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MATH 171 Exam 2 Fall 2021

Sections 503 Solutions P. Yasskin

1-10	/55	12	/10
11	/25	13	/15
		Total	/105

Multiple Choice: (5 points each. No part credit.)

1. If $f(x) = e^{3x} - 4e^{2x} + 2x^4 + \sin(2x) + \ln(1-3x)$, find $f'(0)$.

Solution: $f'(x) = 3e^{3x} - 8e^{2x} + 8x^3 + 2\cos(2x) + \dfrac{-3}{1-3x}$

So $f'(0) = 3e^0 - 8e^0 + 8 \cdot 0^3 + 2\cos(0) + \dfrac{-3}{1} = 3 - 8 + 2 - 3 = \underline{\quad -6 \quad}$.

2. If $F(x) = f'(g(x))$, where

$$\begin{array}{llll} f(2) = 2 & f'(2) = 3 & g(2) = 4 & g'(2) = 5 \\ f(4) = 4 & f'(4) = 5 & g(4) = 6 & g'(4) = 7 \end{array}$$

find $F'(2)$.

Solution: $F'(x) = f'(g(x))g'(x) \qquad F'(2) = f'(g(2))g'(2) = f'(4)g'(2) = 5 \cdot 5 = \underline{\quad 25 \quad}$.

3. If $g(x) = x\cos(\pi x)$, find $g'\left(\frac{1}{2}\right)$. (Type "pi" for π. Type "sqrt(3)" for $\sqrt{3}$.)

Solution: $g'(x) = \cos(\pi x) - x\pi\sin(\pi x) \qquad g'\left(\frac{1}{2}\right) = \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = \underline{\quad -\dfrac{\pi}{2} \quad}$.

4. Find the slope of the curve $xy^2 + x^2y^3 = 6$ at the point $(x,y) = (2,1)$.

Solution: We apply $\dfrac{d}{dx}$ to both sides of the equation and then evaluate at $(2,1)$:

$$y^2 + 2xy\frac{dy}{dx} + 2xy^3 + 3x^2y^2\frac{dy}{dx} = 0$$

$$1 + 4\frac{dy}{dx} + 4 + 12\frac{dy}{dx} = 0$$

So $\left.\dfrac{dy}{dx}\right|_{(2,1)} = \underline{\quad -\dfrac{5}{16} \quad}$.

5. (10 points) Consider the parametric curve $\vec{r}(t) = \langle t^3 + 3t, t^3 - 3t \rangle$.

a. Find the position at time $t = 2$.

Solution: $\vec{r}(2) = \langle 2^3 + 3 \cdot 2, 2^3 - 3 \cdot 2 \rangle = \langle$__14___, ___2__$\rangle$

b. Find the velocity at time $t = 2$.

Solution: $\vec{v}(t) = \langle 3t^2 + 3, 3t^2 - 3 \rangle$ $\quad \vec{v}(2) = \langle 12 + 3, 12 - 3 \rangle = \langle$__15___, ___9__$\rangle$

c. Find the parametric tangent line at $t = 2$.

Solution:
$X = P + t\vec{v} = \vec{r}(2) + t\vec{v}(2)$ $\quad (x,y) = \langle 14,2 \rangle + t\langle 15,9 \rangle = \langle$__$14 + 15t$___, ___$2 + 9t$__$\rangle$

d. Find the slopes at times $t = 2$ and $t = 1$.

Solution: $m = \frac{v_2}{v_1}$ $\quad m(2) == \frac{9}{15} =$ __$\frac{3}{5}$__ $\quad m(1) == \frac{0}{6} =$ ___0___

e. Find the time(s) at which the curve is horizontal.

Solution: At a general time the slope is $m = \frac{v_2}{v_1} = \frac{3t^2 - 3}{3t^2 + 3} = \frac{t^2 - 1}{t^2 + 1}$

The slope is 0 when $t^2 - 1 = 0$ or $t =$ ___-1___ and $t =$ ___1___

6. If $f(x) = \sqrt{25 - x^2} + \arcsin\left(\frac{x}{5}\right)$, then $f'(3) =$

a. 1

b. $\frac{1}{2}$

c. $-\frac{1}{4}$

d. $-\frac{1}{2}$ Correct Choice

e. $-\frac{3}{4}$

Solution: We use $\frac{d}{du}\sqrt{u} = \frac{1}{2\sqrt{u}}$ and $\frac{d}{du}\arcsin u = \frac{1}{\sqrt{1 - u^2}}$. So by the chain rule

$$f'(x) = \frac{-2x}{2\sqrt{25 - x^2}} + \frac{\frac{1}{5}}{\sqrt{1 - \left(\frac{x}{5}\right)^2}} = \frac{-x}{\sqrt{25 - x^2}} + \frac{1}{\sqrt{25 - x^2}} = \frac{1 - x}{\sqrt{25 - x^2}}$$

$$f'(3) = \frac{1 - 3}{\sqrt{25 - 3^2}} = \frac{-2}{4} = -\frac{1}{2}$$

7. Notice that the derivative of $f(x) = x + x^3 + x^5$ is always positive. So it is always increasing and is 1-to-1. So it has an inverse $g(x)$. Find $g'(3)$.

HINT: $f(-1) = -3 \quad f(0) = 0 \quad f(1) = 3 \quad f(2) = 42 \quad f(3) = 273$

Solution: Notice that $f(1) = 3$. So $g(3) = 1$ and so $g'(3) = \dfrac{1}{f'(g(3))} = \dfrac{1}{f'(1)}$.

Now $f'(x) = 1 + 3x^2 + 5x^4$. So $f'(1) = 1 + 3 + 5 = 9$. So $g'(3) = \dfrac{1}{f'(1)} = \underline{\dfrac{1}{9}}$.

8. The distance from Houston to Dallas is 240 miles. The highest speed limit for the entire trip is $75 \dfrac{\text{miles}}{\text{hour}}$. An Aggie makes the trip in 3 hours. Which theorem says that the Aggie was speeding at some point along the trip?

a. The Squeeze Theorem
b. The Mean Value Theorem Correct Choice
c. The Intermediate Value Theorem
d. Rolle's Theorem

Solution: The Aggie's average velocity is $v_{\text{ave}} = \dfrac{x_{\text{final}} - x_{\text{initial}}}{t_{\text{final}} - t_{\text{initial}}} = \dfrac{240 \text{ miles}}{3 \text{ hours}} = 80 \dfrac{\text{miles}}{\text{hour}}$.
On a graph of position, x, as a function of time, t, the slope of a tangent line is the velocity, $v(t) = \dfrac{dx}{dt}$, while the average velocity is the slope of the secant line.
The Mean Value Theorem says that at some time $t = c$ between t_{initial} and t_{final} the slope of the tangent line, $v(c)$, equals the slope of the secant line, $v_{\text{ave}} = 80 \dfrac{\text{miles}}{\text{hour}}$. So at $t = c$ the Aggie was speeding.

9. The side of a cube is measured to be $s = 20\,\text{cm} \pm 0.05\,\text{cm}$. So the volume of the cube is $V = s^3 \pm \Delta V = 8000\,\text{cm}^3 \pm \Delta V$. Using the linear approximation, what is the error ΔV in this computation of the volume.

Solution: $\Delta V \approx dV = \dfrac{dV}{ds} ds = 3s^2 \Delta s = 3(20)^2(0.05) = 1200 \cdot .05 = \underline{\quad 60 \quad}\ \text{cm}^3$

10. If the position function is $x(t) = \sin(t^2)$, find the jerk at $t = 1$.
Note: the jerk is $j(t) = \dfrac{d^3x}{dt^3}$.

Solution: $v(t) = \dfrac{dx}{dt} = 2t\cos(t^2) \qquad a(t) = \dfrac{d^2x}{dt^2} = 2\cos(t^2) - 4t^2\sin(t^2)$

$j(t) = \dfrac{d^3x}{dt^3} = -4t\sin(t^2) - 8t\sin(t^2) - 8t^3\cos(t^2)$

$j(1) = -4\sin(1) - 8\sin(1) - 8\cos(1) = \underline{\quad -12 \quad} \sin 1 + \underline{\quad -8 \quad} \cos 1 \underline{\quad}$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (25 points) Consider the function $f(x) = \frac{1}{5}x^5 - x^4 + 3$.

a. (3 pts) Find $f'(x) =$ ______________

Solution: $f'(x) = x^4 - 4x^3$

b. (3 pts) Find $f''(x) =$ ______________

Solution: $f''(x) = 4x^3 - 12x^2$

c. (2 pts) Find all critical points of f, i.e all values of x at which $f'(x) = 0$.

Solution: $f'(x) = x^4 - 4x^3 = x^3(x-4)$ $\quad x = 0, 4$

d. (3 pts) Find the intervals where f is increasing and decreasing.

Solution: We test $f'(x)$ at a point in each of the intervals $(-\infty, 0)$, $(0, 4)$, $(4, \infty)$:
$f'(-1) = (-1)^3(-1-4) = 5 > 0$ $\quad f'(1) = (1)^3(1-4) = -3 < 0$ $\quad f'(5) = (5)^3(5-4) = 5^3 > 0$
increasing on: $(-\infty, 0)$ and $(4, \infty)$ decreasing on: $(0, 4)$

e. (2 pts) Find all secondary critical points of f, i.e all values of x at which $f''(x) = 0$.

Solution: $f''(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ $\quad x = 0, 3$

f. (3 pts) Find the intervals where f is concave up and concave down.

Solution: We test $f''(x)$ at a point in each of the intervals $(-\infty, 0)$, $(0, 3)$, $(3, \infty)$:
$f''(-1) = 4(-1)^2(-4) = -16 < 0$ $\quad f''(1) = 4(1)^2(-2) = -8 < 0$ $\quad f''(4) = 4 \cdot 4^2(1) = 4^3 > 0$
concave up on: $(3, \infty)$ concave down on: $(-\infty, 0)$ and $(0, 3)$

g. (4 pts) What does the Second Derivative Test say about each critical point?

Solution: $f''(0) = 4 \cdot 0^2(0-3) = 0$ Test FAILS at $x = 0$.
$f''(4) = 4 \cdot 4^2(4-3) = 4^3 > 0$ $\quad x = 4$ is a local minimum

h. (3 pts) Find the x location of all local minima and local maxima of f.

Solution: We already know $x = 4$ is a local minimum.
We apply the First Derivative Test to $x = 0$:
f is increasing on: $(-\infty, 0)$ and decreasing on: $(0, 4)$
So $x = 0$ is a local maximum.

i. (2 pts) Find the x location of all inflection points of f.

Solution: f is concave down on: $(0, 3)$ and concave up on: $(3, \infty)$
So $x = 3$ is an inflection point.

12. (10 points) A weather balloon is currently at $x_0 = 2490$ meters from the weather station and currently has velocity $v = 4 \frac{\text{meters}}{\text{hour}}$. The balloon measures the temperature is currently $T_0 = 78^\circ$F and has derivative $\frac{dT}{dx} = 0.2 \frac{^\circ\text{F}}{\text{meter}}$.

a. (4 pts) What is $\frac{dT}{dt}$, i.e. the current time rate of change of the temperature?

Solution: Since $v = \frac{dx}{dt}$, we have $\frac{dT}{dt} = \frac{dT}{dx}\frac{dx}{dt} = 0.2 \cdot 4 = 0.8 \frac{^\circ\text{F}}{\text{meter}}$

b. (3 pts) What will be the approximate position x_1 of the balloon after $\frac{1}{2}$ hour?

Solution: $x_1 = x_o + \frac{dx}{dt}\Delta t = 2490 + 4 \cdot \frac{1}{2} = 2492$ meters

c. (3 pts) What will be the approximate temperature T_1 at the location of the balloon after $\frac{1}{2}$ hour?

Solution: $T_1 = T_0 + \frac{dT}{dt}\Delta t = 78 + 0.8 \cdot \frac{1}{2} = 78.4$

13. (15 points) Compute the derivatives of the following functions.

a. $p(t) = \sin^3(\cos(t^2))$

Solution: The outer function is cubing. Inside that there is $\sin$. Inside that is $\cos$ and inside that is squaring. So:
$p'(t) = 3\sin^2(\cos(t^2))[\cos(\cos(t^2))][-\sin(t^2)]2t$

b. $g(x) = \csc(\arcsin(x^2))$ (There cannot be any trig or arctrig functions in your answer.)

Solution: This can be done by chain rule, but it is easier to simplify first:
Let $\theta = \arcsin(x^2)$. then $\sin\theta = x^2$. Draw a triangle with opposite side x^2 and hypotenuse 1. Then the adjacent side is $\sqrt{1-x^4}$. So $\csc\theta = \frac{1}{x^2}$. Therefore
$g(x) = \csc(\arcsin(x^2)) = \csc\theta = \frac{1}{x^2}$ and $g'(x) = \frac{-2}{x^3}$.

c. $f(x) = \frac{(x-2)^{12}(x+1)^{10}}{(x-3)^8}$ Find $f'(1)$.

Solution: We use logarithmic differentiation:

$$\ln f(x) = 12\ln(x-2) + 10\ln(x+1) - 8\ln(x-3)$$

$$\frac{f'(x)}{f(x)} = \frac{12}{x-2} + \frac{10}{x+1} - \frac{8}{x-3}$$

$$f'(x) = \frac{(x-2)^{12}(x+1)^{10}}{(x-3)^8}\left(\frac{12}{x-2} + \frac{10}{x+1} - \frac{8}{x-3}\right)$$

$$f'(1) = \frac{(-1)^{12}(2)^{10}}{(-2)^8}\left(\frac{12}{-1} + \frac{10}{2} - \frac{8}{-2}\right) = 4(-12+5+4) = -12$$