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MATH 172 Exam 1 Spring 2019

Sections 501 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the area between $y = x^2 - 8$ and $y = 2x$.

a. 24

b. $\frac{80}{3}$

c. 36 correct choice

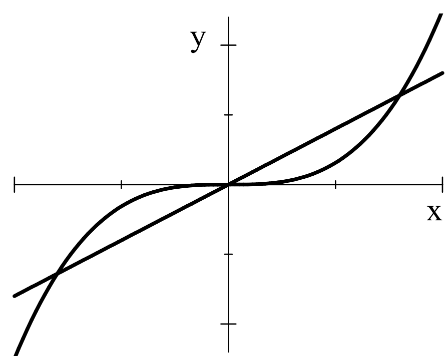
d. $\frac{124}{3}$

e. 48

Solution: The curves intersect when $x^2 - 8 = 2x$ or $0 = x^2 - 2x - 8 = (x+2)(x-4)$ or $x = -2, 4$.

$$A = \int_{-2}^{4} (2x - x^2 + 8)\, dx = \left[x^2 - \frac{x^3}{3} + 8x \right]_{-2}^{4} = \left(16 - \frac{64}{3} + 32\right) - \left(4 - \frac{-8}{3} - 16\right) = 60 - \frac{72}{3} = 36$$

2. Find the area between $y = x^3$ and $y = 16x$.



a. 32

b. 36

c. 48

d. 64

e. 128 correct choice

Solution: The curves intersect when $x^3 = 16x$ or $0 = x^3 - 16x = x(x^2 - 16)$ or $x = 0, \pm 4$.
Since the region is symmetric, we can double the right half:

$$A = 2\int_0^4 16x - x^3\, dx = 2\left[8x^2 - \frac{x^4}{4} \right]_0^4 = 2(8 \cdot 16 - 64) = 128$$

1-13	/65	15	/10
14	/20	16	/15
		Total	/110

3. Find the area between $x = 36 - y^2$ and the y-axis

a. 108
b. 216
c. 144
d. 288 correct choice
e. 432

Solution: The parabola intersects the y-axis at $y = \pm 6$. So
$$A = \int_{-6}^{6} (36 - y^2)\, dy = \left[36y - \frac{y^3}{3} \right]_{-6}^{6} = 2(36 \cdot 6 - 36 \cdot 2) = 288$$

4. Compute $\int_0^{\sqrt{\pi}} x \sin(x^2)\, dx$.

a. 1 correct choice
b. 2
c. 3
d. 4
e. 6

Solution: Let $u = x^2$. Then $du = 2x\, dx$ and
$$\int_0^{\sqrt{\pi}} x \sin(x^2)\, dx = \frac{1}{2} \int_0^{\pi} \sin u\, du = \left[-\frac{1}{2} \cos u \right]_0^{\pi} = \frac{1}{2} + \frac{1}{2} = 1$$

5. Compute $\int (x^2 + 1) e^{2x}\, dx$.

a. $\frac{1}{2}(x^2 + 1)e^{2x} - \frac{1}{4}xe^{2x} + \frac{1}{4}e^{2x} + C$
b. $\frac{1}{2}(x^2 + 1)e^{2x} - \frac{1}{2}xe^{2x} - \frac{1}{2}e^{2x} + C$
c. $\frac{1}{2}(x^2 + 1)e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$ correct choice
d. $\frac{1}{2}(x^2 + 1)e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}e^{2x} + C$
e. $\frac{1}{2}(x^2 + 1)e^{2x} - \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

Solution: Use parts with $u = x^2 + 1$, $dv = e^{2x}\, dx$, $du = 2x\, dx$, $v = \frac{1}{2}e^{2x}$
$$I = \frac{1}{2}(x^2 + 1)e^{2x} - \int xe^{2x}\, dx$$

Now use parts with $u = x$, $dv = e^{2x}\, dx$, $du = dx$, $v = \frac{1}{2}e^{2x}$
$$I = \frac{1}{2}(x^2 + 1)e^{2x} - \left[\frac{1}{2}xe^{2x} - \frac{1}{2} \int e^{2x}\, dx \right]$$

$$I = \frac{1}{2}(x^2 + 1)e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$$

6. Find the average value of the function $f(x) = 9 - x^2$ on the interval $[0, 3]$.

a. $\frac{27}{4}$

b. 6 correct choice

c. 5

d. $\frac{9}{2}$

e. 3

Solution: $f_{\text{ave}} = \frac{1}{3}\int_0^3 (9 - x^2)\,dx = \frac{1}{3}\left[9x - \frac{x^3}{3}\right]_0^3 = \frac{1}{3}(27 - 9) = 6$

7. Find the length of the parametric curve $x = t^4$ and $y = \frac{1}{2}t^6$ for $0 \le t \le 1$.

a. $\frac{13}{6}$

b. $\frac{13}{3}$

c. $\frac{13}{2}$

d. $\frac{1}{54}$

e. $\frac{61}{54}$ correct choice

Solution: $\frac{dx}{dt} = 4t^3 \qquad \frac{dy}{dt} = 3t^5$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 \sqrt{(4t^3)^2 + (3t^5)^2}\,dt = \int_0^1 \sqrt{16t^6 + 9t^{10}}\,dt = \int_0^1 t^3\sqrt{16 + 9t^4}\,dt$$

Let $u = 16 + 9t^4$. Then $du = 36t^3 dt$ and $\frac{1}{36}du = t^3 dt$. So

$$L = \frac{1}{36}\int_{16}^{25} \sqrt{u}\,du = \frac{1}{36}\left[\frac{2u^{3/2}}{3}\right]_{16}^{25} = \frac{1}{54}(25^{3/2} - 16^{3/2}) = \frac{1}{54}(125 - 64) = \frac{61}{54}$$

8. The curve $y = x^3$ for $0 \le x \le 2$ is rotated about the x-axis. Find the surface area.

a. $\frac{\pi}{27}2^{3/2}$

b. $\frac{\pi}{12}(2^{3/2} - 1)$

c. $\frac{\pi}{27}(145^{3/2} - 1)$ correct choice

d. $\frac{\pi}{12}(145^{3/2} - 1)$

e. $\frac{\pi}{12}145^{3/2}$

Solution: $\frac{dy}{dx} = 3x^2$. The radius is $r = y = x^3$. So the surface area is:

$$A = \int 2\pi r\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 2\pi x^3\sqrt{1 + (3x^2)^2}\,dx = \int_0^2 2\pi x^3\sqrt{1 + 9x^4}\,dx$$

Let $u = 1 + 9x^4$. Then $du = 36x^3 dx$ and $\frac{1}{36}du = x^3 dx$. So

$$A = \frac{1}{36}\int_1^{145} 2\pi\sqrt{u}\,du = \left[\frac{\pi}{18}\frac{2u^{3/2}}{3}\right]_1^{145} = \frac{\pi}{27}(145^{3/2} - 1)$$

9. Compute $\int_0^{\pi} \sin^3\theta \cos^2\theta \, d\theta$.

a. $\frac{2}{5}$

b. $\frac{2}{3}$

c. $\frac{2}{15}$

d. $\frac{4}{15}$ correct choice

e. $\frac{8}{15}$

Solution: Let $u = \cos\theta$. Then $du = -\sin\theta \, d\theta$ and $\sin^2\theta = 1 - \cos^2\theta = 1 - u^2$. So
$\int_0^{\pi} \sin^3\theta \cos^2\theta \, d\theta = -\int_1^{-1} (1 - u^2)u^2 \, du = -\left[\frac{u^3}{3} - \frac{u^5}{5} \right]_1^{-1} = -2\left(\frac{-1}{3} - \frac{-1}{5} \right) = \frac{4}{15}$

10. Compute $\int_{-\pi/4}^{\pi/4} \tan^4\theta \sec^2\theta \, d\theta$.

a. $\frac{2}{5}$ correct choice

b. $\frac{2}{3}$

c. $\frac{2}{15}$

d. $\frac{4}{15}$

e. $\frac{8}{15}$

Solution: Let $u = \tan\theta$. Then $du = \sec^2\theta \, d\theta$. So
$\int_{-\pi/4}^{\pi/4} \tan^4\theta \sec^2\theta \, d\theta = \int_{-1}^{1} u^4 \, du = \left[\frac{u^5}{5} \right]_{-1}^{1} = \left(\frac{1}{5} - \frac{-1}{5} \right) = \frac{2}{5}$

11. Compute $\int_0^{\pi/4} \tan^3\theta \sec^3\theta \, d\theta$.

a. $\frac{2}{15}\left(\sqrt{2} - 1\right)$

b. $\frac{2}{15}\left(\sqrt{2} + 1\right)$ correct choice

c. $\frac{1}{15}\left(\sqrt{2} + 1\right)$

d. $\frac{1}{15}\left(\sqrt{2} - 1\right)$

e. $\frac{2}{15}\left(1 - \sqrt{2}\right)$

Solution: Let $u = \sec\theta$. Then $du = \sec\theta \tan\theta \, d\theta$ and $\tan^2\theta = \sec^2\theta - 1 = u^2 - 1$. So
$\int_0^{\pi/4} \tan^3\theta \sec^3\theta \, d\theta = \int_1^{\sqrt{2}} (u^2 - 1)u^2 \, du = \left[\frac{u^5}{5} - \frac{u^3}{3} \right]_1^{\sqrt{2}} = \left(\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} \right) - \left(\frac{1}{5} - \frac{1}{3} \right)$
$= \frac{12 - 10}{15}\sqrt{2} - \frac{3 - 5}{15} = \frac{2}{15}\left(\sqrt{2} + 1\right)$

12. Compute $\int \frac{1}{(9+x^2)^{3/2}}\,dx$

a. $\frac{1}{9\sqrt{9+x^2}} + C$

b. $\frac{x}{9\sqrt{9+x^2}} + C$ correct choice

c. $\frac{1}{3\sqrt{9+x^2}} + C$

d. $\frac{\sqrt{9+x^2}}{9x} + C$

e. $\frac{\sqrt{9+x^2}}{3x} + C$

Solution: Let $x = 3\tan\theta$. Then $dx = 3\sec^2\theta\,d\theta$. So

$$I = \int \frac{1}{(9+x^2)^{3/2}}\,dx = \int \frac{1}{(9+9\tan^2\theta)^{3/2}}\,3\sec^2\theta\,d\theta = \frac{1}{9}\int \frac{1}{(1+\tan^2\theta)^{3/2}}\,\sec^2\theta\,d\theta = \frac{1}{9}\int \frac{1}{\sec\theta}\,d\theta$$
$$= \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C$$

Draw a triangle with opposite side x, adjacent side 3 and hypotenous $\sqrt{9+x^2}$. So

$$I = \frac{x}{9\sqrt{9+x^2}} + C$$

13. Compute $\int \frac{1}{\sqrt{x^2-4}}\,dx$.

a. $\frac{1}{x}\ln\left|\sqrt{x^2-4}\right| + C$

b. $\ln\left|\sqrt{x^2-4}\right| + C$

c. $\ln\left|x - \sqrt{x^2-4}\right| + C$

d. $\ln\left|x + \sqrt{x^2-4}\right| + C$ correct choice

e. $\ln\left|\frac{2}{x} + \frac{\sqrt{x^2-4}}{2}\right| + C$

Solution: Let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$. So

$$I = \int \frac{1}{\sqrt{x^2-4}}\,dx = \int \frac{1}{\sqrt{4\sec^2\theta - 4}}\,2\sec\theta\tan\theta\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

Draw a triangle with hypotenous x, adjacent side 2 and opposite side $\sqrt{x^2-4}$.

Then $\sec\theta = \frac{x}{2}$ and $\tan\theta = \frac{\sqrt{x^2-4}}{2}$. So

$$I = \ln\left|\frac{x+\sqrt{x^2-4}}{2}\right| + C = \ln\left|x + \sqrt{x^2-4}\right| + C - \ln 2$$

Work Out: (Points indicated. Part credit possible. Show all work.)

14. $(20$ points$)$ A 10 cm bar has linear density $\delta = e^{-x}$ g/cm where x is measured from one end.

a. Find the total mass of the bar.

Solution: $M = \int \delta\, dx = \int_0^{10} e^{-x}\, dx = [-e^{-x}]_0^{10} = -e^{-10} + e^0 = 1 - e^{-10}$

b. Find the center of mass of the bar.

Solution: $M_1 = \int x\delta\, dx = \int_0^{10} xe^{-x}\, dx$ Use parts with $\begin{array}{ll} u = x & dv = e^{-x}\, dx \\ du = dx & v = -e^{-x} \end{array}$

$$M_1 = \left[-xe^{-x} + \int e^{-x}\, dx\right]_0^{10} = [-xe^{-x} - e^{-x}]_0^{10} = (-10e^{-10} - e^{-10}) - (-e^0) = 1 - 11e^{-10}$$

$$\bar{x} = \frac{M_1}{M} = \frac{1 - 11e^{-10}}{1 - e^{-10}}$$

15. $(10$ points$)$ Compute $\int x \arctan x\, dx$.

HINT: To complete the last integral, add and subtract 1 in the numerator.

Solution: Parts with $\begin{array}{ll} u = \arctan x & dv = x\, dx \\ du = \dfrac{1}{1+x^2}\, dx & v = \dfrac{1}{2}x^2 \end{array}$. Then

$$\int x \arctan x\, dx = \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\, dx = \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\, dx$$

$$= \frac{1}{2}x^2 \arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C$$

16. $(15$ points$)$ Compute $\int e^{3x} \cos 4x\, dx$.

Solution: Use parts with $\begin{array}{ll} u = \cos 4x & dv = e^{3x}\, dx \\ du = -4 \sin 4x\, dx & v = \dfrac{1}{3}e^{3x} \end{array}$. Then

$$I = \int e^{3x} \cos 4x\, dx = \frac{1}{3}e^{3x} \cos 4x + \frac{4}{3}\int e^{3x} \sin 4x\, dx$$

Next use parts with $\begin{array}{ll} u = \sin 4x & dv = e^{3x}\, dx \\ du = 4 \cos 4x\, dx & v = \dfrac{1}{3}e^{3x} \end{array}$

$$I = \frac{1}{3}e^{3x} \cos 4x + \frac{4}{3}\left[\frac{1}{3}e^{3x} \sin 4x - \frac{4}{3}\int e^{3x} \cos 4x\, dx\right] = \frac{1}{3}e^{3x} \cos 4x + \frac{4}{9}e^{3x} \sin 4x - \frac{16}{9}I$$

$$I + \frac{16}{9}I = \frac{1}{3}e^{3x} \cos 4x + \frac{4}{9}e^{3x} \sin 4x$$

$$I = \frac{9}{25}\left(\frac{1}{3}e^{3x} \cos 4x + \frac{4}{9}e^{3x} \sin 4x\right) + C = \frac{3}{25}e^{3x} \cos 4x + \frac{4}{25}e^{3x} \sin 4x + C$$