Name________________________Section__________

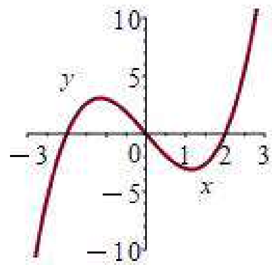
MATH 171 Exam 3B Fall 2022
Section 502/504 Solutions P. Yasskin

Short Answer: Points indicated.

Show your work in case there is part credit.

1-4	/40	7	/20
5	/10	8	/10
6	/10	9	/15
		Total	/105

1. (20 points) Consider a function, $y = f(x)$.
At the right is the graph of its derivative, $y = f'(x)$.
Give answers to the nearest integer.



a. (5 points) Find the interval(s) where $f(x)$ is increasing.

Solution: $f(x)$ is increasing where $f'(x)$ is positive which is on $[-2,0]$ and $[2,3]$.

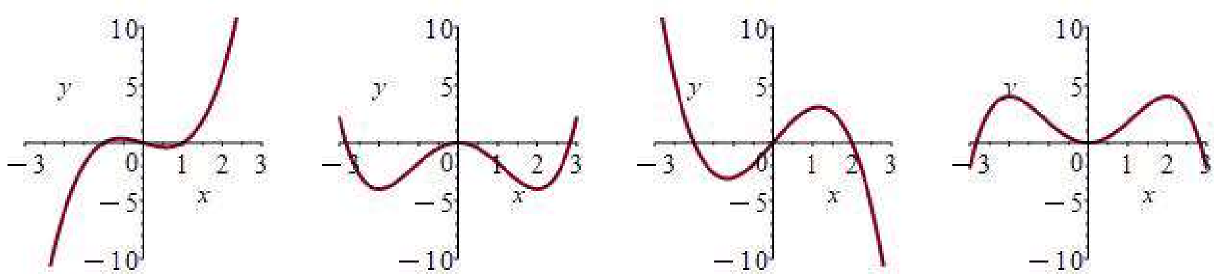
b. (5 points) Find the location(s) of all local maxima of $f(x)$.

Solution: $f(x)$ has a local maximum where $f'(x)$ changes from positive to negative, which is at $x = 0$.

c. (5 points) Find the interval(s) where $f(x)$ is concave down.

Solution: $f(x)$ is concave down when $f''(x)$ is negative or $f'(x)$ is decreasing which is on $[-1,1]$.

d. (5 points) Which of these is the graph of $y = f(x)$?



A B C D

Solution: The derivative is negative, positive, negative, positive.
So the function is decreasing, increasing, decreasing, increasing, which is plot B.

2. (9 points) Find the general antiderivative of $p(x) = 12x^3 + \sin x + \dfrac{x}{1+x^2}$.

Solution: Since $\dfrac{d}{dx}x^4 = 4x^3 \qquad \dfrac{d}{dx}\cos x = -\sin x \qquad \dfrac{d}{dx}\ln(1+x^2) = \dfrac{2x}{1+x^2}$,
the general antiderivative of $p(x)$ is:

$$P(x) = 3x^4 - \cos x + \frac{1}{2}\ln(1+x^2) + C$$

3. (5 points) Find the area under the curve $y = \sec^2 x$ above the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.
(Evaluate all trig functions.)

Solution: $A = \displaystyle\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx = \Big[\tan x\Big]_{-\pi/4}^{\pi/4} = \tan\left(\frac{\pi}{4}\right) - \tan\left(-\frac{\pi}{4}\right) = 1 - -1 = 2$

4. (6 points) Use a right Riemann sum with 3 equal width intervals to estimate $\displaystyle\int_1^7 \frac{1}{1+x}\,dx$.

Solution: The width of each interval is $\Delta x = \dfrac{7-1}{3} = 2$.
The partition points are $x_i = 1, 3, 5, 7$.
The function values at right endpoints are $f(x_i) = \dfrac{1}{4}, \dfrac{1}{6}, \dfrac{1}{8}$.
The right Riemann sum approximation is

$$\int_1^7 \frac{1}{1+x}\,dx \approx \sum_{i=1}^{3} f(x_i)\Delta x = \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8}\right)2 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{6+4+3}{12} = \frac{13}{12}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

5. (10 points) A right triangle has sides $a = 5$ cm and $b = 12$ cm and hypotenuse $c = 13$ cm. If b is increasing at $\frac{db}{dt} = 3\ \frac{\text{cm}}{\text{sec}}$ while c is increasing at $\frac{dc}{dt} = 2\ \frac{\text{cm}}{\text{sec}}$, at what rate is a changing? Is it increasing or decreasing?

Solution: The sides satisfy $a^2 + b^2 = c^2$. We differentiate this, simplify, plug in numbers and solve for $\frac{da}{dt}$.

$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$ $\qquad a\frac{da}{dt} + b\frac{db}{dt} = c\frac{dc}{dt}$ $\qquad 5\frac{da}{dt} + 12(3) = 13(2)$

$5\frac{da}{dt} = 26 - 36 = -10$ $\qquad \frac{da}{dt} = -2$ $\qquad$ decreasing

6. (10 points) If $g(x) = \int_{e^{-x}}^{e^x} \frac{1}{1 + \ln t}\,dt$, find $g'(x)$ and $g'(0)$.

Solution: Let $G(t)$ be an antiderivative of $\frac{1}{1 + \ln t}$. So $G'(t) = \frac{1}{1 + \ln t}$.
Then

$$g(x) = \Big[G(t)\Big]_{e^{-x}}^{e^x} = G(e^x) - G(e^{-x})$$

By the chain rule:

$$\begin{aligned} g'(x) &= \frac{d}{dx}\Big[G(e^x) - G(e^{-x})\Big] \\ &= G'(e^x)e^x + G'(e^{-x})e^{-x} \\ &= \frac{1}{1 + \ln e^x}e^x + \frac{1}{1 + \ln e^{-x}}e^{-x} \\ &= \frac{1}{1 + x}e^x + \frac{1}{1 - x}e^{-x} \end{aligned}$$

Finally:

$$\begin{aligned} g'(0) &= \frac{1}{1 + 0}e^0 + \frac{1}{1 - 0}e^{-0} \\ &= 1 + 1 = 2 \end{aligned}$$

7. (20 points) For each limit, identify the indeterminate form and then compute the limit:

a. (10 points) $\lim_{x\to 3} \dfrac{x\ln x - x - x\ln 3 + 3}{(x-3)^2}$

Solution: The limit of the numerator is $3\ln 3 - 3 - 3\ln 3 + 3 = 0$ and of the denominator is 0. So the limit has the indeterminate form $\dfrac{0}{0}$.

$$\lim_{x\to 3} \frac{x\ln x - x - x\ln 3 + 3}{(x-3)^2} \overset{l'H}{=} \lim_{x\to 3} \frac{x\frac{1}{x} + \ln x - 1 - \ln 3}{2(x-3)} = \lim_{x\to 3} \frac{\ln x - \ln 3}{2(x-3)} \overset{l'H}{=} \lim_{x\to 3} \frac{\frac{1}{x}}{2} = \frac{1}{6}$$

b. (10 points) $\lim_{x\to 0^+} \left(1 + \dfrac{2x}{3}\right)^{8/x}$

Solution: The limit of the base is 1 and of the exponent is ∞. So the limit has the indeterminate form 1^∞. We do the limit by inserting $e^{\ln}$ and using the identity $\ln a^b = b\ln a$.

$$\lim_{x\to 0^+} \left(1 + \frac{2x}{3}\right)^{8/x} = \lim_{x\to 0^+} e^{\ln\left(1+\frac{2x}{3}\right)^{8/x}} = \lim_{x\to 0^+} e^{\frac{8}{x}\ln\left(1+\frac{2x}{3}\right)} \overset{l'H}{=} \lim_{x\to 0^+} e^{\frac{\frac{8}{1+\frac{2x}{3}}\frac{2}{3}}{1}} = e^{\frac{16}{3}}$$

8. (10 points) Find the smallest value of $f = 6x + y$ on the curve $x^3 y = 2$ in the first quadrant. How do you know this is the minimum?

Solution: The curve is $y = \dfrac{2}{x^3}$. So we need to minimize $f = 6x + \dfrac{2}{x^3}$.

We set the derivative equal to 0 and solve for x:

$f' = 6 - \dfrac{6}{x^4} = 0 \qquad 6 = \dfrac{6}{x^4} \qquad x^4 = 1 \qquad x = 1 \qquad y = 2 \qquad f = 6(1) + (2) = 8$

We check the second derivative: $f'' = \dfrac{24}{x^5}$ At the critical point $f''(1) = 24 > 0$.

This is concave up, so minimum.

9. (15 points) Evaluate each integral.

a. (5 points) $\int \cos x \sin^5 x\, dx$

Solution: Let $u = \sin x$. Then $du = \cos x\, dx$.
$\int \cos x \sin^5 x\, dx = \int u^5\, du = \frac{1}{6}u^6 + C = \frac{1}{6}\sin^6 x + C$

b. (5 points) $\int_0^1 x^2 e^{6x^3}\, dx$

Solution: Let $u = 6x^3$. Then $du = 18x^2\, dx$ and $x^2\, dx0 = \frac{1}{18}\, du$.
If $x = 0$, then $u = 0$. If $x = 1$, then $u = 6$.
$\int_0^1 x^2 e^{6x^3}\, dx = \frac{1}{18}\int_0^6 e^u\, du = \frac{1}{18}e^u \Big|_0^6 = \frac{1}{18}(e^6 - e^0) = \frac{1}{18}(e^6 - 1)$ OR
$\int_0^1 x^2 e^{6x^3}\, dx = \frac{1}{18}\int_{x=0}^1 e^u\, du = \frac{1}{18}e^u \Big|_{x=0}^1 = \frac{1}{18}e^{6x^3} \Big|_0^1 = \frac{1}{18}(e^6 - e^0) = \frac{1}{18}(e^6 - 1)$

c. (5 points) $\int x^5 \sqrt{1 + x^3}\, dx$

Solution: Let $u = 1 + x^3$. Then $du = 3x^2\, dx$ and $\frac{1}{3}\, du = x^2\, dx$ and $x^3 = u - 1$.
$\int x^5 \sqrt{1 + x^3}\, dx = \frac{1}{3}\int (u-1)\sqrt{u}\, du = \frac{1}{3}\int (u^{3/2} - u^{1/2})\, du = \frac{1}{3}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C$
$= \frac{1}{3}\left[\frac{2}{5}(1 + x^3)^{5/2} - \frac{2}{3}(1 + x^3)^{3/2}\right] + C$