Name________________________Section____________

1-7	/70	9	/ 5
8	/10	10	/15
		Total	/100

MATH 171 Exam 1B Fall 2022

Section 502/504 Solutions P. Yasskin

Multiple Choice and Short Answer:

(Show your work in case there is part credit.)

1. (5 points) Find the angle between the vectors $\vec{u} = \left\langle \sqrt{3}, -3 \right\rangle$ and $\vec{v} = \left\langle 1, \sqrt{3} \right\rangle$.

$\theta =$ ______

Solution: $|\vec{u}| = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$ $\quad |\vec{v}| = \sqrt{1+3} = 2$ $\quad \vec{u} \cdot \vec{v} = \sqrt{3} - 3\sqrt{3} = -2\sqrt{3}$

$\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{-2\sqrt{3}}{2\sqrt{3} \cdot 2} = \dfrac{-1}{2}$ $\quad \theta = 120°$

2. (10 points) Write $\vec{a} = \langle 3,7 \rangle$ as the sum of two vectors $\vec{p}$ and $\vec{q}$ where $\vec{p}$ is parallel to $\vec{b} = \langle 10,4 \rangle$ and $\vec{q}$ is perpendicular to $\vec{b} = \langle 10,4 \rangle$.

$\vec{a} = \vec{p} + \vec{q}$ where...................................... $\vec{p} = \langle$______, ______$\rangle$ and $\vec{q} = \langle$______, ______$\rangle$

Solution: $\vec{p} = \text{proj}_{\vec{b}}\vec{a} = \dfrac{\vec{b} \cdot \vec{a}}{|\vec{b}|^2}\vec{b} = \dfrac{30+28}{100+16}\langle 10,4 \rangle = \dfrac{58}{116}\langle 10,4 \rangle = \langle 5,2 \rangle$

$\vec{q} = \text{proj}_{\perp\vec{b}}\vec{a} = \vec{a} - \text{proj}_{\vec{b}}\vec{a} = \langle 3,7 \rangle - \langle 5,2 \rangle = \langle -2,5 \rangle$

Check: $\vec{p} + \vec{q} = \langle 5,2 \rangle + \langle -2,5 \rangle = \langle 3,7 \rangle = \vec{a}$

$\vec{p} = \langle 5,2 \rangle$ is a multiple of $\vec{b} = \langle 10,4 \rangle$ $\quad \vec{q} \cdot \vec{b} = \langle -2,5 \rangle \cdot \langle 10,4 \rangle = -20 + 20 = 0$

3. (5 points) A line passes through the point $P = (4,-1)$ and is tangent to the direction $\vec{v} = \langle 6,4 \rangle$. Which of the following points are on the line? (Circle your one answer.)

- **a**. $(10,6)$
- **b**. $(7,3)$
- **c**. $(-2,-5)$
- **d**. $(1,3)$

Solution: The general point on the line is $X(t) = P + t\vec{v} = (4,-1) + t\langle 6,4 \rangle$.
We try the values of t which make the x-components work.

$X(1) = (4,-1) + 1\langle 6,4 \rangle = (10,3)$ which agrees with the x-component in (a) but not the y–component.

$X\left(\frac{1}{2}\right) = (4,-1) + \frac{1}{2}\langle 6,4 \rangle = (7,1)$ which agrees with the x-component in (b) but not the y–component.

$X(-1) = (4,-1) - 1\langle 6,4 \rangle = (-2,-5)$ which agrees with (c).

$X\left(-\frac{1}{2}\right) = (4,-1) - \frac{1}{2}\langle 6,4 \rangle = (1,-3)$ which agrees with the x-component in (d) but not the y-component.

4. (5 points) Find the part of the real line where the function $f(x) = \sqrt{16 - x^2} + \dfrac{1}{\sqrt{9 - x^2}}$ is continuous.

Continuous on: ________________________

Solution: Since the function is constructed using elementary operations, it is continuous where it is defined. The square roots require their arguments to be non-negative. So we need both $|x| \le 4$ and $|x| \le 3$. Since the denominator must be non-zero, we need $x \ne 3$. All together, these say $|x| < 3$ or $-3 < x < 3$ or $(-3, 3)$ or draw a number line with round brackets at -3 and 3.

5. Compute each of the following limits:

a. (5 points) $\lim_{x\to 3} \dfrac{x^2 - x - 6}{x^2 - 5x + 6}$ .... = ______

Solution: Factor and cancel.

$$\lim_{x\to 3} \frac{x^2 - x - 6}{x^2 - 5x + 6} = \lim_{x\to 3} \frac{(x-3)(x+2)}{(x-3)(x-2)} = \lim_{x\to 3} \frac{(x+2)}{(x-2)} = 5$$

b. (5 points) $\lim_{h\to 0} \dfrac{e^{2+h} - e^2}{h}$ .... = ______

Solution: You should recognize this as the derivative of $f(x) = e^x$ at $x = 2$. Since $f'(x) = e^x$,
$\lim_{h\to 0} \dfrac{e^{2+h} - e^2}{h} = f'(2) = e^2$.

c. (5 points) $\lim_{x\to\infty} \left(\sqrt{x^2 - 3x} - \sqrt{x^2 + 2x}\right)$ .... = ______

Solution: Multiply by the conjugate. Then divide by the largest term in the denominator.

$$\lim_{x\to\infty} \left(\sqrt{x^2 - 3x} - \sqrt{x^2 + 2x}\right) \frac{\left(\sqrt{x^2 - 3x} + \sqrt{x^2 + 2x}\right)}{\left(\sqrt{x^2 - 3x} + \sqrt{x^2 + 2x}\right)} = \lim_{x\to\infty} \frac{(x^2 - 3x) - (x^2 + 2x)}{\sqrt{x^2 - 3x} + \sqrt{x^2 + 2x}}$$

$$= \lim_{x\to\infty} \frac{-5x}{\sqrt{x^2 - 3x} + \sqrt{x^2 + 2x}} \cdot \frac{x^{-1}}{x^{-1}} = \lim_{x\to\infty} \frac{-5}{\sqrt{1 - \frac{3}{x}} + \sqrt{1 + \frac{2}{x}}} = \frac{-5}{2}$$

d. (5 points) $\lim_{x\to 0} \dfrac{8x^4 + 6x^3}{4x^4 + 2x^3}$ .... = ______

Solution: Divide by the largest term in the denominator.
Near $x = 0$, $x^3 > x^4$. So we divide top and bottom by x^3:

$$\lim_{x\to 0} \frac{8x^4 + 6x^3}{4x^4 + 2x^3} \frac{x^{-3}}{x^{-3}} = \lim_{x\to 0} \frac{8x + 6}{4x + 2} = 3$$

6. (5 points) Find the horizontal asymptotes of the function $g(x) = \dfrac{3 \cdot 2^x + 4 \cdot 3^x}{2^x + 2 \cdot 3^x}$.

The horizontal asymptote as $x \to \infty$ is $y =$ ______.

The horizontal asymptote as $x \to -\infty$ is $y =$ ______.

Solution: $\lim\limits_{x\to\infty} \dfrac{3 \cdot 2^x + 4 \cdot 3^x}{2^x + 2 \cdot 3^x} \dfrac{\left(\frac{1}{3^x}\right)}{\left(\frac{1}{3^x}\right)} = \lim\limits_{x\to\infty} \dfrac{3 \cdot \left(\frac{2}{3}\right)^x + 4}{\left(\frac{2}{3}\right)^x + 2} = 2$ Asymptote is $y = 2$.

$\lim\limits_{x\to-\infty} \dfrac{3 \cdot 2^x + 4 \cdot 3^x}{2^x + 2 \cdot 3^x} \dfrac{\left(\frac{1}{2^x}\right)}{\left(\frac{1}{2^x}\right)} = \lim\limits_{x\to-\infty} \dfrac{3 + 4 \cdot \left(\frac{3}{2}\right)^x}{1 + 2 \cdot \left(\frac{3}{2}\right)^x} = 3$ Asymptote is $y = 3$.

7. Compute each of the following derivatives:

a. (5 points) $f(x) = 4x^3 + \dfrac{3}{x^4}$ .... $f'(x) =$ ____________________

Solution: $f'(x) = 12x^2 - \dfrac{12}{x^5}$

b. (5 points) $g(x) = (\sin x + \cos x)^5$ .... $g'(x) =$ ____________________

Solution: By the Extended Power Rule:
$g'(x) = 5(\sin x + \cos x)^4(\cos x - \sin x)$

c. (5 points) $p(x) = x^\pi + \pi^x$ .... $p'(x) =$ ____________________

Solution: The first term is a power rule. The second term is an exponential with base π.
$p'(x) = \pi x^{\pi-1} + (\ln\pi)\pi^x$

d. (5 points) Find $f'(1)$, if $f(x) = \dfrac{p(x)q(x)}{r(x)}$, given that

$$p(1) = 7, \quad p'(1) = 6, \quad q(1) = 9, \quad q'(1) = 6, \quad r(1) = 3, \quad r'(1) = 2$$

.... $f'(1) =$ ____________________

Solution: By the Product and Quotient Rules: $f' = \dfrac{r[p'q + pq'] - pqr'}{r^2}$

$$f'(1) = \frac{r(1)[p'(1)q(1) + p(1)q'(1)] - p(1)q(1)r'(1)}{r(1)^2} = \frac{3[6 \cdot 9 + 7 \cdot 6] - 7 \cdot 9 \cdot 2}{3^2} = 18 + 14 - 14 = 18$$

Work Out: (Points indicated. Part credit possible. Show all work.)

8. (10 points) Find the tangent line to the graph of $y = g(x) = \tan x$ at $x = \frac{\pi}{3}$.

$y =$ ____________________

Solution: $g\left(\frac{\pi}{3}\right) = \tan\frac{\pi}{3} = \sqrt{3}$ $\quad g'(x) = \sec^2 x$ $\quad g'\left(\frac{\pi}{3}\right) = \sec^2\left(\frac{\pi}{3}\right) = 2^2 = 4$

$$y = g\left(\frac{\pi}{3}\right) + g'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) = \sqrt{3} + 4\left(x - \frac{\pi}{3}\right) = 4x + \sqrt{3} - \frac{4\pi}{3}$$

9. (5 points) Use the limit definition of the derivative to prove $\frac{d}{dx}\cos x = -\sin x$.

Solution: We use the limit definition of a derivative and the identities

$$\cos(x+h) = \cos x\cos h - \sin x\sin h, \qquad \lim_{h\to 0}\frac{\sin h}{h} = 1, \qquad \lim_{h\to 0}\frac{1-\cos h}{h} = 0$$

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h}$$

$$= -\sin x\lim_{h\to 0}\frac{\sin h}{h} + \cos x\lim_{h\to 0}\frac{\cos h - 1}{h} = -\sin x(1) + \cos x(0) = -\sin x$$

10. (15 points) Prove $\lim_{x\to 3}(2+4x) = 14$ by completing the following three steps.

a. Write out the definition.

Solution: $\lim_{x\to 3}(2+4x) = 14$ means:

For all $\epsilon > 0$ there is a $\delta > 0$ such that

if $0 < |x-3| < \delta$ then $|(2+4x) - 14| < \epsilon$.

b. Work backwards to find δ in terms of ε.

Solution: $|(2+4x)-14| < \epsilon \quad\Leftarrow\quad |4x-12| < \epsilon \quad\Leftarrow\quad 4|x-3| < \epsilon \quad\Leftarrow\quad |x-3| < \frac{\epsilon}{4}$

So we need $\delta = \frac{\epsilon}{4}$.

c. Complete the proof.

Solution: Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{4}$. Then if $0 < |x-3| < \delta$ then

$$|x-3| < \frac{\epsilon}{4} \quad\Rightarrow\quad 4|x-3| < \epsilon \quad\Rightarrow\quad |4x-12| < \epsilon \quad\Rightarrow\quad |(7-3x)+5| < \epsilon$$