Name___________________________Section_____________

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| 1-10 | /50 | 13 | /10 |
| 11 | /10 | 14 | /25 |
| 12 | /10 | Total | /105 |

MATH 171 Exam 2B Fall 2022

Section 502/504 Solutions P. Yasskin

Multiple Choice and Short Answer:

(5 points each. Show your work in case there is part credit.)

**1**. Use the linear approximation to approximate $\sqrt{3.8}$.

**a**. $1.949$

**b**. $1.95$ Correct

**c**. $1.951$

**d**. $1.975$

**e**. $1.98$

**f.** $2.02$

**g.** $2.025$

**h.** $2.049$

**i.** $2.05$

**j**. $2.051$

**Solution**: Let $f(x) = \sqrt{x}$. Since we know $f(4) = \sqrt{4} = 2$, we approximate near $x = 4$.
Then $f'(x) = \dfrac{1}{2\sqrt{x}}, f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$.

$f_{\tan}(x) = f(4) + f'(4)(x-4) = 2 + \frac{1}{4}(x-4) \qquad \sqrt{3.8} \approx f_{\tan}(3.8) = 2 + \frac{1}{4}(3.8-4) = 2 - \frac{.2}{4} = 1.95$

A calculator says: $\sqrt{3.8} = 1.9494$

**2**. Notice that the point $(x,y) = (2,1)$ lies on the curve $3x^3y^4 + 4x^2y^3 = 40$.
What is the slope of the curve, $\dfrac{dy}{dx}$, at $(2,1)$?

**Solution**: We implicitly differentiate the equation regarding $y$ as a function of $x$.
We use the Product and Chain rule:

$$9x^2y^4 + 12x^3y^3\frac{dy}{dx} + 8xy^3 + 12x^2y^2\frac{dy}{dx} = 0$$

We plug in $(x,y) = (2,1)$ and solve for the slope:

$$36 + 96\frac{dy}{dx} + 16 + 48\frac{dy}{dx} = 0 \qquad 144\frac{dy}{dx} = -52 \qquad \frac{dy}{dx} = -\frac{52}{144} = -\frac{13}{36}$$

**3**. For the function $f = x^3 - 3x$, the Mean Value Theorem says:
There is a number $c$ in $[1,3]$ where $f'(c) =$

**Solution**: Notice $f(3) = 27 - 9 = 18$ and $f(1) = 1 - 3 = -2$.
So the slope between the endpoints is $m = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{18 - (-2)}{2} = 10$.
The MVT says there is a number $c$ in $[1,3]$ where $f'(c) = 10$.

**4**. If $g(x) = \arcsin(x)$, then $g'\left(\frac{3}{5}\right) =$

**a**. $\frac{3}{4}$

**b**. $\frac{4}{3}$

**c**. $\frac{3}{5}$

**d**. $\frac{5}{3}$

**e**. $\frac{4}{5}$

**f**. $\frac{5}{4}$ Correct

**g**. $\frac{9}{16}$

**h**. $\frac{16}{9}$

**i**. $\frac{9}{25}$

**j**. $\frac{25}{9}$

**k**. $\frac{16}{25}$

**l**. $\frac{25}{16}$

**Solution**: $g'(x) = \dfrac{1}{\sqrt{1-x^2}}$ $\quad g'\left(\frac{3}{5}\right) = \dfrac{1}{\sqrt{1-\left(\frac{3}{5}\right)^2}} = \dfrac{1}{\sqrt{1-\frac{9}{25}}} = \dfrac{1}{\sqrt{\frac{16}{25}}} = \dfrac{5}{4}$

**5**. Suppose $f(x) = x^5$ and $g(x) = f^{-1}(x)$ is the inverse of $f(x)$. What is $g(32)$?

**Solution**: We solve $x^5 = 32$ to get $x = 2$. Since $f(2) = 32$, we have $g(32) = 2$.

**6**. Suppose $f(x) = 4x^3 + \dfrac{1}{x^3}$ and $g(x) = f^{-1}(x)$ is the inverse of $f(x)$. Also notice $f(1) = 5$. The inverse function theorem allows us to easily compute either $g'(1)$ or $g'(5)$. Which one and what is its value?

**Solution**: If $f(a) = b$ and $g(b) = a$, the inverse function theorem says $g'(b) = \dfrac{1}{f'(a)} = \dfrac{1}{f'(g(b))}$. We know $f(1) = 5$. So $g(5) = 1$ and we can compute $g'(5) = \dfrac{1}{f'(1)}$. Since $f'(x) = 12x^2 - \dfrac{3}{x^4}$, we know $f'(1) = 12 - 3 = 9$ and $g'(5) = \dfrac{1}{9}$.

**7**. The point $x = 1$ is a critical point of the function $f(x) = x^4 - 4x^3 + 6x^2 - 4x$.
Then the Second Derivative Test says $x = 1$ is a

**a**. Local Minimum
**b**. Local Maximum
**c**. Inflection Point
**d**. The Second Derivative Test FAILS. Correct

**Solution**: $f'(x) = 4x^3 - 12x^2 + 12x - 4$ Note: $f'(1) = 4 - 12 + 12 - 4 = 0$
$f''(x) = 12x^2 - 24x + 12$ $f''(1) = 12 - 24 + 12 = 0$
So the Second Derivative Test FAILS.

**8**. The point $x = 2$ is a critical point of the function $f(x) = x^4 - 4x^3 + 4x^2$.
Then the Second Derivative Test says $x = 2$ is a

**a**. Local Minimum Correct
**b**. Local Maximum
**c**. Inflection Point
**d**. The Second Derivative Test FAILS.

**Solution**: $f'(x) = 4x^3 - 12x^2 + 8x$ Note: $f'(2) = 32 - 48 + 16 = 0$
$f''(x) = 12x^2 - 24x + 8$ $f''(2) = 48 - 48 + 8 = 8 > 0$ Concave up.
So the Second Derivative Test says $x = 2$ is a local minimum.

**9**. If $p(t) = \ln(t^5)$, what is $p'(10)$?

**Solution**: Rewrite $p(t) = 5\ln(t)$. So $p'(t) = 5\frac{1}{t}$ and $p'(10) = 5\frac{1}{10} = \frac{1}{2}$

**10**. If $q(s) = (2 + s^{1/3})^{3/2}$, what is $q'(8)$? (Simplify to a rational number.)

**Solution**:
$q'(s) = \frac{3}{2}(2 + s^{1/3})^{1/2}\frac{1}{3}s^{-2/3}$ $q'(8) = \frac{3}{2}(2 + 8^{1/3})^{1/2}\frac{1}{3}8^{-2/3} = \frac{3}{2}(2 + 2)^{1/2}\frac{1}{3}\frac{1}{4} = \frac{1}{8}2 = \frac{1}{4}$

Work Out: (Points indicated. Part credit possible. Show all work.)

**11**. (10 points) A conical cup is filled with water to a height $h = 27$ cm and radius $r = 9$ cm, but it is leaking. If $3$ cubic cm leaks out, estimate the change in the height of the water.
(Note: The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.)

**Solution**: Since $\frac{h}{r} = \frac{27}{9} = 3$, we can write $r = \frac{h}{3}$. Plugging that into the volume of the cone we get $V = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{1}{27}\pi h^3$. To get the change in the height, we first approximate the change in the volume using differentials:

$$\Delta V \approx dV = \frac{dV}{dh}dh = \frac{1}{9}\pi h^2 \Delta h$$

We now solve for $\Delta h$ and plug in numbers:

$$\Delta h = \frac{9\Delta V}{\pi h^2} = \frac{9(3)}{\pi(27)^2} = \frac{1}{27\pi}$$

**12**. (10 points) A rod is heating up and expanding. The length $L$ and the temperature $T$ are related by $\frac{L-L_0}{T-T_0} = \frac{L_0}{100}$ where $L_0 = 10$ m is the original length and $T_0 = 30°$C is the original temperature. When $L = 12$ m and $T = 50°$C, what is $\frac{dL}{dT}$?

**Solution**: We apply $\frac{d}{dT}$ to both sides using the Quotient rule and solve for $\frac{dL}{dT}$:

$$\frac{(T-T_0)\frac{dL}{dT} - (L-L_0)1}{(T-T_0)^2} = 0 \qquad (T-T_0)\frac{dL}{dT} - (L-L_0) = 0$$

$$\frac{dL}{dT} = \frac{L-L_0}{T-T_0}$$

Finally we evaluate at $(L,T) = (12,50)$ with $(L_0,T_0) = (10,30)$:

$$\left.\frac{dL}{dT}\right|_{(12,50)} = \frac{12-10}{50-30} = \frac{2}{20} = \frac{1}{10}$$

**13**. (10 points) Find all horizontal and vertical tangents of the parametric curve $\vec{r}(t) = \left(\frac{1}{4}t^4 - \frac{4}{3}t^3 + 2t^2,\ \ \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t\right)$.

**Solution**: $x(t) = \frac{1}{4}t^4 - \frac{4}{3}t^3 + 2t^2 \qquad y(t) = \frac{1}{3}t^3 - \frac{3}{2}t^2 + 2t$

$y'(t) = t^2 - 3t + 2 = (t-2)(t-1)$ Potential horizontal tangents are at $t = 1,2$

$x'(t) = t^3 - 4t^2 + 4t = t(t^2 - 4t + 4) = t(t-2)^2$ Potential vertical tangents are at $t = 0,2$

But, $t = 2$ can't be both. We look at the slope:

$$m = \frac{(t-1)(t-2)}{t(t-2)^2} = \frac{(t-1)}{t(t-2)}$$

So horizontal tangents is at $t = 1$.
Vertical tangents are is at $t = 0,2$.

**14**. (25 points) Find the first and second derivatives of each of the following functions:
(You do not need to simplify, but you may want to simplify the first derivative if it makes it easier to compute the second derivative.)

**a**. (7 points) $f(x) = \sin(x^4)$

**Solution**:

$$f'(x) = \cos(x^4)4x^3$$

$$f''(x) = -\sin(x^4)4x^3 4x^3 + \cos(x^4)12x^2$$

**b**. (7 points) $g(x) = \ln(x^3 + 6)$

**Solution**:

$$g'(x) = \frac{3x^2}{x^3 + 6}$$

$$g''(x) = \frac{(x^3 + 6)6x - 3x^2(3x^2)}{(x^3 + 6)^2} = \frac{36x - 3x^4}{(x^3 + 6)^2}$$

**c**. (7 points) $p(x) = \arctan(3x)$

**Solution**:

$$p'(x) = \frac{3}{1 + 9x^2} = 3(1 + 9x^2)^{-1}$$

$$p''(x) = 3(-1)(1 + 9x^2)^{-2}(18x) = \frac{-54x}{(1 + 9x^2)^2}$$

**d**. (4 points) $q(x) = x^{(x^2)}$

HINT: In the base, let $x = e^{(\ln x)}$.

ON THIS ONE YOU ONLY NEED THE FIRST DERIVATIVE.

**Solution**: We rewrite the function as $q(x) = x^{(x^2)} = (e^{(\ln x)})^{(x^2)} = e^{x^2 \ln x}$

We calculate the derivative using the Chain and Product rules:

$$q'(x) = e^{x^2 \ln x}\left[2x\ln x + x^2\frac{1}{x}\right] = (2x\ln x + x)x^{(x^2)}$$