Name__________________________Section_____________

1-4	/40	7	/20
5	/10	8	/10
6	/10	9	/15
		Total	/105

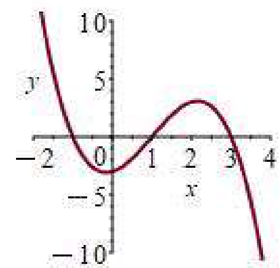
MATH 171 Exam 3A Fall 2022

Section 502/504 Solutions P. Yasskin

Short Answer: Points indicated.

Show your work in case there is part credit.

1. (20 points) Consider a function, $y = f(x)$.
At the right is the graph of its derivative, $y = f'(x)$.
Give answers to the nearest integer.



a. (5 points) Find the interval(s) where $f(x)$ is decreasing.

Solution: $f(x)$ is decreasing where $f'(x)$ is negative which is on $[-1,1]$ and $[3,4]$.

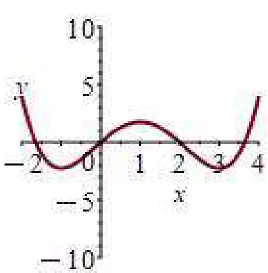
b. (5 points) Find the location(s) of all local minima of $f(x)$.

Solution: $f(x)$ has a local minimum where $f'(x)$ changes from negative to positive, which is at $x = 1$.

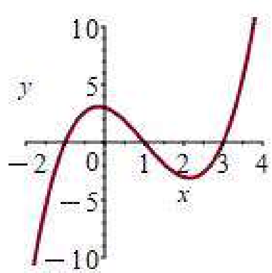
c. (5 points) Find the interval(s) where $f(x)$ is concave up.

Solution: $f(x)$ is concave up when $f''(x)$ is positive or $f'(x)$ is increasing which is on $[0,2]$.

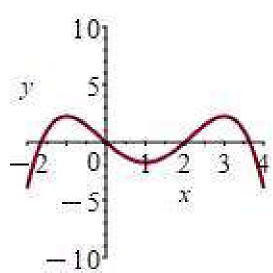
d. (5 points) Which of these is the graph of $y = f(x)$?



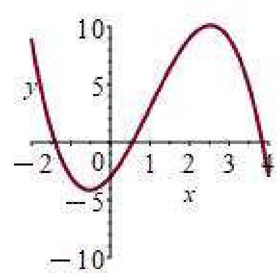
A



B



C



D

Solution: The derivative is positive, negative, positive, negative.
So the function is increasing, decreasing, increasing, decreasing, which is plot C.

2. (9 points) Find the general antiderivative of $p(x) = 6x^2 + \sec^2 x + xe^{x^2}$.

Solution: Since $\frac{d}{dx}x^3 = 3x^2 \qquad \frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}e^{x^2} = 2xe^{x^2}$,
the general antiderivative of $p(x)$ is:

$$P(x) = 2x^3 + \tan x + \frac{1}{2}e^{x^2} + C$$

3. (5 points) Find the area under the curve $y = \frac{2x}{1+x^2}$ above the interval $[1,3]$.

Solution: $A = \int_1^3 \frac{2x}{1+x^2}\,dx = \Big[\ln(1+x^2)\Big]_1^3 = \ln 10 - \ln 2 = \ln 5$

4. (6 points) Use a right Riemann sum with 3 equal width intervals to estimate $\int_3^9 \frac{1}{x-1}\,dx$.

Solution: The width of each interval is $\Delta x = \frac{9-3}{3} = 2$.
The partition points are $x_i = 3, 5, 7, 9$.
The function values at right endpoints are $f(x_i) = \frac{1}{4}, \frac{1}{6}, \frac{1}{8}$.
The right Riemann sum approximation is

$$\int_3^9 \frac{1}{x-1}\,dx \approx \sum_{i=1}^{3} f(x_i)\Delta x = \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8}\right)2 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{6+4+3}{12} = \frac{13}{12}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

5. (10 points) The volume of a square pyramid is $V = \frac{1}{3}s^2h$ where s is the length of the side of the square base and h is the height. Currently, $s = 40$ cm and $h = 30$ cm. If the volume is held fixed while the height decreases at $\frac{dh}{dt} = -3\,\frac{\text{cm}}{\text{sec}}$, how fast is the side, s, changing? Is it increasing or decreasing?

Solution: We differentiate the volume formula using the product rule:

$$\frac{dV}{dt} = \frac{1}{3}2s\frac{ds}{dt}h + \frac{1}{3}s^2\frac{dh}{dt}$$

We plug in $s = 40$, $h = 30$, $\frac{dV}{dt} = 0$ and $\frac{dh}{dt} = -3$, and solve for $\frac{ds}{dt}$:

$$0 = \frac{1}{3}2(40)\frac{ds}{dt}(30) + \frac{1}{3}(40)^2(-3) \qquad \frac{1}{3}2(40)\frac{ds}{dt}(30) = \frac{1}{3}(40)^2(3)$$

$$2\frac{ds}{dt}(30) = (40)(3) \qquad \frac{ds}{dt} = \frac{120}{60} = 2 \qquad \text{increasing}$$

6. (10 points) If $g(x) = \int_{\sin x}^{\cos x} \frac{1}{1+t^4}\,dt$, find $g'(x)$ and $g'(0)$.

Solution: Let $G(t)$ be an antiderivative of $\frac{1}{1+t^4}$. So $G'(t) = \frac{1}{1+t^4}$.

Then

$$g(x) = \Big[G(t)\Big]_{\sin x}^{\cos x} = G(\cos x) - G(\sin x)$$

By the chain rule:

$$\begin{aligned} g'(x) &= \frac{d}{dx}\Big[G(\cos x) - G(\sin x)\Big] \\ &= -G'(\cos x)\sin x - G'(\sin x)\cos x \\ &= -\frac{1}{1+\cos^4 x}\sin x - \frac{1}{1+\sin^4 x}\cos x \end{aligned}$$

Finally

$$\begin{aligned} g'(0) &= -\frac{1}{1+\cos^4 0}\sin 0 - \frac{1}{1+\sin^4 0}\cos 0 \\ &= 0 - \frac{1}{1+0}1 = -1 \end{aligned}$$

7. (20 points) For each limit, identify the indeterminate form and then compute the limit:

a. (10 points) $\lim_{x\to\pi} \dfrac{x\cos x - \sin x + \pi}{(x-\pi)^2}$

Solution: The limit of the numerator is $\pi\cos\pi - \sin\pi + \pi = -\pi - 0 + \pi = 0$ and of the denominator is 0. So the limit has the indeterminate form $\dfrac{0}{0}$.

$$\lim_{x\to\pi} \frac{x\cos x - \sin x + \pi}{(x-\pi)^2} \overset{l'H}{=} \lim_{x\to\pi} \frac{\cos x - x\sin x - \cos x}{2(x-\pi)} = \lim_{x\to\pi} \frac{-x\sin x}{2(x-\pi)} \overset{l'H}{=} \lim_{x\to\pi} \frac{-\sin x - x\cos x}{2} = \frac{\pi}{2}$$

b. (10 points) $\lim_{x\to 0^+} (1-5x)^{3/x}$

Solution: The limit of the base is 1 and of the exponent is ∞. So the limit has the indeterminate form 1^∞. We do the limit by inserting $e^{\ln}$ and using the identity $\ln a^b = b\ln a$.

$$\lim_{x\to 0^+} (1-5x)^{3/x} = \lim_{x\to 0^+} e^{\ln(1-5x)^{3/x}} = \lim_{x\to 0^+} e^{\frac{3}{x}\ln(1-5x)} \overset{l'H}{=} \lim_{x\to 0^+} e^{\frac{\frac{3}{1-5x}(-5)}{1}} = e^{-15}$$

8. (10 points) Find the smallest value of $f = 8x + y$ on the curve $x^2 y = 4$ in the first quadrant. How do you know this is the minimum?

Solution: The curve is $y = \dfrac{4}{x^2}$. So we need to minimize $f = 8x + \dfrac{4}{x^2}$.

We set the derivative equal to 0 and solve for x:

$f' = 8 - \dfrac{8}{x^3} = 0 \qquad 8 = \dfrac{8}{x^3} \qquad x^3 = 1 \qquad x = 1 \qquad y = 4 \qquad f = 8(1) + (4) = 12$

We check the second derivative: $f'' = \dfrac{24}{x^4}$ At the critical point $f''(1) = 24 > 0$.

This is concave up, so minimum.

9. (15 points) Evaluate each integral.

a. (5 points) $\int \frac{(\ln x)^3}{x}\,dx$

Solution: Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$.

$$\int \frac{(\ln x)^3}{x}\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}(\ln x)^4 + C$$

b. (5 points) $\int_0^1 x\sin(\pi x^2)\,dx$

Solution: Let $u = \pi x^2$. Then $du = 2\pi x\,dx$ and $x\,dx = \frac{1}{2\pi}\,du$.

If $x = 0$, then $u = 0$. If $x = 1$, then $u = \pi$.

$$\int_0^1 x\sin(\pi x^2)\,dx = \frac{1}{2\pi}\int_0^{\pi} \sin(u)\,du = \frac{-1}{2\pi}\cos u\Big|_0^{\pi} = \frac{-1}{2\pi}(\cos\pi - \cos 0) = \frac{-1}{2\pi}(-2) = \frac{1}{\pi} \quad \text{OR}$$

$$\int_0^1 x\sin(\pi x^2)\,dx = \frac{1}{2\pi}\int_{x=0}^{1} \sin(u)\,du = \frac{-1}{2\pi}\cos u\Big|_{x=0}^{1} = \frac{-1}{2\pi}\cos(\pi x^2)\Big|_0^1 = \frac{-1}{2\pi}(\cos\pi - \cos 0) = \frac{1}{\pi}$$

c. (5 points) $\int x^3(1+x^2)^{499}\,dx$

Solution: Let $u = 1 + x^2$. Then $du = 2x\,dx$ and $\frac{1}{2}\,du = x\,dx$ and $x^2 = u - 1$.

$$\int x^3(1+x^2)^{499}\,dx = \frac{1}{2}\int (u-1)u^{499}\,du = \frac{1}{2}\int u^{500} - u^{499}\,du = \frac{1}{2}\left(\frac{u^{501}}{501} - \frac{u^{500}}{500}\right) + C$$

$$= \frac{1}{2}\left(\frac{1}{501}(1+x^2)^{501} - \frac{1}{500}(1+x^2)^{500}\right) + C$$