Name________________________Section____________

1-8	/51	11	/12
9	/10	12	/10
10	/10	13	/12
		Total	/105

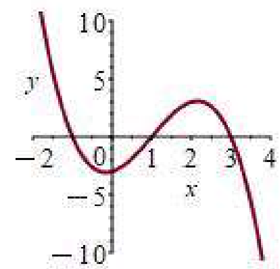
MATH 171 Final Exam A Fall 2022

Section 502/504 Solutions P. Yasskin

Multiple Choice and Short Answer: Points indicated.

Show your work in case there is part credit.

1. (16 points) Consider a function, $y = f(x)$.
At the right is the graph of its **second** derivative, $y = f''(x)$.
Give answers to the nearest integer.



a. (4 points) Find the interval(s) where $f(x)$ is concave up.

Solution: $f(x)$ is concave up where $f''(x)$ is positive which is on $[-2,-1]$ and $[1,3]$.

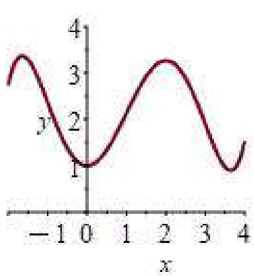
b. (4 points) Find the location(s) of the inflection point(s) of $f(x)$.

Solution: $f(x)$ has an inflection point where $f''(x)$ changes sign, which is at $x = -1, 1, 3$.

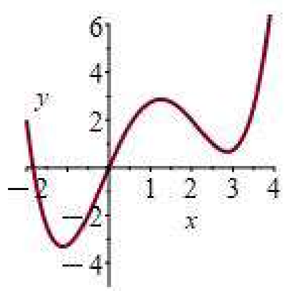
c. (4 points) Find the interval(s) where the first derivative $f'(x)$ is decreasing.

Solution: $f'(x)$ is decreasing where $f''(x)$ is negative which is on $[-1,1]$ and $[3,4]$.

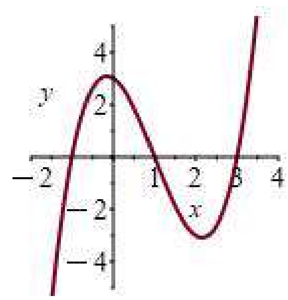
d. (4 points) Which of these is the graph of $y = f(x)$?



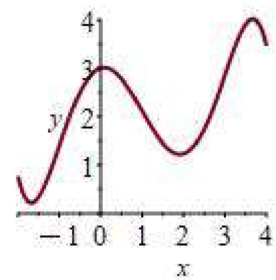
A



B



C



D

Solution: The second derivative is positive, negative, positive, negative.
So the function is concave up, down, up, down, which is plot D.

2. (5 points) Find the length of the projection of the vector $\vec{u} = \langle 8, -4 \rangle$ onto the vector $\vec{v} = \langle 3, 1 \rangle$.

Solution: $\text{comp}_{\vec{v}}\vec{u} = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \dfrac{24 - 4}{\sqrt{9+1}} = \dfrac{20}{\sqrt{10}} = 2\sqrt{10}$

3. (5 points) The point $x = -1$ is a critical point of the function $f(x) = x^4 + \frac{8}{3}x^3 + 2x^2$.
What does the Second Derivative Test say about this point?

a. Local Minimum
b. Local Maximum
c. Inflection Point
d. The test fails

Solution: $f'(x) = 4x^3 + 8x^2 + 4x$ $\quad f''(x) = 12x^2 + 16x + 4$ $\quad f''(-1) = 12 - 16 + 4 = 0$ Test Fails.

4. (5 points) Find the locations of the global minimum and maximum of the function $f(x) = x^3 - 3x^2 - 9x$ on the interval $[0, 4]$.

Solution: $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x+1)(x-3)$ Critical points are $x = -1, 3$.
$x = -1$ is not in the interval. We evaluate the function at the critical points and the endpoints:

$f(0) = 0 \quad f(3) = 27 - 27 - 27 = -27 \quad f(4) = 64 - 48 - 36 = -20$

Global minimum at $x = 3$ Global maximum at $x = 0$

5. (5 points) Find the equation of the line tangent to the curve $x^2y^2 + x^3y^3 = 12$ at $(x, y) = (1, 2)$.
Then find its y-intercept.

Solution: Implicit differentiation:

$2xy^2 + x^2 2y\dfrac{dy}{dx} + 3x^2y^3 + x^3 3y^2\dfrac{dy}{dx} = 0 \quad 8 + 4\dfrac{dy}{dx} + 24 + 12\dfrac{dy}{dx} = 0 \quad 16\dfrac{dy}{dx} = -32 \quad \dfrac{dy}{dx} = -2$

$y = f(x) \quad f(1) = 2 \quad f'(1) = -2$ tangent line is $y = f(1) + f'(1)(x-1)$

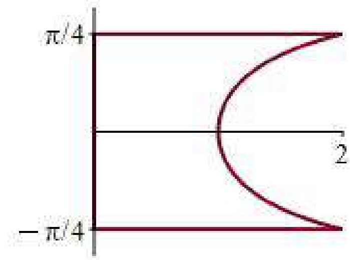
$y = 2 - 2(x-1) = -2x + 4$ $\quad y$-intercept is $b = 4$.

6. (5 points) Find the average value of the function $f(x) = \frac{1}{x}$ on the interval $[1, 4]$. Simplify your answer.

Solution: $f_{\text{ave}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx = \frac{1}{3}\int_1^4 \frac{1}{x}\,dx = \frac{1}{3}\ln x\Big|_1^4 = \frac{1}{3}(\ln 4 - \ln 1) = \frac{1}{3}\ln 4$

7. (5 points) Find the area between the graph of $x = \sec^2 y$ and the y-axis for $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$.

Simplify your answer.



Solution: $A = \displaystyle\int_{-\pi/4}^{\pi/4} \sec^2 y\,dy = \tan y\Big|_{-\pi/4}^{\pi/4} = \left[\tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right)\right] = 2$

8. (5 points) Find the mass and center of mass of a 4 cm bar whose linear mass density is $\delta = 12 + 6x$ where x is measured from one end.

Solution: $M = \displaystyle\int_0^4 \delta\,dx = \int_0^4 (12 + 6x)\,dx = \left[12x + 3x^2\right]_0^4 = 48 + 48 = 96$

$M_1 = \displaystyle\int_0^4 x\delta\,dx = \int_0^4 x(12 + 6x)\,dx = \left[6x^2 + 2x^3\right]_0^4 = 96 + 128 = 224$

$\bar{x} = \dfrac{M_1}{M} = \dfrac{224}{96} = \dfrac{7}{3}$

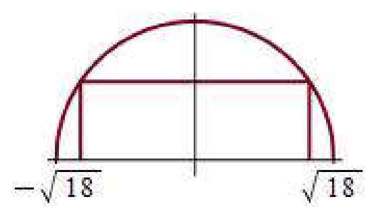
Work Out: (Points indicated. Part credit possible. Show all work.)

9. (10 points) In an alternate universe, objects fall with constant **jerk** instead of constant acceleration. On planet X, objects fall with constant jerk $j = -12\ \frac{\text{m}}{\text{sec}^3}$. If an object is dropped from a height $y = 81$ m with no initial velocity but an initial acceleration of $a(0) = -6\ \frac{\text{m}}{\text{sec}^2}$, how long does it take to hit the ground?

Find the time to the nearest integer.

Solution: The jerk is $j = -12.$
The acceleration is an antiderivative of the jerk. So, $a(t) = -12t + C.$
To find C, we use the initial condition $a(0) = C = -6.$ So $a(t) = -12t - 6.$
The velocity is an antiderivative of the acceleration. So, $v(t) = -6t^2 - 6t + D.$
To find D, we use the initial condition $v(0) = D = 0.$ So $v(t) = -6t^2 - 6t.$
The position is an antiderivative of the velocity. So, $y(t) = -2t^3 - 3t^2 + E.$
To find E, we use the initial condition $y(0) = E = 81.$ So $y(t) = -2t^3 - 3t^2 + 81.$
It hits the ground when $y(t) = 0.$ We try some integers:
$y(1) = -2 - 3 + 81 = 76$ $\quad y(2) = -16 - 12 + 81 = 53$ $\quad y(3) = -54 - 27 + 81 = 0$
So it hits the ground at $t = 3$ sec.

10. (10 points) Find the area of the largest rectangle whose base is on the x-axis and whose upper two vertices are on the semi-circle $y = \sqrt{18 - x^2}$.



Solution: $A = 2xy = 2x\sqrt{18 - x^2}$

$$A'(x) = 2\sqrt{18 - x^2} + 2x\frac{-2x}{2\sqrt{18 - x^2}} = 0 \qquad \frac{2(18 - x^2)}{\sqrt{18 - x^2}} - \frac{2x^2}{\sqrt{18 - x^2}} = 0 \qquad \frac{2(18 - x^2) - 2x^2}{\sqrt{18 - x^2}} = 0$$

$$(18 - x^2) - x^2 = 0 \qquad 18 - 2x^2 = 0 \qquad x = 3 \qquad y = \sqrt{18 - 3^2} = 3 \qquad A = 2 \cdot 3 \cdot 3 = 18$$

11. (12 points) Compute the limits:

a. (6 points) $L = \lim_{x\to 4} \dfrac{\sqrt{5+x} - \sqrt{13-x}}{x-4}$

Solution:

$$L = \lim_{x\to 4} \frac{\sqrt{5+x} - \sqrt{13-x}}{x-4} \cdot \frac{\sqrt{5+x} + \sqrt{13-x}}{\sqrt{5+x} + \sqrt{13-x}} = \lim_{x\to 4} \frac{(5+x)-(13-x)}{(x-4)\left(\sqrt{5+x} + \sqrt{13-x}\right)}$$

$$= \lim_{x\to 4} \frac{2x-8}{(x-4)\left(\sqrt{5+x} + \sqrt{13-x}\right)} = \lim_{x\to 4} \frac{2}{\left(\sqrt{5+x} + \sqrt{13-x}\right)} = \frac{2}{\sqrt{9} + \sqrt{9}} = \frac{2}{6} = \frac{1}{3}$$

b. (6 points) $L = \lim_{x\to\infty} (e^x)^{e^{-x}}$

Solution: $\ln L = \lim_{x\to\infty} \ln\left((e^x)^{e^{-x}}\right) = \lim_{x\to\infty} e^{-x}\ln(e^x) = \lim_{x\to\infty} \dfrac{\ln(e^x)}{e^x} = \lim_{x\to\infty} \dfrac{x}{e^x} \overset{l'H}{=} \lim_{x\to\infty} \dfrac{1}{e^x} = 0$

$L = e^0 = 1$

12. (10 points) If $g(x) = \displaystyle\int_{e^{2x}}^{e^{3x}} \frac{1}{\ln(t)}\,dt$, use Leibniz's method to find $g'(x)$ and $g'(1)$. Simplify exponentials and logs.

Solution: Let $F(t)$ be an antiderivative of $\dfrac{1}{\ln(t)}$. So $F'(t) = \dfrac{1}{\ln(t)}$.

Then

$$g(x) = \left[F(t)\right]_{e^{2x}}^{e^{3x}} = F(e^{3x}) - F(e^{2x})$$

By the chain rule:

$$g'(x) = \frac{d}{dx}\left[F(e^{3x}) - F(e^{2x})\right] = F'(e^{3x})3e^{3x} - F'(e^{2x})2e^{2x}$$

$$= \frac{1}{\ln(e^{3x})}3e^{3x} - \frac{1}{\ln(e^{2x})}2e^{2x} = \frac{1}{3x}3e^{3x} - \frac{1}{2x}2e^{2x}$$

$$= \frac{1}{x}e^{3x} - \frac{1}{x}e^{2x}$$

Finally

$$g'(1) = \frac{1}{1}e^3 - \frac{1}{1}e^2 = e^3 - e^2$$

13. (12 points) Consider the parametric curve $\vec{r} = (t^2 - 4t + 4, 2t^3 - 9t^2 + 12t + 3)$.

a. (6 points) Find the values of t at all horizontal and vertical tangents to the curve.

Solution: $x = t^2 - 4t + 4 \qquad x' = 2t - 4 = 2(t - 2)$
Potential vertical tangent at $t = 2$.
$y = 2t^3 - 9t^2 + 12t + 3 \qquad y' = 6t^2 - 18t + 12 = 6(t^2 - 3t + 2) = 6(t - 2)(t - 1)$
Potential horizontal tangent at $t = 2$ and $t = 1$. However, $t = 2$ cannot be both.
The slope is $m = \dfrac{y'}{x'} = \dfrac{6(t-2)(t-1)}{2(t-2)} = 3(t - 1)$.
So $t = 1$ is a horizontal tangent. There are no vertical tangents.
The point with $t = 2$ has slope $m(2) = 3(2 - 1) = 3$.

b. (6 points) Find a parametric tangent line to the curve at $t = 0$.

Solution: $\vec{r} = (t^2 - 4t + 4, 2t^3 - 9t^2 + 12t + 3) \qquad \vec{r}(0) = (4, 3)$
$\vec{v} = (2t - 4, 6t^2 - 18t + 12) \qquad \vec{v}(0) = (-4, 12)$
$\vec{r}_{\tan}(s) = \vec{r}(0) + s\vec{v}(0) = (4, 3) + s(-4, 12) = (4 - 4s, 3 + 12s)$