Name________________________Section____________

MATH 171 Exam 1A Fall 2022

Section 502/504 Solutions P. Yasskin

Multiple Choice and Short Answer:

(Show your work in case there is part credit.)

1-7	/70	9	/ 5
8	/10	10	/15
		Total	/100

1. (5 points) Find the angle between the vectors $\vec{u} = \left\langle \sqrt{3}, 1 \right\rangle$ and $\vec{v} = \left\langle \sqrt{3}, 3 \right\rangle$.

$\theta =$ ______

Solution: $|\vec{u}| = \sqrt{3+1} = 2$ $\quad |\vec{v}| = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$ $\quad \vec{u} \cdot \vec{v} = 3 + 3 = 6$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \frac{6}{2 \cdot 2\sqrt{3}} = \frac{\sqrt{3}}{2} \qquad \theta = 30°$$

2. (10 points) Write $\vec{a} = \langle -1, 7 \rangle$ as the sum of two vectors $\vec{p}$ and $\vec{q}$ where $\vec{p}$ is parallel to $\vec{b} = \langle 8, -6 \rangle$ and $\vec{q}$ is perpendicular to $\vec{b} = \langle 8, -6 \rangle$.

$\vec{a} = \vec{p} + \vec{q}$ where....................................... $\vec{p} = \langle$ ______ , ______ $\rangle$ and $\vec{q} = \langle$ ______ , ______ $\rangle$

Solution: $\vec{p} = \text{proj}_{\vec{b}}\vec{a} = \dfrac{\vec{b} \cdot \vec{a}}{\left|\vec{b}\right|^2}\vec{b} = \dfrac{-8-42}{64+36}\langle 8, -6 \rangle = \dfrac{-50}{100}\langle 8, -6 \rangle = \langle -4, 3 \rangle$

$\vec{q} = \text{proj}_{\perp\vec{b}}\vec{a} = \vec{a} - \text{proj}_{\vec{b}}\vec{a} = \langle -1, 7 \rangle - \langle -4, 3 \rangle = \langle 3, 4 \rangle$

Check: $\vec{p} + \vec{q} = \langle -4, 3 \rangle + \langle 3, 4 \rangle = \langle -1, 7 \rangle = \vec{a}$

$\vec{p} = \langle -4, 3 \rangle$ is a multiple of $\vec{b} = \langle 8, -6 \rangle$ $\qquad \vec{q} \cdot \vec{b} = \langle 3, 4 \rangle \cdot \langle 8, -6 \rangle = 24 - 24 = 0$

3. (5 points) A line passes through the point $P = (2, -6)$ and is tangent to the direction $\vec{v} = \langle 4, 2 \rangle$. Which of the following points are on the line? (Circle your one answer.)

a. $(6, 2)$
b. $(4, -5)$
c. $(-2, -6)$
d. $(0, -5)$

Solution: The general point on the line is $X(t) = P + t\vec{v} = (2, -6) + t\langle 4, 2 \rangle$.

We try the values of t which make the x-components work.

$X(1) = (2, -6) + 1\langle 4, 2 \rangle = (6, -4)$ which agrees with the x-component in (a) but not the y-component.

$X\left(\frac{1}{2}\right) = (2, -6) + \frac{1}{2}\langle 4, 2 \rangle = (4, -5)$ which is answer (b).

$X(-1) = (2, -6) - 1\langle 4, 2 \rangle = (-2, -8)$ which agrees with the x-component in (c) but not the y-component.

$X\left(-\frac{1}{2}\right) = (2, -6) - \frac{1}{2}\langle 4, 2 \rangle = (0, -7)$ which agrees with the x-component in (d) but not the y-component.

4. (5 points) Find the part of the real line where the function $f(x) = \sqrt{9 - x^2} + \dfrac{1}{\sqrt{4 - x^2}}$ is continuous.

Continuous on: ______________________

Solution: Since the function is constructed using elementary operations, it is continuous where it is defined. The square roots require their arguments to be non-negative. So we need both $|x| \le 3$ and $|x| \le 2$. Since the denominator must be non-zero, we need $x \ne 2$. All together, these say $|x| < 2$ or $-2 < x < 2$ or $(-2, 2)$ or draw a number line with round brackets at -2 and 2.

5. Compute each of the following limits:

a. (5 points) $\lim\limits_{x \to 3} \dfrac{(x+3)^2 - 36}{x - 3}$ .... = ______

Solution: Expand and cancel:

$$\lim_{x \to 3} \frac{(x+3)^2 - 36}{x - 3} = \lim_{x \to 3} \frac{x^2 + 6x - 27}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+9)}{x - 3} = 12$$

b. (5 points) $\lim\limits_{x \to 2} \dfrac{e^x - e^2}{x - 2}$ .... = ______

Solution: You should recognize this as the derivative of $f(x) = e^x$ at $x = 2$. Since $f'(x) = e^x$, $\lim\limits_{x \to 2} \dfrac{e^x - e^2}{x - 2} = f'(2) = e^2$.

c. (5 points) $\lim\limits_{x \to 4} \dfrac{\sqrt{20 - x} - 4}{x - 4}$ .... = ______

Solution: Multiply by the conjugate.

$$\lim_{x \to 4} \frac{\sqrt{20 - x} - 4}{x - 4} \frac{\sqrt{20 - x} + 4}{\sqrt{20 - x} + 4} = \lim_{x \to 4} \frac{20 - x - 16}{(x - 4)\left(\sqrt{20 - x} + 4\right)} = \lim_{x \to 4} \frac{-(x - 4)}{(x - 4)\left(\sqrt{20 - x} + 4\right)}$$

$$= \frac{-1}{\sqrt{20 - 4} + 4} = \frac{-1}{8}$$

d. (5 points) $\lim\limits_{x \to -\infty} \dfrac{3 \cdot 2^x + 4 \cdot 3^x}{2^x + 2 \cdot 3^x}$ .... = ______

Solution: Divide by the largest term in the denominator.

$$\lim_{x \to -\infty} \frac{3 \cdot 2^x + 4 \cdot 3^x}{2^x + 2 \cdot 3^x} \frac{\left(\frac{1}{2^x}\right)}{\left(\frac{1}{2^x}\right)} = \lim_{x \to -\infty} \frac{3 + 4 \cdot \left(\frac{3}{2}\right)^x}{1 + 2 \cdot \left(\frac{3}{2}\right)^x} = 3$$

6. (5 points) Find the smallest interval with integer endpoints in which there is a solution of the equation $x^5 + 3x^3 = 200$.

There is a solution in the interval $I = [\_\_\_\_\_\_, \_\_\_\_\_\_]$.

Solution: Let $f(x) = x^5 + 3x^3$. We plug in values:

$$f(0) = 0 \qquad f(1) = 4 \qquad f(2) = 32 + 24 = 56 \qquad f(3) = 243 + 81 = 324$$

Since $f(x)$ is continuous and $56 < 200 < 324$, the Intermediate Value Theorem guarantees there is a solution to $f(x) = 200$ in the interval $I = [2, 3]$.

7. Compute each of the following derivatives:

a. (5 points) $f(x) = 3x^5 + 4x^\pi$ .... $f'(x) =$ ____________________

Solution: $f'(x) = 15x^4 + 4\pi x^{\pi-1}$

b. (5 points) $g(x) = \dfrac{1}{(\sin x + \cos x)^3}$ .... $g'(x) =$ ____________________

Solution: $g(x) = (\sin x + \cos x)^{-3}$ By the Extended Power Rule:

$$g'(x) = -3(\sin x + \cos x)^{-4}(\cos x - \sin x) = -3\frac{\cos x - \sin x}{(\sin x + \cos x)^4}$$

c. (5 points) $p(x) = e^\pi + \pi^e$ .... $p'(x) =$ ____________________

Solution: Everything is constant. So $p'(x) = 0$.

d. (5 points) Find $f'(1)$, if $f(x) = \dfrac{p(x)q(x)}{r(x)}$, given that

$$p(1) = 2, \quad p'(1) = 3, \quad q(1) = 4, \quad q'(1) = 6, \quad r(1) = 4, \quad r'(1) = 2$$

.... $f'(1) =$ ____________________

Solution: By the Product and Quotient Rules: $f' = \dfrac{r[p'q + pq'] - pqr'}{r^2}$

$$f'(1) = \frac{r(1)[p'(1)q(1) + p(1)q'(1)] - p(1)q(1)r'(1)}{r(1)^2} = \frac{4[3 \cdot 4 + 2 \cdot 6] - 2 \cdot 4 \cdot 2}{4^2} = \frac{12 + 12 - 4}{4} = 5$$

Work Out: (Points indicated. Part credit possible. Show all work.)

8. (10 points) Find the tangent line to the graph of $y = g(x) = \sec x$ at $x = \frac{\pi}{4}$.

$y =$ ____________________

Solution: $g\left(\frac{\pi}{4}\right) = \sec\frac{\pi}{4} = \sqrt{2}$ $\quad g'(x) = \sec x \tan x$ $\quad g'\left(\frac{\pi}{4}\right) = \sec\frac{\pi}{4}\tan\frac{\pi}{4} = \sqrt{2}$

$$y = g\left(\frac{\pi}{4}\right) + g'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) = \sqrt{2} + \sqrt{2}\left(x - \frac{\pi}{4}\right) = \sqrt{2}\,x + \sqrt{2} - \frac{\pi}{4}\sqrt{2}$$

9. (5 points) Use the limit definition of the derivative to prove $\frac{d}{dx}\sin x = \cos x$.

Solution: We use the limit definition of a derivative and the identities

$$\sin(x+h) = \sin x\cos h + \cos x\sin h, \qquad \lim_{h\to 0}\frac{\sin h}{h} = 1, \qquad \lim_{h\to 0}\frac{1-\cos h}{h} = 0$$

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \sin x\lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h} = \sin x(0) + \cos x(1) = \cos x$$

10. (15 points) Prove $\lim\limits_{x\to 4}(7 - 3x) = -5$ by completing the following three steps.

a. Write out the definition.

Solution: $\lim\limits_{x\to 4}(7 - 3x) = -5$ means:

For all $\epsilon > 0$ there is a $\delta > 0$ such that
if $0 < |x - 4| < \delta$ then $|(7 - 3x) + 5| < \epsilon$.

b. Work backwards to find δ in terms of ε.

Solution: $|(7 - 3x) + 5| < \epsilon \quad\Leftarrow\quad |12 - 3x| < \epsilon \quad\Leftarrow\quad 3|4 - x| < \epsilon \quad\Leftarrow\quad |x - 4| < \frac{\epsilon}{3}$

So we need $\delta = \frac{\epsilon}{3}$.

c. Complete the proof.

Solution: Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{3}$. Then if $0 < |x - 4| < \delta$ then

$$|x - 4| < \frac{\epsilon}{3} \quad\Rightarrow\quad 3|4 - x| < \epsilon \quad\Rightarrow\quad |12 - 3x| < \epsilon \quad\Rightarrow\quad |(7 - 3x) + 5| < \epsilon$$