Name________________________________

MATH 172 Exam 2 Spring 2018

Sections 501/502 (circle one) Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. Find the general partial fraction expansion of $f(x) = \dfrac{(x+2)^2}{(x^4-16)(x-2)}$.

a. $\dfrac{A}{(x-2)^2} + \dfrac{Bx+C}{x^2+4}$

b. $\dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{Cx+D}{x^2+4}$ correct choice

c. $\dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4}$

d. $\dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4} + \dfrac{Dx+E}{x^2-4}$

e. $\dfrac{A}{(x-2)^2} + \dfrac{Bx+C}{x^2+4} + \dfrac{Dx+E}{(x^2+4)^2}$

Solution: $f(x) = \dfrac{(x+2)^2}{(x^2+4)(x+2)(x-2)^2} = \dfrac{(x+2)}{(x^2+4)(x-2)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{Cx+D}{x^2+4}$

2. Given the partial fraction expansion:

$$\frac{x^2+32x-4}{x^4-16} = \frac{2}{x-2} + \frac{2}{x+2} + \frac{-4x+1}{x^2+4}$$

which term in the following integral is INCORRECT?

$$\int \frac{x^2+32x-4}{x^4-16}\,dx = \underbrace{\ln|x-2|^2}_{\text{A}} + \underbrace{\ln|x+2|^2}_{\text{B}} - \underbrace{\ln|x^2+4|^2}_{\text{C}} + \underbrace{\frac{1}{2}\arctan\left(\frac{x}{2}\right)}_{\text{D}}$$

a. A

b. B

c. C

d. D

e. They are all correct. correct choice

Solution: Simplfy using a log identity and then differentiate:

$$\frac{d}{dx}\ln|x-2|^2 = \frac{d}{dx}2\ln|x-2| = \frac{2}{x-2}$$

$$\frac{d}{dx}\ln|x+2|^2 = \frac{d}{dx}2\ln|x+2| = \frac{2}{x+2}$$

$$\frac{d}{dx} - \ln|x^2+4|^2 = \frac{d}{dx} - 2\ln|x^2+4| = \frac{-2(2x)}{x^2+4} = \frac{-4x}{x^2+4}$$

$$\frac{d}{dx}\frac{1}{2}\arctan\left(\frac{x}{2}\right) = \frac{1}{2}\frac{1}{1+\left(\frac{x}{2}\right)^2}\frac{1}{2} = \frac{1}{4+x^2}$$

They are all correct.

1-9	/54	11	/15
10	/15	12	/20
		Total	/104

3. A spring has a rest length of $x_0 = 5$ m. It requires 12 N of force to hold the spring at $x = 7$ m. Find the work done to stretch the spring from $x = 6$ m to $x = 8$ m.

a. 6

b. 8

c. 12

d. 18

e. 24 correct choice

Solution: $F = k(x - x_0)$ $\quad 12 = k(7-5)$ $\quad k = 6$ $\quad F = 6(x-5)$

$W = \int F\,dx = \int_6^8 6(x-5)\,dx = 3(x-5)^2\Big|_6^8 = 3(3)^2 - 3(1)^2 = 24$

4. A 40 ft rope hangs from the top of a building. It's linear weight density is $\rho = 3$ lb/ft. How much work is done to lift the rope to the top of the building?

a. 2400 ft-lb correct choice

b. 1800 ft-lb

c. 1200 ft-lb

d. 600 ft-lb

e. 300 ft-lb

Solution: The piece of rope of length dy which is y ft from the top of the building is lifted $D = y$ ft and has weight $dF = 3\,dy$. So the work is

$$W = \int D\,dF = \int_0^{40} y3\,dy = 3\frac{y^2}{2}\Big|_0^{40} = 3\frac{40^2}{2} = 2400 \text{ ft-lb}$$

5. The region between $y = \sin x$ and the x-axis for $0 \le x \le \pi$ is rotated about the x-axis. Find the volume swept out.

a. $\frac{\pi}{4}$

b. $\frac{\pi}{2}$

c. π^2

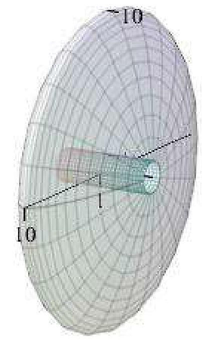
d. $\frac{\pi^2}{2}$ correct choice

e. $\frac{\pi^2}{4}$

Solution: This is an x-integral. The slices are vertical and rotate about the x-axis to give disks. So the volume is:

$$V = \int_0^\pi \pi R^2\,dx = \int_0^\pi \pi\sin^2 x\,dx = \int_0^\pi \pi\frac{1-\cos 2x}{2}\,dx = \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{\pi^2}{2}$$

6. The region between $y = 10 - x^2$ and $y = 1$ is rotated about the x-axis. Which integral gives the volume swept out?



a. $V = 2\pi \int_0^3 (9x - x^3)\, dx$

b. $V = \pi \int_0^3 (9x - x^3)\, dx$

c. $V = 2\pi \int_{-3}^3 (9x - x^3)\, dx$

d. $V = 2\pi \int_{-3}^3 (x^4 - 20x^2 + 99)\, dx$

e. $V = \pi \int_{-3}^3 (x^4 - 20x^2 + 99)\, dx$ correct choice

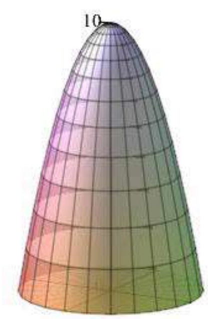
Solution: Each y is a function of x. So we do an x-integral. They intersect when

$$10 - x^2 = 1 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = \pm 3$$

The slices are vertical and rotate about the x-axis into washers.
The inner radius is $r = 1$ and the outer radius is $R = 10 - x^2$.
So the volume is

$$V = \int_{-3}^3 \pi R^2 - \pi r^2\, dx = \pi \int_{-3}^3 (10 - x^2)^2 - (1)^2\, dx = \pi \int_{-3}^3 (x^4 - 20x^2 + 99)\, dx$$

7. The region between $y = 10 - x^2$ and $y = 1$ is rotated about the y-axis. Find the volume swept out.



a. $\dfrac{81\pi}{4}$

b. $\dfrac{81\pi}{2}$ correct choice

c. 18π

d. 36π

e. 81π

Solution: Each y is a function of x. So we do an x-integral.
The slices are vertical and rotate about the y-axis into cylinders.
The radius is $r = x$ and the height is $h = (10 - x^2) - 1 = 9 - x^2$.
The functions intersect when

$$10 - x^2 = 1 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = \pm 3$$

However, if we integrate from $x = -3$ to $x = 3$, we are double counting the volume.
So we integrate from $x = 0$ to $x = 3$. So the volume is

$$V = \int_0^3 2\pi r h\, dx = 2\pi \int_0^3 x(9 - x^2)\, dx = 2\pi \int_0^3 (9x - x^3)\, dx = 2\pi \left[9\frac{x^2}{2} - \frac{x^4}{4} \right]_0^3 = \frac{81\pi}{2}$$

8. Compute $\int_0^2 \frac{1}{(x-1)^2}\,dx$.

a. -2

b. 0

c. 2

d. diverges to $+\infty$ correct choice

e. diverges to $-\infty$

Solution: $\int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx = \left[\frac{-1}{x-1}\right]_0^1 + \left[\frac{-1}{x-1}\right]_1^2$

$= \lim_{x\to 1^-}\left[\frac{-1}{x-1}\right] - \left[\frac{-1}{0-1}\right] + \left[\frac{-1}{2-1}\right] - \lim_{x\to 1^+}\left[\frac{-1}{x-1}\right] = " \frac{-1}{0^-} " - 1 - 1 - " \frac{-1}{0^+} " = \infty - 2 + \infty = \infty$

9. Compute $\int_2^\infty \frac{1}{(x-1)^2}\,dx$.

a. -1

b. 0

c. 1 correct choice

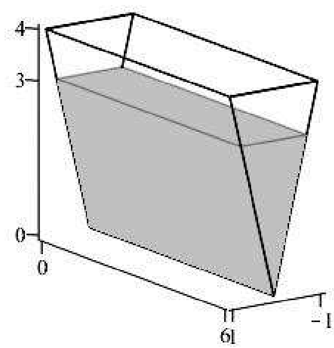
d. diverges to $+\infty$

e. diverges to $-\infty$

Solution: $\int_2^\infty \frac{1}{(x-1)^2}\,dx = \left[\frac{-1}{x-1}\right]_2^\infty = \lim_{x\to\infty}\left[\frac{-1}{x-1}\right] - \left[\frac{-1}{2-1}\right] = " \frac{-1}{\infty} " - \frac{-1}{1} = 0 + 1 = 1$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (15 points) The tank shown is 6 m long, 2 m wide at the top and 4 m high. It is filled with water to a depth of 3 m. How much work is done to pump the water out the top of the tank? Take the density of water to be ρ kg/m^3 and the acceleration of gravity to be g m/sec^2. (You don't need numbers for ρ and g.)



Solution: Put the 0 of the y-axis at the bottom of the tank and measure y upward.
The slice at height y has to be lifted a distance $D = 4 - y$.
The slice at height y is a rectangle of length 6 and width w, and so area $A = 6w$.
By similar triangles $\frac{w}{y} = \frac{2}{4} = \frac{1}{2}$. So $w = \frac{y}{2}$ and $A = 6\frac{y}{2} = 3y$.
The slice at height y with thickness dy has volume $dV = A\,dy = 3y\,dy$ and weight $dF = \rho g\,dV = 3\rho g y\,dy$.
There is water for $0 \le y \le 3$. (This is the tricky part.) So the work is:

$$W = \int_0^3 D\,dF = \int_0^3 (4-y)\,3\rho g y\,dy = \rho g\left[6y^2 - y^3\right]_0^3 = \rho g(54-27) = 27\rho g$$

11. (15 points) Find the coefficients in the partial fraction expansion

$$\frac{10}{(x^2+4)(x^2-1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+1} + \frac{D}{x-1}$$

Solution: Clear the denominator:

$$10 = (Ax+B)(x^2-1) + C(x^2+4)(x-1) + D(x^2+4)(x+1)$$

Plug in $x = 1$: $10 = (A+B)(0) + C(1+4)(0) + D(1+4)(1+1) = 10D$

$\Rightarrow \quad D = 1$

Plug in $x = -1$: $10 = (-A+B)(0) + C(1+4)(-2) + D(1+4)(0) = -10C$

$\Rightarrow \quad C = -1$

Plug in $x = 0$: $10 = B(-1) + C(4)(-1) + D(4)(1) = -B - 4C + 4D = -B + 8$

$\Rightarrow \quad B = -2$

Coeff of x^3: $0 = A + C + D = A - 1 + 1 = A$

$\Rightarrow \quad A = 0$

$A = 0$
$B = -2$
$C = -1$
$D = 1$

12. (20 points) Use a Comparison Theorem to determine whether each of the following integrals converges or diverges. Clearly state the comparison integral, why the comparison integral converges or diverges and why the original integral converges or diverges.

a. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}+x^2}\,dx$

Solution: For large x, we know $x^2 \gg \sqrt{x}$. So we take the comparison integral to be $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$.

We compute: $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \left[\frac{-1}{x}\right]_1^\infty = 0 - \frac{-1}{1} = 1$ which converges.

Further, $\sqrt{x}+x^2 > x^2$. So $\displaystyle\frac{1}{\sqrt{x}+x^2} < \frac{1}{x^2}$. So $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}+x^2}\,dx < \int_1^\infty \frac{1}{x^2}\,dx = 1$.

So $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}+x^2}\,dx$ also converges.

b. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}+x^2}\,dx$

Solution: For small x, we know $\sqrt{x} \gg x^2$. So we take the comparison integral to be $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$.

We compute: $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = [2\sqrt{x}]_0^1 = 2 - 2\sqrt{0} = 2$ which converges.

Further, $\sqrt{x}+x^2 > \sqrt{x}$. So $\displaystyle\frac{1}{\sqrt{x}+x^2} < \frac{1}{\sqrt{x}}$. So $\displaystyle\int_0^1 \frac{1}{\sqrt{x}+x^2}\,dx < \int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$.

So $\displaystyle\int_0^1 \frac{1}{\sqrt{x}+x^2}\,dx$ also converges.

c. What do parts (a) and (b) say about the convergence or divergence of $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}+x^2}\,dx$?

Solution: $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}+x^2}\,dx = \int_0^1 \frac{1}{\sqrt{x}+x^2}\,dx + \int_1^\infty \frac{1}{\sqrt{x}+x^2}\,dx$ also converges.