Name_________________________Section____________

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12	/10	Total	/105

MATH 171 Exam 2A Fall 2022

Section 502/504 Solutions P. Yasskin

Multiple Choice and Short Answer:

(5 points each. Show your work in case there is part credit.)

1. Use the linear approximation to approximate $\sqrt{4.2}$.

a. 1.949 **f.** 2.02

b. 1.95 **g.** 2.025

c. 1.951 **h.** 2.049

d. 1.975 **i.** 2.05 Correct

e. 1.98 **j**. 2.051

Solution: Let $f(x) = \sqrt{x}$. Since we know $f(4) = \sqrt{4} = 2$, we approximate near $x = 4$.
Then $f'(x) = \dfrac{1}{2\sqrt{x}}, f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$.

$f_{\tan}(x) = f(4) + f'(4)(x-4) = 2 + \frac{1}{4}(x-4)$ $\quad \sqrt{4.2} \approx f_{\tan}(4.2) = 2 + \frac{1}{4}(4.2-4) = 2 + \frac{.2}{4} = 2.05$

A calculator says: $\sqrt{4.2} \approx 2.0494$

2. Notice that the point $(x,y) = (1,2)$ lies on the curve $3x^4y^3 + 4x^3y^2 = 40$.
What is the slope of the curve, $\dfrac{dy}{dx}$, at $(1,2)$?

Solution: We implicitly differentiate the equation regarding y as a function of x.
We use the product rule and the chain rule:

$$12x^3y^3 + 9x^4y^2\frac{dy}{dx} + 12x^2y^2 + 8x^3y\frac{dy}{dx} = 0$$

We plug in $(x,y) = (1,2)$ and solve for the slope:

$$96 + 36\frac{dy}{dx} + 48 + 16\frac{dy}{dx} = 0 \qquad 52\frac{dy}{dx} = -144 \qquad \frac{dy}{dx} = -\frac{144}{52} = -\frac{36}{13}$$

3. For the function $f = x^4 - 4x$, the Mean Value Theorem says:
There is a number c in $[2,4]$ where $f'(c) =$

Solution: Notice $f(4) = 256 - 16 = 240$ and $f(2) = 16 - 8 = 8$.

So the slope between the endpoints is $m = \dfrac{f(4) - f(2)}{4-2} = \dfrac{240-8}{2} = 116$.

The MVT says there is a number c in $[1,3]$ where $f'(c) = 116$.

4. If $g(x) = \arctan(x)$, then $g'\left(\frac{3}{4}\right) =$

a. $\frac{3}{4}$

b. $\frac{4}{3}$

c. $\frac{3}{5}$

d. $\frac{5}{3}$

e. $\frac{4}{5}$

f. $\frac{5}{4}$

g. $\frac{9}{16}$

h. $\frac{16}{9}$

i. $\frac{9}{25}$

j. $\frac{25}{9}$

k. $\frac{16}{25}$ Correct

l. $\frac{25}{16}$

Solution: $g'(x) = \frac{1}{1+x^2}$ $\quad g'\left(\frac{3}{4}\right) = \frac{1}{1+\left(\frac{3}{4}\right)^2} = \frac{1}{1+\frac{9}{16}} = \frac{1}{\frac{25}{16}} = \frac{16}{25}$

5. Suppose $f(x) = \frac{1}{x^3}$ and $g(x) = f^{-1}(x)$ is the inverse of $f(x)$. What is $g(8)$?

Solution: We solve $\frac{1}{x^3} = 8$ to get $x = \frac{1}{2}$. Since $f\left(\frac{1}{2}\right) = 8$, we have $g(8) = \frac{1}{2}$.

6. Suppose $f(x) = 2x^5 + \frac{1}{x^5}$ and $g(x) = f^{-1}(x)$ is the inverse of $f(x)$. Also notice $f(1) = 3$.
The inverse function theorem allows us to easily compute either $g'(1)$ or $g'(3)$.
Which one and what is its value?

Solution: If $f(a) = b$ and $g(b) = a$, the inverse function theorem says
$g'(b) = \frac{1}{f'(a)} = \frac{1}{f'(g(b))}$. We know $f(1) = 3$. So $g(3) = 1$ and we can compute $g'(3) = \frac{1}{f'(1)}$.
Since $f'(x) = 10x^4 - \frac{5}{x^6}$, we know $f'(1) = 10 - 5 = 5$ and $g'(3) = \frac{1}{5}$.

7. The point $x = 2$ is a critical point of the function $f(x) = x^4 - 8x^3 + 24x^2 - 32x$.
Then the Second Derivative Test says $x = 2$ is a

a. Local Minimum
b. Local Maximum
c. Inflection Point
d. The Second Derivative Test FAILS. Correct

Solution: $f'(x) = 4x^3 - 24x^2 + 48x - 32$ Note: $f'(2) = 32 - 96 + 96 - 32 = 0$
$f''(x) = 12x^2 - 48x + 48$ $f''(2) = 48 - 96 + 48 = 0$
So the Second Derivative Test FAILS.

8. The point $x = 1$ is a critical point of the function $f(x) = x^4 - 4x^3 + 4x^2$.
Then the Second Derivative Test says $x = 1$ is a

a. Local Minimum
b. Local Maximum Correct
c. Inflection Point
d. The Second Derivative Test FAILS.

Solution: $f'(x) = 4x^3 - 12x^2 + 8x$ Note: $f'(1) = 4 - 12 + 8 = 0$
$f''(x) = 12x^2 - 24x + 8$ $f''(1) = 12 - 24 + 8 = -4 < 0$ Concave down.
So the Second Derivative Test says $x = 1$ is a local maximum.

9. If $p(t) = \ln(t^4)$, what is $p'(2)$?

Solution: Rewrite $p(t) = 4\ln(t)$. So $p'(t) = 4\frac{1}{t}$ and $p'(2) = 4\frac{1}{2} = 2$

10. If $q(s) = (5 + s^{2/3})^{3/2}$, what is $q'(8)$? (Simplify to a rational number.)

Solution:
$q'(s) = \frac{3}{2}(5 + s^{2/3})^{1/2}\frac{2}{3}s^{-1/3}$ $q'(8) = \frac{3}{2}(5 + 8^{2/3})^{1/2}\frac{2}{3}8^{-1/3} = \frac{3}{2}(5 + 4)^{1/2}\frac{2}{3}\frac{1}{2} = 9^{1/2}\frac{1}{2} = \frac{3}{2}$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (10 points) A conical cup is filled with water to a height $h = 16$ cm and radius $r = 4$ cm, but it is leaking. If 2 cubic cm leaks out, estimate the change in the height of the water.
(Note: The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.)

Solution: Since $\frac{h}{r} = \frac{16}{4} = 4$, we can write $r = \frac{h}{4}$. Plugging that into the volume of the cone we get $V = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h = \frac{1}{48}\pi h^3$. To get the change in the height, we first approximate the change in the volume using differentials:

$$\Delta V \approx dV = \frac{dV}{dh}dh = \frac{1}{16}\pi h^2 \Delta h$$

We now solve for Δh and plug in numbers:

$$\Delta h = \frac{16\Delta V}{\pi h^2} = \frac{16(2)}{\pi(16)^2} = \frac{1}{8\pi}$$

12. (10 points) For a pendulum, the period T and the acceleration of gravity g are related by $T\sqrt{g} = 2\pi\sqrt{L}$ where L is the (constant) length of the pendulum. The pendulum is put on a rocket ship so that g and hence T are changing. When $g = 2\ \frac{\text{m}}{\text{sec}^2}$ and $T = 1$ sec, what is $\frac{dT}{dg}$?

Solution: We apply $\frac{d}{dg}$ to both sides using the Product and Power rules and solve for $\frac{dT}{dg}$:

$$\frac{T}{2\sqrt{g}} + \sqrt{g}\frac{dT}{dg} = 0 \qquad \frac{dT}{dg} = -\frac{T}{2g}$$

Finally we evaluate at $(g,T) = (2,1)$: $\left.\frac{dT}{dg}\right|_{(2,1)} = -\frac{1}{4}$

13. (10 points) Find all horizontal and vertical tangents of the parametric curve $\vec{r}(t) = \left(\frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t,\ \ \frac{1}{4}t^4 - \frac{4}{3}t^3 + 2t^2\right)$.

Solution: $x(t) = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t \qquad y(t) = \frac{1}{4}t^4 - \frac{4}{3}t^3 + 2t^2$

$y'(t) = t^3 - 4t^2 + 4t = t(t^2 - 4t + 4) = t(t-2)^2$ Potential horizontal tangents are at $t = 0,2$

$x'(t) = t^2 - 3t + 2 = (t-2)(t-1)$ Potential vertical tangents are at $t = 1,2$

But, $t = 2$ can't be both. We look at the slope:

$$m = \frac{t(t-2)^2}{(t-1)(t-2)} = \frac{t(t-2)}{(t-1)}$$

So horizontal tangents are at $t = 0,2$.
Vertical tangents is at $t = 1$.

14. (25 points) Find the first and second derivatives of each of the following functions:
(You do not need to simplify, but you may want to simplify the first derivative if it makes it easier to compute the second derivative.)

a. (7 points) $f(x) = \cos(x^3)$

Solution:

$$f'(x) = -\sin(x^3)3x^2$$

$$f''(x) = -\cos(x^3)3x^2 3x^2 - \sin(x^3)6x$$

b. (7 points) $g(x) = \ln(x^2 + 4)$

Solution:

$$g'(x) = \frac{2x}{x^2 + 4}$$

$$g''(x) = \frac{(x^2 + 4)2 - 2x(2x)}{(x^2 + 4)^2} = \frac{8 - 2x^2}{(x^2 + 4)^2}$$

c. (7 points) $p(x) = \arcsin(4x)$

Solution:

$$p'(x) = \frac{4}{\sqrt{1 - 16x^2}} = 4(1 - 16x^2)^{-1/2}$$

$$p''(x) = 4\left(-\frac{1}{2}\right)(1 - 16x^2)^{-3/2}(-32x) = \frac{64x}{(1 - 16x^2)^{3/2}}$$

d. (4 points) $q(x) = x^{(x^3)}$
HINT: In the base, let $x = e^{(\ln x)}$.
ON THIS ONE YOU ONLY NEED THE FIRST DERIVATIVE.

Solution: We rewrite the function as $q(x) = x^{(x^3)} = (e^{(\ln x)})^{(x^3)} = e^{x^3 \ln x}$
We calculate the derivative using the Chain and Product rules:

$$q'(x) = e^{x^3 \ln x}\left[3x^2 \ln x + x^3 \frac{1}{x}\right] = (3x^2 \ln x + x^2)x^{(x^3)}$$