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1-11	/55	14	/10
12	/20	15	/10
13	/10	Total	/105

MATH 171 Final Exam Fall 2021

Sections 503 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. For what value(s) of p are the vectors $\vec{a} = (3,p)$ and $\vec{b} = (4,6)$, perpendicular?

a. $\frac{1}{2}$ only

b. $-\frac{1}{2}$ only

c. 2 only

d. -2 only Correct Choice

e. 0 only

f. $\frac{1}{2}$ or $-\frac{1}{2}$ only

g. 2 or -2 only

h. 2 or $\frac{1}{2}$ only

i. -2 or $-\frac{1}{2}$ only

j. no value of p

Solution: $\vec{a}$ and $\vec{b}$ are perpendicular iff $\vec{a} \cdot \vec{b} = 0$. In this case, $\vec{a} \cdot \vec{b} = 12 + 6p = 0$. So $p = -2$.

2. Compute $\lim\limits_{x\to 3} \dfrac{x^2 - 4x + 3}{x^2 - 9}$

a. $-\frac{1}{9}$

b. $-\frac{1}{3}$

c. $-\frac{1}{6}$

d. 0

e. $\frac{1}{6}$

f. $\frac{1}{3}$ Correct Choice

g. $\frac{1}{2}$

h. $\frac{2}{3}$

i. 1

j. undefined

Solution: $\lim\limits_{x\to 3} \dfrac{x^2 - 4x + 3}{x^2 - 9} = \lim\limits_{x\to 3} \dfrac{(x-3)(x-1)}{(x-3)(x+3)} = \lim\limits_{x\to 3} \dfrac{(x-1)}{(x+3)} = \dfrac{2}{6} = \dfrac{1}{3}$

3. As $x \to \infty$, the function $f(x) = \sqrt{x^2 + 5x} - \sqrt{x^2 + 2x}$ has a horizontal asymptote at

a. $y = -\infty$

b. $y = -3$

c. $y = -\frac{3}{2}$

d. $y = -\frac{1}{2}$

e. $y = 0$

f. $y = \frac{1}{2}$

g. $y = \frac{3}{2}$ Correct Choice

h. $y = 1$

i. $y = 3$

j. $y = \infty$

Solution: $\lim\limits_{x\to\infty}\left(\sqrt{x^2+5x} - \sqrt{x^2+2x}\right) = \lim\limits_{x\to\infty}\left(\sqrt{x^2+5x} - \sqrt{x^2+2x}\right) \cdot \dfrac{\sqrt{x^2+5x} + \sqrt{x^2+2x}}{\sqrt{x^2+5x} + \sqrt{x^2+2x}}$

$$= \lim_{x\to\infty} \frac{(x^2+5x) - (x^2+2x)}{\sqrt{x^2+5x} + \sqrt{x^2+2x}} = \lim_{x\to\infty} \frac{3x}{\sqrt{x^2+5x} + \sqrt{x^2+2x}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{3}{\sqrt{1+\frac{5}{x}} + \sqrt{1+\frac{2}{x}}} = \frac{3}{2}$$

4. The limit $\lim_{h\to 0}\dfrac{4(2+h)^3-32}{h}$ can be interpreted as which of the following?

a. $f'(2)$ where $f(x) = x^3$ f. $f'(4)$ where $f(x) = x^3$ i. $f'(32)$ where $f(x) = x^3$

b. $f'(2)$ where $f(x) = 4x^3$ Correct g. $f'(4)$ where $f(x) = 4x^3$ j. $f'(32)$ where $f(x) = 4x^3$

c. $f'(2)$ where $f(x) = x^4$ h. $f'(4)$ where $f(x) = x^4$ k. $f'(32)$ where $f(x) = x^4$

d. $f'(2)$ where $f(x) = 12x^2$ i. $f'(4)$ where $f(x) = 12x^2$ l. $f'(32)$ where $f(x) = 12x^2$

Solution: $f(x+h) = 4(2+h)^3$ So $x = 2$ and $f(x) = 4x^3$ and $f(2) = 32$. So the limit is $f'(2)$.

5. Find the line tangent to $y = \frac{1}{4}x^4$ at $x = 2$. Its y-intercept is:

a. -20 f. 4

b. -16 g. 12

c. -12 Correct Choice h. 16

d. -4 i. 20

e. 0 j. none of these

Solution: $f(x) = \frac{1}{4}x^4$ $f'(x) = x^3$ $f(2) = 4$ $f'(2) = 8$

Tan Line: $y = f(2) + f'(2)(x-2) = 4 + 8(x-2) = 8x - 12$ $b = -12$

6. A spacecraft is being sent to Mars. Its distance from the earth is given by $p(t) = 7t^3 + 1$. At time $t = 2$ the position is measured, but the error in the time measurement is ± 0.1. What is the resulting error in the calculated position?

a. ± 7.3 f. ± 73

b. ± 7.4 g. ± 74

c. ± 8.4 Correct Choice h. ± 84

d. ± 8.5 i. ± 85

e. 0 j. Impossible to determine.

Solution: Using differentials, we find $dp = 21t^2 dt = 21 \cdot 2^2 \cdot (\pm 0.1) = \pm 8.4$.

7. Find the line tangent to the curve $y^3 = x^2 - xy$ at $(x,y) = (-2,2)$. Its y-intercept is:

a. $-\frac{3}{5}$ f. $\frac{3}{5}$

b. $-\frac{4}{5}$ g. $\frac{4}{5}$ Correct Choice

c. $-\frac{5}{4}$ h. $\frac{5}{4}$

d. $-\frac{5}{3}$ i. $\frac{5}{3}$

e. 0 j. 1

Solution: Since $(x,y) = (-2,2)$, we have $f(-2) = 2$. By implicit differentiation:

$$3y^2\frac{dy}{dx} = 2x - y - x\frac{dy}{dx} \qquad 12\frac{dy}{dx} = -4 - 2 + 2\frac{dy}{dx} \qquad 10\frac{dy}{dx} = -6 \qquad \left.\frac{dy}{dx}\right|_{(-2,2)} = \frac{-3}{5}$$

Tan Line: $y = f(-2) + f'(-2)(x+2) = 2 - \frac{3}{5}(x+2) = -\frac{3}{5}x + \frac{4}{5}$ $b = \frac{4}{5}$

8. Compute $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

a. $-\frac{1}{6}$ Correct Choice

b. $-\frac{1}{3}$

c. $-\frac{1}{2}$

d. 0

e. $\frac{1}{6}$

f. $\frac{1}{3}$

g. $\frac{1}{2}$

h. undefined

Solution: $\lim_{x\to 0} \frac{\sin x - x}{x^3} \overset{l'H}{=} \lim_{x\to 0} \frac{\cos x - 1}{3x^2} \overset{l'H}{=} \lim_{x\to 0} \frac{-\sin x}{6x} = \frac{-1}{6}$

9. Find the area under $y = \sqrt{x}$ between $x = 1$ and $x = 4$.

a. 1

b. 3

c. 4

d. 6

e. 12

f. $\frac{14}{3}$ Correct Choice

g. $\frac{16}{3}$

h. $\frac{28}{3}$

i. $\frac{21}{2}$

j. $\frac{21}{4}$

Solution: $A = \int_1^4 x^{1/2}\,dx = \frac{2x^{3/2}}{3}\Big|_1^4 = \frac{2}{3}(8-1) = \frac{14}{3}$

10. Compute $\int_0^\pi e^{\cos x}\sin x\,dx$

a. 0

b. $\frac{1}{e} - e$

c. $e - \frac{1}{e}$ Correct Choice

d. $-\frac{1}{e}$

e. $-e$

f. 1

g. $1-e$

h. $e-1$

i. $1-\frac{1}{e}$

j. $\frac{1}{e}-1$

Solution: $u = \cos x \qquad du = -\sin x\,dx$

$$\int_0^\pi e^{\cos x}\sin x\,dx = -\int e^u\,du = -e^u = \left[-e^{\cos x}\right]_0^\pi = -e^{-1} + e^1 = e - \frac{1}{e}$$

11. Calculate $\lim_{x\to 0^+} \frac{1}{x}\int_0^x e^{t^2}\,dt$

a. $-\infty$

b. $-e$

c. -1

d. $-\frac{1}{e}$

e. 0

f. $\frac{1}{e}$

g. 1 Correct Choice

h. e

i. e^2

j. ∞

Solution: By l'Hopital's rule and the FTC, $\lim_{x\to 0^+} \frac{1}{x}\int_0^x e^{t^2}\,dt = \lim_{x\to 0^+} \frac{e^{x^2}}{1} = e^0 = 1.$

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (20 points) Consider the function $g(x) = \frac{3}{3+x^2}$. Find each of the following and say why. (If an item does not exist, say NONE and say why not.) Then graph the function.

a. horizontal asymptote as $x \to \infty$:

Solution: $\lim_{x\to\infty} \frac{3}{3+x^2} = 0 \qquad y = 0$

b. horizontal asymptote as $x \to -\infty$:

Solution: $\lim_{x\to-\infty} \frac{3}{3+x^2} = 0 \qquad y = 0$

c. $g'(x)$ and critical points:

Solution: $g'(x) = \frac{-3(2x)}{(3+x^2)^2} = \frac{-6x}{(3+x^2)^2} = 0$ critical point at $x = 0$

d. Intervals where g is increasing and decreasing:

Solution: We test each interval:

$(-\infty, 0] \qquad g'(-1) = \frac{-6(-1)}{(3+1)^2} > 0$ increasing

$[0, \infty) \qquad g'(1) = \frac{-6(1)}{(3+1)^2} < 0$ decreasing

e. $g''(x)$ and secondary critical points:

Solution: $g''(x) = \frac{(3+x^2)^2(-6) + 6x2(3+x^2)2x}{(3+x^2)^4} = \frac{(3+x^2)(-6) + 24x^2}{(3+x^2)^3} = \frac{-18+18x^2}{(3+x^2)^3} = 0$

secondary critical points at $x = \pm 1$

f. Intervals where g is concave up and down:

Solution: We test each interval:

$(-\infty, -1] \qquad g''(-2) = \frac{-18+18(4)}{(3+4)^2} > 0$ concave up

$[-1, 1] \qquad g''(0) = \frac{-18}{(3)^2} < 0$ concave down

$[1, \infty) \qquad g''(2) = \frac{-18+18(4)}{(3+4)^2} > 0$ concave up

g. x-coordinate at each local maximum

Solution: Critical point at $x = 0$ is a local maximum
either because: $g'(x)$ changes from positive to negative
or because: $g''(0) = \dfrac{-18}{(3)^2} < 0.$

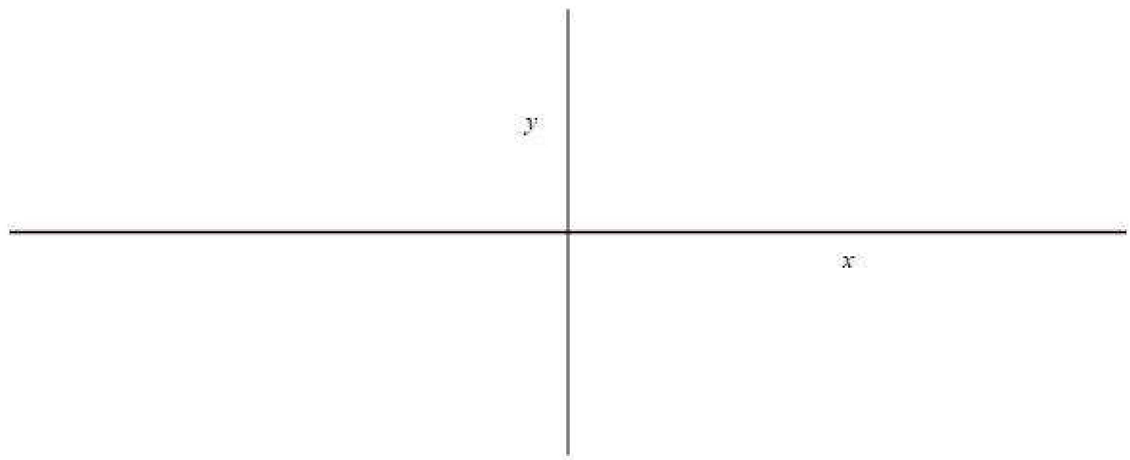
h. x-coordinate at each local minimum

Solution: No local minimum because no other critical point.

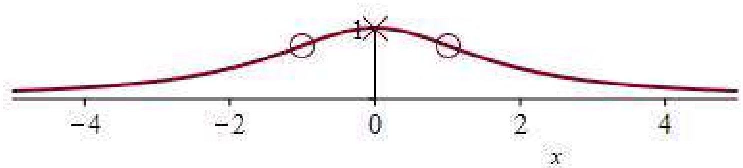
i. x-coordinate at each inflection point

Solution: The concavity $g''(x)$ changes at -1 and 1. Inflection points at $x = \pm 1$.

j. Plot: (Put an $\times$ at each local minimum or maximum. Put an ○ at each inflection point.)



Solution:



13. (10 points) Find the equation of the line tangent to $y = x^2$ at the general point $x = a$. For what value(s) of a does the tangent line pass through the point $(3,8)$?

Solution: $f(x) = x^2 \qquad f(a) = a^2 \qquad f'(x) = 2x \qquad f'(a) = 2a$

$y = f_{\tan}(x) = f(a) + f'(a)(x-a) = a^2 + 2a(x-a) = 2ax - a^2$

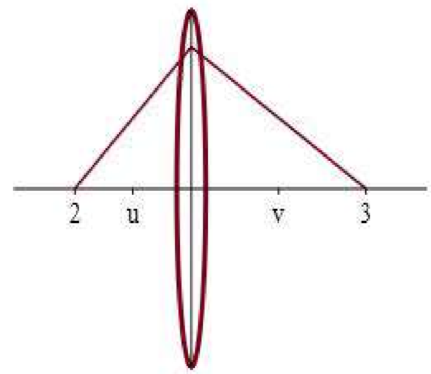
$(3,8)$ lies on the tangent line if $8 = 2a3 - a^2$, or $a^2 - 6a + 8 = 0$, or $(a-2)(a-4) = 0$

So $a = 2$ or $a = 4$.

14. (10 points) When light passes through a lens with focal length f the distance to the object, u, is related to the distance to the image, v, by the equation

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f}.$$

Here f is a constant. As u changes, v changes. If $f = \frac{6}{5}$, $u = 2$, and $\frac{du}{dt} = -0.4$, find v and $\frac{dv}{dt}$. Is v getting longer or shorter?

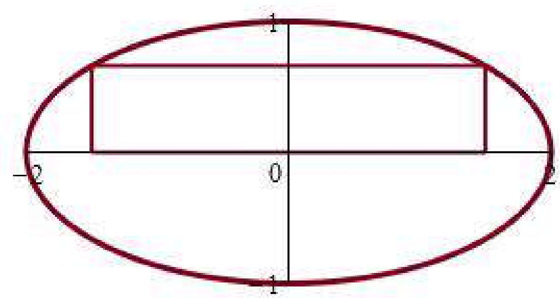


Solution: $\frac{1}{v} = \frac{1}{f} - \frac{1}{u} = \frac{5}{6} - \frac{1}{2} = \frac{1}{3} \qquad v = 3$

$-\frac{1}{u^2}\frac{du}{dt} - \frac{1}{v^2}\frac{dv}{dt} = 0 \qquad \frac{1}{v^2}\frac{dv}{dt} = -\frac{1}{u^2}\frac{du}{dt} \qquad \frac{dv}{dt} = -\frac{v^2}{u^2}\frac{du}{dt} = -\frac{9}{4}(-0.4) = 0.9$

v is getting longer.

15. (10 points) A rectangle is inscribed in the upper half of the ellipse $\frac{x^2}{4} + y^2 = 1$ with its base on the x-axis. Find the maximum area of such a rectangle.



Solution: $A = 2xy$ with $y = \sqrt{1 - \frac{x^2}{4}}$. So $A = 2x\sqrt{1 - \frac{x^2}{4}}$.

$A' = 2\sqrt{1 - \frac{x^2}{4}} + 2x\frac{1}{2}\frac{\frac{-2x}{4}}{\sqrt{1 - \frac{x^2}{4}}} = 0 \qquad$ Multiply by $\sqrt{1 - \frac{x^2}{4}}$

$2\left(1 - \frac{x^2}{4}\right) - \frac{x^2}{2} = 0$ or $2 - x^2 = 0 \qquad x = \sqrt{2}$

$y = \sqrt{1 - \frac{2}{4}} = \frac{1}{\sqrt{2}} \qquad A = 2xy = 2\left(\sqrt{2}\right)\left(\frac{1}{\sqrt{2}}\right) = 2$