MATH 151 Honors, Fall 2013

FINAL EXAMINATION - SOLUTIONS

Part 1 – Multiple Choice (12 questions, 4 points each, No Calculators)

Write your name and section number on the ScanTron form.
Mark your responses on the ScanTron form and on the exam itself

1. Find the 2017-th derivative of $f(x) = x\ln x$

a. $2014!\,x^{-2015}$
b. $-2016!\,x^{-2017}$
c. $2015!\,x^{-2016}$
d. $2016!\,x^{-2017}$
e. $-2015!\,x^{-2016}$ Correct Choice

SOLUTION: $f'(x) = \ln x + 1$ $f''(x) = x^{-1}$ $f'''(x) = -x^{-2}$ $f^{(4)}(x) = 2x^{-3}$
$f^{(5)}(x) = -3 \cdot 2x^{-4}$ $f^{(2017)}(x) = -2015!\,x^{-2016}$

2. Evaluate $\lim\limits_{x\to\infty}\left(\sqrt{x^2-4x+5} - \sqrt{x^2+4x+5}\right)$

a. $-\infty$
b. -4 Correct Choice
c. 0
d. 2
e. 4

SOLUTION: $\lim\limits_{x\to\infty}\left(\sqrt{x^2-4x+5} - \sqrt{x^2+4x+5}\right)\dfrac{\left(\sqrt{x^2-4x+5} + \sqrt{x^2+4x+5}\right)}{\left(\sqrt{x^2-4x+5} + \sqrt{x^2+4x+5}\right)}$

$$= \lim_{x\to\infty}\frac{(x^2-4x+5)-(x^2+4x+5)}{\left(\sqrt{x^2-4x+5} + \sqrt{x^2+4x+5}\right)} = \lim_{x\to\infty}\frac{-8x}{\left(\sqrt{x^2-4x+5} + \sqrt{x^2+4x+5}\right)}\frac{x^{-1}}{x^{-1}} = \frac{-8}{2} = -4$$

3. Find the line tangent to the curve $y^3 = x^2 - xy$ at $(x,y) = (-2,2)$. Its y-intercept is

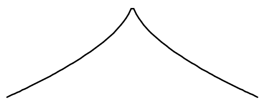
a. $-\frac{4}{5}$
b. $-\frac{3}{5}$
c. $\frac{3}{5}$
d. $\frac{4}{5}$ Correct Choice
e. 2

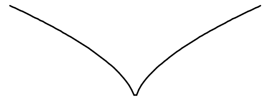
SOLUTION: By implicit differentiation:

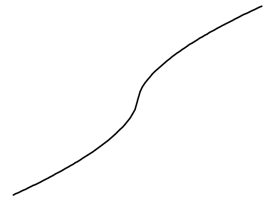
$$3y^2\frac{dy}{dx} = 2x - y - x\frac{dy}{dx} \qquad 12\frac{dy}{dx} = -4-2+2\frac{dy}{dx} \qquad 10\frac{dy}{dx} = -6 \qquad \left.\frac{dy}{dx}\right|_{(-2,2)} = \frac{-3}{5}$$

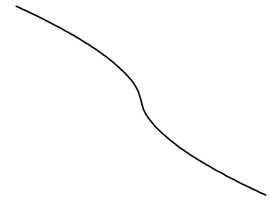
Tan Line: $y = f(-2) + f'(-2)(x+2) = 2 - \frac{3}{5}(x+2) = -\frac{3}{5}x + \frac{4}{5}$ $\quad b = \frac{4}{5}$

4. Near $x = 2$, the graph of $f(x) = (x-2)^{3/5}$ has the shape:

a. 

b. 

c.  Correct Choice

d. 

e None of these

SOLUTION: $f'(x) = \frac{3}{5}(x-2)^{-2/5}$

$\lim_{x \to 2^-} f'(x) = \text{“}\frac{3}{5(0^-)^{2/5}}\text{”} = \infty \qquad \lim_{x \to 2^+} f'(x) = \text{“}\frac{3}{5(0^+)^{2/5}}\text{”} = \infty$

5. $\lim_{x \to 0} \frac{x\cos x - x}{\sin x - x} =$

a. -3
b. $-\frac{3}{2}$
c. $\frac{1}{2}$
d. $\frac{3}{2}$
e. 3 Correct Choice

SOLUTION:

$\lim_{x \to 0} \frac{x\cos x - x}{\sin x - x} \overset{\text{l'H}}{=} \lim_{x \to 0} \frac{\cos x - x\sin x - 1}{\cos x - 1} \overset{\text{l'H}}{=} \lim_{x \to 0} \frac{-\sin x - \sin x - x\cos x}{-\sin x} \overset{\text{l'H}}{=} \lim_{x \to 0} \frac{-2\cos x - \cos x + x\sin x}{-\cos x} = 3$

6. When a dead body was found, its temperature was $85°F$. After 2 hours its temperature dropped to $80°F$ while the ambient temperature was $75°F$. How long before the victim was found was its temperature $95°F$.

a. 6 hours
b. 4 hours
c. 2 hours Correct Choice
d. 1 hour
e. $.5$ hours

SOLUTION: $T(t) = T_0 + Ae^{-kt} \qquad T_0 = 75 \qquad T(0) = 85 = 75 + Ae^0 \qquad A = 10$

$T(t) = 75 + 10e^{-kt} \qquad T(2) = 80 = 75 + 10e^{-k2} \qquad e^{-k2} = \frac{1}{2} \qquad k = \frac{-1}{2}\ln\frac{1}{2} = \frac{\ln 2}{2}$

$T(t) = 75 + 10e^{-t(\ln 2)/2} \qquad T(t_M) = 95 = 75 + 10e^{-t_M(\ln 2)/2} \qquad e^{-t_M(\ln 2)/2} = 2$

$-t_M(\ln 2)/2 = \ln 2 \qquad t_M = -2$

7. Find all horizontal asymptotes of $f(x) = \tanh(x)$.

a. $y = 1$ and $y = -1$ only   Correct Choice
b. $y = 1$ only
c. $y = \frac{\pi}{2}$ only
d. $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ only
e. $y = -\frac{\pi}{2}$ only

SOLUTION: $\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1$
$\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to-\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = -1$

8. If the initial position and velocity of a car are $x(0) = 2$ and $v(0) = 3$ and its accereration is $a(t) = e^{-t} + 6t$ find its position at $t = 2$.

a. $-e^{-2} + 16$
b. $-e^{-2} + 17$
c. $e^{-2} + 13$
d. $e^{-2} + 16$
e. $e^{-2} + 17$ Correct Choice

SOLUTION: $v(t) = -e^{-t} + 3t^2 + C$ $v(0) = -1 + C = 3$ $C = 4$ $v(t) = -e^{-t} + 3t^2 + 4$
$x(t) = e^{-t} + t^3 + 4t + K$ $x(0) = 1 + K = 2$ $K = 1$ $x(t) = e^{-t} + t^3 + 4t + 1$
$x(2) = e^{-2} + 8 + 8 + 1 = e^{-2} + 17$

9. Write the integral $\int_2^5 \sqrt{4+x}\,dx$ as a limit of Riemann sums with intervals of equal length and right endpoints.

a. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{4 + \frac{5i}{n}}$
b. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{6 + \frac{3i}{n}}$ Correct Choice
c. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{6 + \frac{5i}{n}}$
d. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{4 + \frac{3i}{n}}$
e. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{n}\sqrt{6 + \frac{5i}{n}}$

SOLUTION: $\Delta x = \frac{b-a}{n} = \frac{5-2}{n} = \frac{3}{n}$ $x_i^* = x_i = a + i\Delta x = 2 + \frac{3i}{n}$ $f(x) = \sqrt{4+x}$
$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ $\int_2^5 \sqrt{4+x}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{6 + \frac{3i}{n}}$

10. Find the points of inflection of the function $f(x) = \frac{1}{4}x^5 - \frac{5}{4}x^4$.

- **a**. $x = 3$ only   Correct Choice
- **b**. $x = 0$ and $x = 3$ only
- **c**. $x = -\sqrt{3}$ and $x = \sqrt{3}$ only
- **d**. $x = 0$ only
- **e**. $x = -3$ and $x = 0$ only

SOLUTION: $f'(x) = \frac{5}{4}x^4 - 5x^3$ $f''(x) = 5x^3 - 15x^2 = 5x^2(x-3) = 0$

Potential inflection points are $x = 0, 3$ but the concavity does not change at $x = 0$.

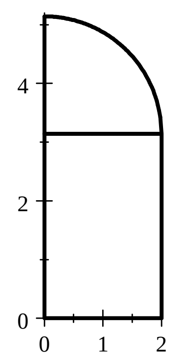
11. Evaluate $\int_0^2 \pi + \sqrt{4 - x^2}\, dx$

- **a**. π
- **b**. 2π
- **c**. 3π Correct Choice
- **d**. 4π
- **e**. 6π

SOLUTION:

$\int_0^2 \pi + \sqrt{4 - x^2}\, dx$ gives the area under the curve at the right which consites of a quarter circle of radius 2 on top of a rectangle of width 2 and height π. So

$\int_0^2 \pi + \sqrt{4 - x^2}\, dx = 2\pi + \frac{1}{4}\pi 2^2 = 3\pi$



12. Find the area under $y = \sqrt{x}$ between $x = 1$ and $x = 4$.

- **a**. $\frac{21}{2}$
- **b**. $\frac{27}{2}$
- **c**. 4
- **d**. $\frac{14}{3}$ Correct Choice
- **e**. 6

SOLUTION: $A = \int_1^4 x^{1/2}\, dx = \frac{2x^{3/2}}{3}\Big|_1^4 = \frac{2}{3}(8 - 1) = \frac{14}{3}$

Part 2 – Work Out Problems (4 questions. Points indicated. No Calculators)

Solve each problem in the space provided. Show all your work neatly and concisely, and indicate your final answer clearly. You will be graded, not merely on the final answer, but also on the quality and correctness of the work leading up to it.

13. (10 points) Let $f(x) = \sinh(x)$ and $g(x) = \operatorname{arcsinh}(x)$ which is the inverse function of $f(x)$.

a. Find $g'\left(\frac{e - \frac{1}{e}}{2}\right)$.

SOLUTION: If $b = g\left(\frac{e - \frac{1}{e}}{2}\right)$ then $\frac{e - \frac{1}{e}}{2} = f(b) = \sinh(b) = \frac{e^b - e^{-b}}{2}$. So $b = 1$.

Further $f'(x) = \cosh(x)$ and $f'(1) = \cosh(1) = \frac{e + \frac{1}{e}}{2}$. So

$g'\left(\frac{e - \frac{1}{e}}{2}\right) = \frac{1}{f'(1)} = \frac{2}{e + \frac{1}{e}}$.

b. Find a formula for $\operatorname{arcsinh}(x)$ which does not involve trig or hyperbolic trig functions or their inverses. The answer can only involve + - * / ^ $\sqrt[n]{}$ log.
HINT: Write the equation $y = \sinh(x)$ as a quadratic equation for $z = e^x$. Solve for z and then for x.

SOLUTION: Let $z = e^x$.

$y = \sinh(x) = \frac{e^x - e^{-x}}{2} = \frac{z - \frac{1}{z}}{2}$ $\qquad 2y = z - \frac{1}{z} \qquad 2yz = z^2 - 1 \qquad z^2 - 2yz - 1 = 0$

$z = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1}$ Since $z = e^x > 0$, we need the $+$ sign.

So $z = e^x = y + \sqrt{y^2 + 1}$ and $x = \ln\left(y + \sqrt{y^2 + 1}\right) = \operatorname{arcsinh}(y)$.

Finally, replace y by x: $\quad \operatorname{arcsinh}(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$

14. (30 points) Analyze the graph of the function $f(x) = \frac{\ln(x)}{x}$. Find each of the following. If none or nowhere or undefined, say so.

a. $\lim_{x \to 0^+} \frac{\ln(x)}{x} = \frac{-\infty}{0^+} = -\infty$ 1

b. $\lim_{x \to \infty} \frac{\ln(x)}{x} = \lim_{x \to \infty} \frac{\ln(x)}{x} \overset{\text{l'H}}{=} \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$ 1

c. x-intercepts: $y = \frac{\ln(x)}{x} = 0$ when $\ln x = 0$, i.e. $x = e^0 = 1$ 1

d. y-intercepts: none 1

e. horizontal asymptotes: $y = 0$ 1

f. vertical asymptotes: $x = 0$ 1

g. $f'(x) = \frac{x\frac{1}{x} - \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2}$ 1

h. $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \frac{1 - \ln(x)}{x^2} = \frac{1 - -\infty}{(0^+)^2} = \infty$ 1

i. $\lim_{x \to \infty} f'(x) = \lim_{x \to \infty} \frac{1 - \ln(x)}{x^2} \overset{\text{l'H}}{=} \lim_{x \to \infty} \frac{-\frac{1}{x}}{2x} = \lim_{x \to \infty} \frac{-1}{2x^2} = 0$ 2

j. intervals where increasing: $f'(x) = \frac{1 - \ln(x)}{x^2} = 0$ when $\ln(x) = 1$ $x = e$

$f'(1) = 1 > 0$ increasing on $(0, e)$ 2

k. intervals where decreasing: $f'(e^2) = \frac{1 - 2}{e^4} < 0$ decreasing on (e, ∞) 1

l. local maxima (x and y coordinates): $f(e) = \frac{\ln(e)}{e} = \frac{1}{e}$

local maximum at $\left(e, \frac{1}{e}\right)$ 2

m. local minima (x and y coordinates): none 1

n. $f''(x) = \dfrac{x^2\left(-\frac{1}{x}\right) - (1-\ln(x))2x}{x^4} = \dfrac{-x - 2x + 2x\ln(x)}{x^4} = \dfrac{-3 + 2\ln(x)}{x^3}$ 2

o. $\lim_{x\to 0^+} f''(x) = \lim_{x\to 0^+} \dfrac{-3+2\ln(x)}{x^3} = \dfrac{-3+2(-\infty)}{(0^+)^3} = -\infty$ 1

p. $\lim_{x\to\infty} f''(x) = \lim_{x\to\infty} \dfrac{-3+2\ln(x)}{x^3} \overset{\text{l'H}}{=} \lim_{x\to\infty} \dfrac{\frac{2}{x}}{3x^2} = 0$ 2

q. intervals where concave down:

$f''(x) = \dfrac{-3+2\ln(x)}{x^3} = 0$ when $\ln(x) = \dfrac{3}{2}$ $\quad x = e^{3/2}$

$f''(e) = \dfrac{-3+2}{e^3} > 0$ So concave down on $(0, e^{3/2})$. 2

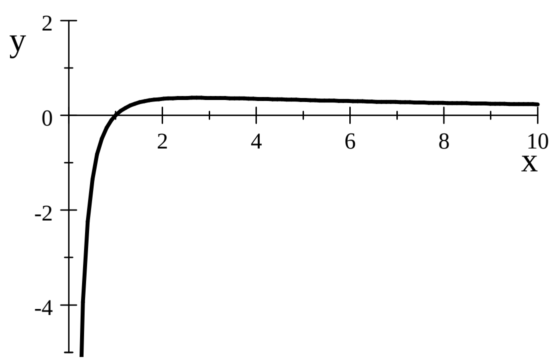
r. intervals where concave up: $f''(e^2) = \dfrac{-3+2\cdot 2}{e^6} > 0$

So concave up on $(e^{3/2}, \infty)$. 1

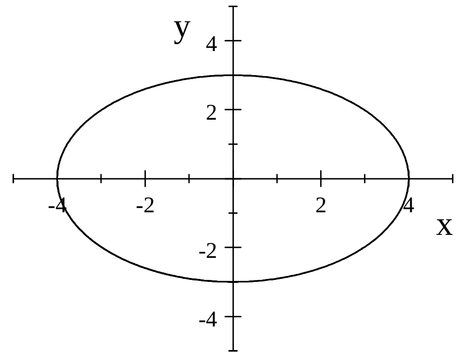
s. inflection points $(x$ and y coordinates$)$: $f(e^{3/2}) = \dfrac{\ln(e^{3/2})}{e^{3/2}} = \dfrac{3}{2e^{3/2}}$

inflection point at $\left(e^{3/2}, \dfrac{3}{2e^{3/2}}\right)$ 2

t. Roughly graph the function: 4



15. (5 points) The ellipse whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ crosses the x-axis at $x = \pm a$ and the y-axis at $y = \pm b$ and has area $A = \pi ab$.



For example, the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ with $a = 4$ cm and $b = 3$ cm is shown at the right and has area $A = 12\pi$.

If a is currently 4 cm and decreasing at $\frac{da}{dt} = -2\ \frac{\text{cm}}{\text{min}}$,

while b is currently 3 cm and increasing at $\frac{db}{dt} = 1\ \frac{\text{cm}}{\text{min}}$,

is the area increasing or decreasing and at what rate?

SOLUTION: $A = \pi ab \qquad \frac{dA}{dt} = \pi\frac{da}{dt}b + \pi a\frac{db}{dt} = \pi(-2)(3) + \pi(4)(1) = -2\pi\ \frac{\text{cm}}{\text{min}}$

A is decreasing.

16. (10 points) A paint can needs to hold a liter of paint (1000 cm^3). The shape will be a cylinder of radius r and height h. The sides and bottom will be made from aluminum which costs 15¢ per cm^2. The top will be made from plastic which costs 5¢ per cm^2. What are the DIMENSIONS and COST of the cheapest such can? (Keep answers in terms of π.)

SOLUTION: $V = \pi r^2 h = 1000 \qquad h = \frac{1000}{\pi r^2}$

$$C = 15(2\pi rh + \pi r^2) + 5\pi r^2 = 30\pi rh + 20\pi r^2 = \frac{30000}{r} + 20\pi r^2$$

$$C' = -\frac{30000}{r^2} + 40\pi r = 0 \qquad 40\pi r = \frac{30000}{r^2} \qquad r^3 = \frac{30000}{40\pi} = \frac{750}{\pi}$$

$$r = \left(\frac{750}{\pi}\right)^{1/3} \qquad h = \frac{1000}{\pi\left(\frac{750}{\pi}\right)^{2/3}} = \frac{1000}{750}\left(\frac{750}{\pi}\right)^{1/3} = \frac{4}{3}\left(\frac{750}{\pi}\right)^{1/3}$$

$$C = \frac{30000}{r} + 20\pi r^2 = \frac{30000}{\left(\frac{750}{\pi}\right)^{1/3}} + 20\pi\left(\frac{750}{\pi}\right)^{2/3} = 40\pi\left(\frac{750}{\pi}\right)^{2/3} + 20\pi\left(\frac{750}{\pi}\right)^{2/3} = 60\pi\left(\frac{750}{\pi}\right)^{2/3}$$

OR $C = 30\pi rh + 20\pi r^2 = 30\pi\left(\frac{750}{\pi}\right)^{1/3}\frac{4}{3}\left(\frac{750}{\pi}\right)^{1/3} + 20\pi\left(\frac{750}{\pi}\right)^{2/3} = 60\pi\left(\frac{750}{\pi}\right)^{2/3}$

To see that C is the absolute minimum, we note there is only one critical point and $C = \frac{30000}{r} + 20\pi r^2$ approaches $+\infty$ as r approaches 0^+ or ∞.