Name____________________________Section______________

1-8	/51	11	/12
9	/10	12	/10
10	/10	13	/12
		Total	/105

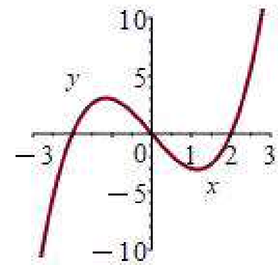
MATH 171 Final Exam B Fall 2022

Section 502/504 Solutions P. Yasskin

Multiple Choice and Short Answer: Points indicated.

Show your work in case there is part credit.

1. (16 points) Consider a function, $y = f(x)$.
At the right is the graph of its **second** derivative, $y = f''(x)$.
Give answers to the nearest integer.



a. (4 points) Find the interval(s) where $f(x)$ is concave down.

Solution: $f(x)$ is concave down where $f''(x)$ is negative which is on $[-3,-2]$ and $[0,2]$.

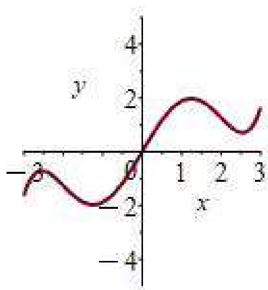
b. (4 points) Find the location(s) of the inflection point(s) of $f(x)$.

Solution: $f(x)$ has an inflection point where $f'(x)$ changes sign, which is at $x = -3, 0, 2$.

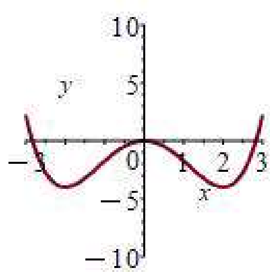
c. (4 points) Find the interval(s) where the first derivative $f'(x)$ is increasing.

Solution: $f'(x)$ is increasing where $f''(x)$ is positive which is on $[-2,0]$ and $[2,3]$.

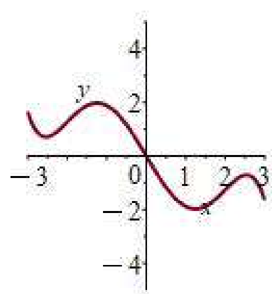
d. (4 points) Which of these is the graph of $y = f(x)$?



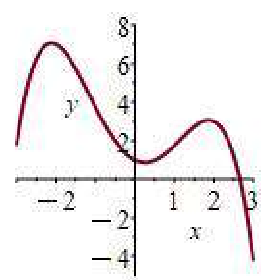
A



B



C



D

Solution: The second derivative is negative, positive, negative, positive.
So the function is concave down, up, down, up, which is plot A.

2. (5 points) A triangle has vertices $A = (1,3)$, $B = (4,4)$ and $C = (5,1)$. Find the angle at A.

Solution: $\overrightarrow{AB} = B - A = (3,1)$ $\quad \overrightarrow{AC} = C - A = (4,-2)$

$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{12-2}{\sqrt{9+1}\,\sqrt{16+4}} = \frac{10}{\sqrt{10}\,\sqrt{20}} = \frac{1}{\sqrt{2}} \qquad \theta = 45°$$

3. (5 points) The point $x = 1$ is a critical point of the function $f(x) = x^4 - \frac{8}{3}x^3 + 2x^2$.
What does the Second Derivative Test say about this point?

- **a**. Local Minimum
- **b**. Local Maximum
- **c**. Inflection Point
- **d**. The test fails

Solution: $f'(x) = 4x^3 - 8x^2 + 4x \quad f''(x) = 12x^2 - 16x + 4 \quad f''(1) = 12 - 16 + 4 = 0$ Test Fails.

4. (5 points) Find the locations of the global minimum and maximum of the function $f(x) = x^4 - 8x^2$ on the interval $[-1,3]$.

Solution: $f'(x) = 4x^3 - 16x = 4x(x^2-4) = 4x(x-2)(x+2)$ Critical points are $x = -2, 0, 2$.
$x = -2$ is not in the interval. We evaluate the function at the critical points and the endpoints:

$$f(-1) = 1 - 8 = -7 \qquad f(0) = 0 \qquad f(2) = 16 - 32 = -16 \qquad f(3) = 81 - 72 = 9$$

Global minimum at $x = 2$ Global maximum at $x = 3$

5. (5 points) Find the equation of the line tangent to the curve $x^4y^4 - 16xy = -16$ at $(x,y) = (2,1)$.
Then find its y-intercept.

Solution: Implicit differentiation:

$$4x^3y^4 + x^4 4y^3\frac{dy}{dx} - 16y - 16x\frac{dy}{dx} = 0 \qquad 32 + 64\frac{dy}{dx} - 16 - 32\frac{dy}{dx} = 0 \qquad 32\frac{dy}{dx} = -16 \qquad \frac{dy}{dx} = -\frac{1}{2}$$

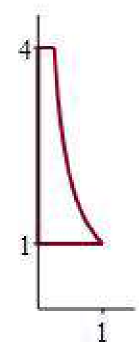
$y = f(x) \quad f(2) = 1 \quad f'(2) = -1$ tangent line is $y = f(2) + f'(2)(x-2)$

$y = 1 - \frac{1}{2}(x-2) = -\frac{1}{2}x + 2$ $\quad$ y-intercept is $b = 2$.

6. (5 points) Find the average value of the function $f(x) = \sec^2 x$ on the interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. Simplify your answer.

Solution: $f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \sec^2 x\,dx = \frac{2}{\pi}\tan x\Big|_{-\pi/4}^{\pi/4} = \frac{2}{\pi}\left[\tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right)\right] = \frac{4}{\pi}$

7. (5 points) Find the area between the graph of $x = \frac{1}{y}$ and the y-axis for $1 \le y \le 4$. Simplify your answer.



Solution: $A = \int_1^4 \frac{1}{y}\,dy = \ln y\Big|_1^4 = (\ln 4 - \ln 1) = \ln 4$

8. (5 points) Find the mass and center of mass of a 3 cm bar whose linear mass density is $\delta = 36 - 6x$ where x is measured from one end.

Solution: $M = \int_0^3 \delta\,dx = \int_0^3 (36-6x)\,dx = \left[36x - 3x^2\right]_0^3 = 108 - 27 = 81$

$M_1 = \int_0^3 x\delta\,dx = \int_0^3 x(36-6x)\,dx = \left[18x^2 - 2x^3\right]_0^3 = 162 - 54 = 108$

$\bar{x} = \frac{M_1}{M} = \frac{108}{81} = \frac{4}{3}$

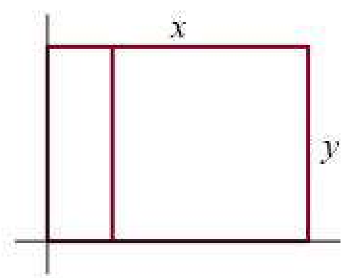
Work Out: (Points indicated. Part credit possible. Show all work.)

9. (10 points) A ball is thrown up from a height of $y(0) = 64$ ft with an initial velocity of $v(0) = 48\ \frac{\text{ft}}{\text{sec}}$. When does it hit the ground?

Note the acceleration of gravity is $a = -32\ \frac{\text{ft}}{\text{sec}^2}$.

Solution: The acceleration is $a = -32$.
The velocity is an antiderivative of the acceleration. So, $v(t) = -32t + C$.
To find C, we use the initial condition $v(0) = C = 48$. So $v(t) = -32t + 48$.
The position is an antiderivative of the velocity. So, $y(t) = -16t^2 + 48t + K$.
To find K, we use the initial condition $y(0) = K = 64$. So $y(t) = -16t^2 + 48t + 64$.
It hits the ground when $y(t) = 0$. We solve:
$-16t^2 + 48t + 64 = 0 \qquad t^2 - 3t - 4 = 0 \qquad (t-4)(t+1) = 0$.
So $t = -1, 4$. The solution $t = -1$ is in the past. So it hits the ground at $t = 4$ sec.

10. (10 points) A farmer wants to fence in a grazing field and split it into two pieces for cows and horses. The field will be a rectangle surrounded by fence with a divider fence parallel to one side. What are the length, width and area of the field which maximize the total area, if the total length of fence is 2400 meters?



Solution: $A = xy \qquad 2x + 3y = 2400 \qquad y = 800 - \frac{2}{3}x \qquad A = 800x - \frac{2}{3}x^2$

$A' = 800 - \frac{4}{3}x = 0 \qquad x = 600 \qquad y = 800 - \frac{2}{3}600 = 400 \qquad A = 600 \cdot 400 = 240\,000$

11. (12 points) Compute the limits:

a. (6 points) $L = \lim_{x \to 2} \left(\frac{1}{x^2 - 2x} - \frac{4}{x^3 - 4x} \right)$

Solution: $L = \lim_{x \to 2} \left(\frac{1}{x(x-2)} - \frac{4}{x(x-2)(x+2)} \right) = \lim_{x \to 2} \left(\frac{x+2}{x(x-2)(x+2)} - \frac{4}{x(x-2)(x+2)} \right)$

$= \lim_{x \to 2} \frac{x-2}{x(x-2)(x+2)} = \lim_{x \to 2} \frac{1}{x(x+2)} = \frac{1}{8}$

b. (6 points) $L = \lim_{x \to \infty} (1 + x^2)^{2/x^2}$

Solution:

$\ln L = \lim_{x \to \infty} \ln(1 + x^2)^{2/x^2} = \lim_{x \to \infty} \frac{2}{x^2} \ln(1 + x^2) = \lim_{x \to \infty} \frac{2\ln(1 + x^2)}{x^2} \overset{l'H}{=} \lim_{x \to \infty} \frac{\frac{2 \cdot 2x}{1 + x^2}}{2x} = \lim_{x \to \infty} \frac{2}{1 + x^2} = 0$

$L = e^0 = 1$

12. (10 points) If $g(x) = \int_{\ln(2x)}^{\ln(3x)} \frac{1}{1 + e^t} dt$, use Leibniz's method to find $g'(x)$ and $g'(1)$. Simplify exponentials and logs.

Solution: Let $F(t)$ be an antiderivative of $\frac{1}{1 + e^t}$. So $F'(t) = \frac{1}{1 + e^t}$.
Then

$$g(x) = \left[F(t) \right]_{\ln(2x)}^{\ln(3x)} = F(\ln(3x)) - F(\ln(2x))$$

By the chain rule:

$$g'(x) = \frac{d}{dx}\left[F(\ln(3x)) - F(\ln(2x)) \right] = F'(\ln(3x))\frac{3}{3x} - F'(\ln(2x))\frac{2}{2x}$$

$$= \frac{1}{1 + e^{\ln(3x)}} \frac{1}{x} - \frac{1}{1 + e^{\ln(2x)}} \frac{1}{x} = \frac{1}{1 + 3x} \frac{1}{x} - \frac{1}{1 + 2x} \frac{1}{x}$$

Finally,

$$g'(1) = \frac{1}{1 + 3} - \frac{1}{1 + 2} = -\frac{1}{12}$$

13. (12 points) Consider the parametric curve $\vec{r} = (2t^3 - 9t^2 + 12t + 3, t^2 - 4t + 4)$.

a. (6 points) Find the values of t at all horizontal and vertical tangents to the curve.

Solution: $x = 2t^3 - 9t^2 + 12t + 3$ $\quad x' = 6t^2 - 18t + 12 = 6(t^2 - 3t + 2) = 6(t-2)(t-1)$
Potential vertical tangent at $t = 2$ and $t = 1$.
$y = t^2 - 4t + 4$ $\quad y' = 2t - 4 = 2(t-2)$
Potential horizontal tangent at $t = 2$. However, $t = 2$ cannot be both.
The slope is $m = \dfrac{y'}{x'} = \dfrac{2(t-2)}{6(t-2)(t-1)} = \dfrac{1}{3(t-1)}$.
So $t = 1$ is a vertical tangent. There are no horizontal tangents.
The point with $t = 2$ has slope $m(2) = \dfrac{1}{3(2-1)} = \dfrac{1}{3}$.

b. (6 points) Find a parametric tangent line to the curve at $t = 0$.

Solution: $\vec{r} = (2t^3 - 9t^2 + 12t + 3, t^2 - 4t + 4)$ $\quad \vec{r}(0) = (3,4)$
$\vec{v} = (6t^2 - 18t + 12, 2t - 4)$ $\quad \vec{v}(0) = (12,-4)$
$\vec{r}_{\tan}(s) = \vec{r}(0) + s\vec{v}(0) = (3,4) + s(12,-4) = (3 + 12s, 4 - 4s)$