14$ up to at least $N = 25$, preferably with each value of N covered by at least two students. (But skip prime values of N, of course.) Once there is agreement among students on a graph have them tape a sheet of paper to the wall showing the graph along with the master number. Keep track of conjectures made along the way that will expedite the process. The goal is to identify enough patterns and make conjectures based on them that they can predict that star graphs arise for $N = 2p$, where p is an odd prime, and that complete graphs arise for $N = p^2$. Hence they should predict that the master numbers for the two graphs presented earlier are $N = 122$ and $N = 121$, respectively.

Related Questions It is entirely possible that all the available time will have elapsed at this point. But depending upon the energy level of the group and the time remaining, it might be possible to walk through rudimentary proofs of some of their conjectures. Some questions for further study might include the following.

1. What sort of graphs arise for $N = 3p$ or $N = p^3$? What about $N = 4p$, $N = pq$, or $N = pqr$ for distinct primes p, q and r?
2. Closer scrutiny reveals that any two vertices in a multiplication graph are either directly connected or else connected via one other vertex. Prove this fact. (Hard)
3. Is it possible to obtain a cycle graph as the multiplication graph of some master number? Answer: NO. (Hard, unless the previous question has been discussed.)
4. Is it the case that different values of N always give rise to different graphs, assuming that we include loops? (Very hard; I'll leave this as an open question.)